Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 36(1), 2009, Pages 122–144
ISSN: 1223-6934

The localization of pseudo MTL - algebras

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Abstract. In this paper we develope a theory of localization for pseudo MTL - algebras. For commutative case see [15] and [16].

2000 Mathematics Subject Classification. Primary 18B35; Secondary 18C05.
Key words and phrases. MTL - algebra, pseudo MTL - algebra, residuated lattice, topology, multiplier, $\mathcal{F}-$ multiplier, strong $\mathcal{F}-$ multiplier, $\wedge$ - closed system, pseudo MTL - algebra of fractions, maximal pseudo MTL - algebra of quotients, pseudo MTL - algebra of localization.

1. Introduction

Basic Fuzzy logic (BL from now on) is the many-valued residuated logic introduced by Hájek in [11] to cope with the logic of continuous t-norms and their residua. Monoidal logic (ML from now on), is a logic whose algebraic counterpart is the class of residuated; MTL-algebras (see [7]) are algebraic structures for the Esteva-Godo monoidal t-norm based logic (MTL), a many-valued propositional calculus that formalizes the structure of the real unit interval $[0, 1]$, induced by a left–continuous t-norm.

Pseudo $BL-$ algebras were introduced by A. Di Nola, G. Georgescu and A. Iorgulescu in [6] as a non-commutative extension of Hájek's $BL-$algebras. Pseudo $BL-$algebras are bounded non-commutative residuated lattices $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ which satisfy the *pseudo-divisibility condition* $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$ and the *pseudo-prelinearity condition* $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

Depending on the above conditions, there are two directions to extend pseudo $BL-$algebras. One direction investigates the (bounded) non-commutative residuated lattices satisfying the pseudo-divisibility condition which were studied under the name *(bounded) divisible pseudo - residuated lattices* or *bounded Rl - monoids*. The second direction deals with (bounded) non-commutative residuated lattices with the pseudo-prelinearity condition, that is *pseudo $MTL-$ algebras*.

Pseudo MTL algebras were in [8] under the name *weak-BL algebras* in order to obtain a structure on $[0, 1]$, since there are not pseudo $BL-$algebras on $[0, 1]$.

So, Pseudo $MTL-$ algebras are non-commutative fuzzy structures which arise from pseudo t-norms, namely, pseudo $BL-$algebras without the pseudo-divisibility condition.

In this paper we develope a theory of localization for pseudo MTL - algebras and we deal with generalizations of results which are obtained in [15] and [16].

This paper is organized as follows: In Section 2 we recall the basic definitions and we put in evidence many rules of calculus in pseudo MTL - algebras and a characterizations for the boolean elements in a pseudo MTL - algebra. In Section 3

Received: 11 March 2009.

we introduce the pseudo MTL - algebra of fractions relative to a $\wedge-$ closed system. In Section 4 we develop a theory for strong multipliers on a pseudo MTL - algebra and in Section 5 we define the notions of pseudo MTL - algebra of fractions and maximal pseudo MTL - algebra of quotients for a pseudo MTL - algebra. In the least part of this section it is proved the existence of the maximal pseudo MTL - algebra of quotients.

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A.

Using the model of localization ring, in [10], G. Georgescu defined for a bounded distributive lattice L *the localization lattice* $L_{\mathcal{F}}$ *of* L *with respect to a topology* $\mathcal{F}$ *on* L and prove that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals); analogous results we have for the lattice of fractions of a bounded distributive lattice relative to a $\wedge-$ closed system.

In Sections 6 and 7 we develop a theory of localization for pseudo MTL - algebras. So, for a pseudo MTL - algebra A we define the notion of localization pseudo MTL - algebra relative to a topology $\mathcal{F}$ on A and in Section 8 we describe the localization pseudo MTL - algebra $A_{\mathcal{F}}$ in some special instances.

Since $MTL-$ algebras are particular classes of pseudo $MTL-$ algebras, the results of this paper generalize a part of the results from [15], [16] for $MTL-$ algebras.

2. Definitions and preliminaries

Definition 2.1. *A* pseudo MTL- algebra *(*[8]*) is an algebra* $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 2, 0, 0)$ *equipped with an order* $\leq$ *satisfying the following axioms:*

(a_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice relative to the order* $\leq$*;*

(a_2) $(A, \odot, 1)$ *is a monoid;*

(a_3) $x \odot y \leq z$ *iff* $x \leq y \rightarrow z$ *iff* $y \leq x \rightsquigarrow z$*, for every* $x, y, z \in A$;

(a_4) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$*, for every* $x, y \in A$ *(pseudo-prelinearity).*

Remark 2.1. *If* A *satisfies only the axioms* a_1, a_2 *and* a_3 *then* A *is called a* residuated lattice.

Remark 2.2. *If additionally for any* $x, y \in A$ *the structure* A *by Definition 2.1 satisfies the axiom*

(a_5)**:** $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ *(pseudo-divisibility), then* A *is a* pseudo BL- algebra.

Remark 2.3. *If* A *satisfies the axioms* a_1, a_2, a_3 *and* a_5 *then it is a* bounded divisible residuated lattice. *These structures were also studied under the name* bounded RL-monoids.

Remark 2.4. *A pseudo MTL- algebra* A *is called commutative if the operation* $\odot$ *is commutative. In this case the operations* $\rightarrow$ *and* $\rightsquigarrow$ *coincide, and thus, a commutative pseudo-*MTL *algebra is a* MTL *algebra.*

A totally ordered pseudo-MTL algebra is called a *chain*.

For examples of pseudo-MTL algebras see [4] and [12].

In [4], [6], [8], [12] it is proved that if A is a residuated lattice and $a, a_1, ..., a_n, b, b_i, c \in A$, $(i \in I)$ then we have the following rules of calculus:

(c_1) $a \odot (a \rightsquigarrow b) \leq b \leq a \rightsquigarrow (a \odot b)$ and $a \odot (a \rightsquigarrow b) \leq a \leq b \rightsquigarrow (b \odot a)$,

(c_2) $(a \rightarrow b) \odot a \leq a \leq b \rightarrow (a \odot b)$ and $(a \rightarrow b) \odot a \leq b \leq a \rightarrow (b \odot a)$,
(c_3) if $a \leq b$ then $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$,
(c_4) if $a \leq b$ then $c \rightsquigarrow a \leq c \rightsquigarrow b$ and $c \rightarrow a \leq c \rightarrow b$,
(c_5) if $a \leq b$ then $b \rightsquigarrow c \leq a \rightsquigarrow c$ and $b \rightarrow c \leq a \rightarrow c$,
(c_6) $a \leq b$ iff $a \rightarrow b = 1$ iff $a \rightsquigarrow b = 1$,
(c_7) $a \rightsquigarrow a = a \rightarrow a = 1$,
(c_8) $1 \rightsquigarrow a = 1 \rightarrow a = a$,
(c_9) $b \leq a \rightsquigarrow b$ and $b \leq a \rightarrow b$,
(c_{10}) $a \odot b \leq a \wedge b$ and $a \odot b \leq a, b$,
(c_{11}) $a \rightsquigarrow 1 = a \rightarrow 1 = 1$,
(c_{12}) $a \rightsquigarrow b \leq (c \odot a) \rightsquigarrow (c \odot b)$,
(c_{13}) $a \rightarrow b \leq (a \odot c) \rightarrow (b \odot c)$,
(c_{14}) if $a \leq b$ then $a \leq c \rightsquigarrow b$ and $a \leq c \rightarrow b$,
(c_{15}) $(b \rightsquigarrow c) \odot a \leq b \rightsquigarrow (c \odot a)$ and $a \odot (b \rightarrow c) \leq b \rightarrow (a \odot c)$,
(c_{16}) if $a \leq b$ then $b \rightsquigarrow 0 \leq a \rightsquigarrow 0$ and $b \rightarrow 0 \leq a \rightarrow 0$,
(c_{17}) $0 \odot a = a \odot 0 = 0$,
(c_{18}) $(a \rightsquigarrow b) \odot (b \rightsquigarrow c) \leq a \rightsquigarrow c$ and $(b \rightarrow c) \odot (a \rightarrow b) \leq a \rightarrow c$,
(c_{19}) $(a_1 \rightsquigarrow a_2) \odot (a_2 \rightsquigarrow a_3) \odot ... \odot (a_{n-1} \rightsquigarrow a_n) \leq a_1 \rightsquigarrow a_n$,
(c_{20}) $(a_{n-1} \rightarrow a_n) \odot ... \odot (a_2 \rightarrow a_3) \odot (a_1 \rightarrow a_2) \leq a_1 \rightarrow a_n$,
(c_{21}) $a \vee b = ((a \rightsquigarrow b) \rightarrow b) \wedge ((b \rightsquigarrow a) \rightarrow a)$,
(c_{22}) $a \vee b = ((a \rightarrow b) \rightsquigarrow b) \wedge ((b \rightarrow a) \rightsquigarrow a)$,
(c_{23}) $a \rightsquigarrow (b \rightsquigarrow c) = (b \odot a) \rightsquigarrow c$ and $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c$,
(c_{24}) $a \rightsquigarrow b = a \rightsquigarrow (a \wedge b)$,
(c_{25}) $a \rightarrow b = a \rightarrow (a \wedge b)$,
(c_{26}) $c \odot (a \wedge b) \leq (c \odot a) \wedge (c \odot b)$ and $(a \wedge b) \odot c \leq (a \odot c) \wedge (b \odot c)$,
(c_{27}) if $a \vee b = 1$ then $a \rightarrow b = a \rightsquigarrow b = b$,
(c_{28}) if $a \vee b = 1$ then, for each natural number $n \geq 1, a^n \vee b^n = 1$,
(c_{29}) for each natural number $n \geq 1, (a \rightarrow b)^n \vee (b \rightarrow a)^n = (a \rightsquigarrow b)^n \vee (b \rightsquigarrow a)^n = 1$,
(c_{30}) $a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$,
$(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$,
$a \rightsquigarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightsquigarrow b_i)$,
$a \rightarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightarrow b_i)$,
$(\bigvee_{i \in I} b_i) \rightsquigarrow a = \bigwedge_{i \in I} (b_i \rightsquigarrow a)$,
$(\bigvee_{i \in I} b_i) \rightarrow a = \bigwedge_{i \in I} (b_i \rightarrow a)$,
(whenever the arbitrary meets and unions exist)

Proposition 2.1. *(*[4], [7], [8]*) If A is a pseudo $MTL-$algebra, then for every $x, y, z \in A$ we have :*
(c_{31}) *if* $x \vee y = 1$ *then* $x \odot y = x \wedge y$;
(c_{32}) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$ *and* $x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$;
(c_{33}) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$ *and* $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$;
(c_{34}) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$ *and* $(y \wedge z) \odot x = (y \odot x) \wedge (z \odot x)$;
(c_{35}) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

In a pseudo $MTL-$algebra A we denote $a^{\sim} = a \rightsquigarrow 0$ and $a^{-} = a \rightarrow 0$, for every $a \in A$. Using these notations we have the following rules of calculus in a pseudo $MTL-$algebra :

(c_{36}) $1^{\sim} = 1^{-} = 0, 0^{\sim} = 0^{-} = 1$,
(c_{37}) $a \odot a^{\sim} = a^{-} \odot a = 0$,
(c_{38}) $b \leq a^{\sim}$ iff $a \odot b = 0$,
(c_{39}) $b \leq a^{-}$ iff $b \odot a = 0$,
(c_{40}) $a \leq a^{-} \rightsquigarrow b, a \leq a^{\sim} \to b$,
(c_{41}) $a \leq (a^{\sim})^{-}, a \leq (a^{-})^{\sim}$,
(c_{42}) $a \rightsquigarrow b \leq b^{\sim} \to a^{\sim}, a \to b \leq b^{-} \rightsquigarrow a^{-}$,
(c_{43}) $a \to b^{\sim} = b \rightsquigarrow a^{-}, a \rightsquigarrow b^{-} = b \to a^{\sim}$,
(c_{44}) $a \leq b$ implies $b^{\sim} \leq a^{\sim}$ and $b^{-} \leq a^{-}$,
(c_{45}) $((a^{\sim})^{-})^{\sim} = a^{\sim}, ((a^{-})^{\sim})^{-} = a^{-}$,
(c_{46}) $a \to a^{\sim} = a \rightsquigarrow a^{-}$,
(c_{47}) $(b \odot a)^{\sim} = a \rightsquigarrow b^{\sim}, (a \odot b)^{-} = a \to b^{-}$,
(c_{48}) $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}, (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$,
(c_{49}) $(a \wedge b)^{-} = a^{-} \vee b^{-}, (a \vee b)^{-} = a^{-} \wedge b^{-}$,
(c_{50}) $(a \vee b)^{-\sim} = a^{-\sim} \vee b^{-\sim}, (a \vee b)^{\sim-} = a^{\sim-} \vee b^{\sim-}$,
(c_{51}) $a^{-} \rightsquigarrow b^{-} = a^{-\sim} \to b^{-\sim} = a \to b^{-\sim}$ and $b^{\sim} \to a^{\sim} = a^{\sim-} \rightsquigarrow b^{\sim-} = a \rightsquigarrow b^{\sim-}$.

2.1. The Boolean center of a pseudo MTL-algebra. Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice. Recall that an element $a \in L$ is called *complemented* if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such element b exists it is called a *complement* of a. We will denote $b = a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.

In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique (following c_{35} in a pseudo $MTL-$ algebra the complements are unique).

Lemma 2.1. *([9]) Suppose that A is a residuated lattice and $a \in A$ have a complement $b \in A$. Then, the following hold:*
(i) *If c is another complement of a in A, then $c = b$;*
(ii) *$a' = b$ and $b' = a$;*
(iii) *$a^2 = a$.*

Remark 2.5. *Since in particular a pseudo $MTL-$ algebra is a residuated lattice, Lemma 2.1 is also true if A is a pseudo $MTL-$ algebra.*

In the following we denote by A the universe of a pseudo $MTL-$ algebra A and by $B(A)$ the set of all complemented elements of A.

Lemma 2.2. *If $e \in B(A)$, then $e' = e^{-} = e^{\sim}$ and $(e^{-})^{\sim} = (e^{\sim})^{-} = e$, where by e' we denote the complement of e.*

Proof. If $e \in B(A)$, and $a = e'$, then $e \vee a = 1$ and $e \wedge a = 0$. Since $e \odot a \leq e \wedge a = 0$, then $e \odot a = 0$, hence $a \leq e \rightsquigarrow 0 = e^{\sim}$ and $a \odot e \leq e \wedge a = 0$, then $a \odot e = 0$, hence $a \leq e \to 0 = e^{-}$. On the another hand, $e^{-} = e^{-} \odot 1 = e^{-} \odot (e \vee a) \overset{c_{30}}{=} (e^{-} \odot e) \vee (e^{-} \odot a) = 0 \vee (e^{-} \odot a) = e^{-} \odot a$, hence $e^{-} \leq a$, and $e^{\sim} = 1 \odot e^{\sim} = (e \vee a) \odot e^{\sim} \overset{c_{30}}{=} (e \odot e^{\sim}) \vee (a \odot e^{\sim}) = 0 \vee (a \odot e^{\sim}) = a \odot e^{\sim}$, hence $e^{\sim} \leq a$, that is $e^{-} = e^{\sim} = a$. The equality $(e^{-})^{\sim} = (e^{\sim})^{-} = e$ follows from Lemma 2.1, (ii).■

Proposition 2.2. *([9]) If $e, f \in B(A)$, then $e \wedge f, e \vee f, e \to f, e \rightsquigarrow f \in B(A)$ and for every $x \in A$,*
(c_{52})**:** *$e \odot x = e \wedge x = x \odot e$.*

Corollary 2.1. *([9]) The set $B(A)$ is the universe of a Boolean subalgebra of A, called the* Boolean center *of A.*

Proposition 2.3. *For $e \in A$ the following are equivalent:*
(i) $e \in B(A)$,
(ii) $e \vee e^{-} = e \vee e^{\sim} = 1$.

Proof. $(i) \Rightarrow (ii)$. Follows from Lemma 2.2.

$(i) \Rightarrow (ii)$. From $e \vee e^{-} = 1$ we deduce that $0 = 1^{\sim} = (e \vee e^{-})^{\sim} \overset{c_{48}}{=} e^{\sim} \wedge (e^{-})^{\sim} \overset{c_{41}}{\geq} e^{\sim} \wedge e$, so $e^{\sim} \wedge e = 0$. We have $e \vee e^{\sim} = 1$ and $e \wedge e^{\sim} = 0$, so $e \in B(A)$.■

Proposition 2.4. *If $e \in B(A)$ then:*
(i) $e^2 = e$ *and* $e = (e^{\sim})^{-} = (e^{-})^{\sim}$,
(ii) $e^{-} \to e = e$ *and* $e \to e^{-} = e^{-}$,
(ii') $e^{\sim} \rightsquigarrow e = e$ *and* $e \rightsquigarrow e^{\sim} = e^{\sim}$,
(iii) $(e \to x) \to e = e$, *for every* $x \in A$,
(iii') $(e \rightsquigarrow x) \rightsquigarrow e = e$, *for every* $x \in A$,
(iv) $e \wedge x = (e \to x) \odot e = (x \to e) \odot x = e \odot (e \rightsquigarrow x) = x \odot (x \rightsquigarrow e)$, *for every* $x \in A$.

Proof. (i). Follows from Lemma 2.1 (iii) and Lemma 2.2.

(ii). If $e \in B(A)$, then $e \vee e^{-} = 1$. Since, by c_{22}, $1 = e \vee e^{-} = [(e \to e^{-}) \rightsquigarrow e^{-}] \wedge [(e^{-} \to e) \rightsquigarrow e]$, we deduce that $(e \to e^{-}) \rightsquigarrow e^{-} = (e^{-} \to e) \rightsquigarrow e = 1$, hence $e \to e^{-} \leq e^{-}$ and $e^{-} \to e \leq e$ that is, $e \to e^{-} = e^{-}$ and $e^{-} \to e = e$.

(ii'). As for (ii) using c_{21}.

(iii). If $x \in A$, then from $0 \leq x$ we deduce using c_4 and c_5 that $e^{-} \leq e \to x$ hence $(e \to x) \to e \leq e^{-} \to e = e$, by (ii). Since $e \leq (e \to x) \to e$ we obtain $(e \to x) \to e = e$.

(iii'). As for (iii).

(iv). For $x \in A$ and $e \in B(A)$, since by c_{52}, $e \wedge x = e \odot x = x \odot e \leq (e \to x) \odot e, (x \to e) \odot x, e \odot (e \rightsquigarrow x), x \odot (x \rightsquigarrow e) \leq x, e$ we deduce that $(e \to x) \odot e = (x \to e) \odot x = e \odot (e \rightsquigarrow x) = x \odot (x \rightsquigarrow e) = e \wedge x$. ■

Proposition 2.5. *For $e \in A$ the following are equivalent:*
(i) $e \in B(A)$,
(ii) $e = (e^{\sim})^{-} = (e^{-})^{\sim}$ *and* $e \wedge x = e \odot x$, *for every* $x \in A$.

Proof. $(i) \Rightarrow (ii)$. By Propositions 2.2 and 2.4.

$(ii) \Rightarrow (i)$. Suppose $e = (e^{\sim})^{-} = (e^{-})^{\sim}$ and $e \wedge x = e \odot x$, for every $x \in A$.

For $x = e^{-}, e^{\sim}$ using c_{37} we obtain $e^{-} \wedge e = e^{-} \odot e = 0$ and $e \wedge e^{\sim} = e \odot e^{\sim} = 0$, so, we have: $1 = 0^{\sim} = (e^{-} \wedge e)^{\sim} \overset{c_{48}}{=} (e^{-})^{\sim} \vee e^{\sim} = e \vee e^{\sim}$, and $1 = 0^{-} = (e \wedge e^{\sim})^{-} \overset{c_{49}}{=} e^{-} \vee (e^{\sim})^{-} = e^{-} \vee e$, hence $e^{\sim} \vee e = e^{-} \vee e = 1$ and using Proposition 2.3 we deduce that $e \in B(A)$. ■

Proposition 2.6. *If $e \in B(A)$ and $x \in A$, then*
(c_{53}) $x \to e = (x \odot e^{\sim})^{-} = x^{-} \vee e$,
(c_{54}) $x \rightsquigarrow e = (e^{-} \odot x)^{\sim} = e \vee x^{\sim}$.

Proof. We have

$$x \to e = x \to (e^{\sim})^{-} \overset{c_{47}}{=} (x \odot e^{\sim})^{-} = (x \wedge e^{\sim})^{-} \overset{c_{49}}{=} x^{-} \vee (e^{\sim})^{-} = x^{-} \vee e,$$

$$x \rightsquigarrow e = x \rightsquigarrow (e^{-})^{\sim} \overset{c_{47}}{=} (e^{-} \odot x)^{\sim} = (e^{-} \wedge x)^{\sim} \overset{c_{48}}{=} (e^{-})^{\sim} \vee x^{\sim} = e \vee x^{\sim}. ■$$

Lemma 2.3. *If $e, f \in B(A)$ and $x, y \in A$, then:*
(c_{55}) $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$,
(c_{56}) $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$,
(c_{57}) $e \odot (x \rightsquigarrow y) = e \odot [(e \odot x) \rightsquigarrow (e \odot y)]$ *and* $(x \to y) \odot e = [(x \odot e) \to (y \odot e)] \odot e$,
(c_{58}) $x \odot (e \rightsquigarrow f) = x \odot [(x \odot e) \rightsquigarrow (x \odot f)]$ *and* $(e \to f) \odot x = [(e \odot x) \to (f \odot x)] \odot x$,

(c_{59}) *$e \to (x \to y) = (e \to x) \to (e \to y)$ and $e \rightsquigarrow (x \rightsquigarrow y) = (e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y)$.*

Proof. (c_{55}). We have

$$(e \vee x) \odot (e \vee y) \overset{c_{30}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] = [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$

$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

(c_{56}). We have

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

(c_{57}). By c_{13} we have $x \to y \leq (x \odot e) \to (y \odot e)$, hence by c_3, $(x \to y) \odot e \leq [(x \odot e) \to (y \odot e)] \odot e$. Conversely, $[(x \odot e) \to (y \odot e)] \odot e \leq e$ and $[(x \odot e) \to (y \odot e)] \odot (x \odot e) \leq y \odot e \leq y$ so $[(x \odot e) \to (y \odot e)] \odot e \leq x \to y$. Hence $[(x \odot e) \to (y \odot e)] \odot e \leq (x \to y) \wedge e = (x \to y) \odot e$.

By c_{12} we have $x \rightsquigarrow y \leq (e \odot x) \rightsquigarrow (e \odot y)$, hence by c_3, $e \odot (x \rightsquigarrow y) \leq e \odot [(e \odot x) \rightsquigarrow (e \odot y)]$. Conversely, $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e$ and $(e \odot x) \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e \odot y \leq y$ so $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq x \rightsquigarrow y$.

Hence $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e \wedge (x \rightsquigarrow y) = e \odot (x \rightsquigarrow y)$.

(c_{58}). We have

$$[(e \odot x) \to (f \odot x)] \odot x = [(e \odot x) \to (f \wedge x)] \odot x$$

$$\overset{c_{30}}{=} [((e \odot x) \to f) \wedge ((e \odot x) \to x)] \odot x$$

$$= [((e \odot x) \to f) \wedge 1] \odot x = [(e \odot x) \to f] \odot x = [(x \odot e) \to f] \odot x$$

$$\overset{c_{23}}{=} [x \to (e \to f)] \odot x = x \wedge (e \to f) = x \odot (e \to f).$$

We have

$$x \odot [(x \odot e) \rightsquigarrow (x \odot f)] = x \odot [(x \odot e) \rightsquigarrow (x \wedge f)]$$

$$\overset{c_{30}}{=} x \odot [((x \odot e) \rightsquigarrow x) \wedge ((x \odot e) \rightsquigarrow f)] = x \odot [1 \wedge ((x \odot e) \rightsquigarrow f)]$$

$$= x \odot [(x \odot e) \rightsquigarrow f] = x \odot [(e \odot x) \rightsquigarrow f] \overset{c_{23}}{=} x \odot [x \rightsquigarrow (e \rightsquigarrow f)]$$

$$= x \wedge (e \rightsquigarrow f) = x \odot (e \rightsquigarrow f).$$

(c_{59}).We have

$$(e \to x) \to (e \to y) \overset{c_{23}}{=} [(e \to x) \odot e] \to y = (e \wedge x) \to y = (e \odot x) \to y \overset{c_{23}}{=} e \to (x \to y),$$

$$(e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y) \overset{c_{23}}{=} [e \odot (e \rightsquigarrow x)] \rightsquigarrow y = (e \wedge x) \rightsquigarrow y = (x \odot e) \rightsquigarrow y \overset{c_{23}}{=} e \rightsquigarrow (x \rightsquigarrow y). \blacksquare$$

3. Pseudo-MTL algebra of fractions relative to a $\wedge-$ closed system

Definition 3.1. *A nonempty subset $S \subseteq A$ is called* $\wedge-$closed system *in A if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.*

We denote by $S(A)$ the set of all $\wedge-$closed system of A (clearly $\{1\}, A \in S(A)$).

For $S \in S(A)$, on the pseudo - MTL algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

Lemma 3.1. *θ_S is a congruence on A.*

Proof. The reflexivity, symmetry and transitivity of θ_S are immediately.

The compatibility of θ_S with the operations $\wedge, \vee, \odot$ is as in the case of MTL algebras. To prove the compatibility of θ_S with the operations $\rightarrow$ and $\rightsquigarrow$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

We obtain using c_{57}:

$$(x \rightarrow z) \wedge g = (x \rightarrow z) \odot g = [(x \odot g) \rightarrow (z \odot g)] \odot g$$
$$= [(y \odot g) \rightarrow (t \odot g)] \odot g = (y \rightarrow t) \odot g = (y \rightarrow t) \wedge g,$$

hence $(x \rightarrow z, y \rightarrow t) \in \theta_S$ and

$$(x \rightsquigarrow z) \wedge g = g \odot (x \rightsquigarrow z) = g \odot [(g \odot x) \rightsquigarrow (g \odot z)]$$
$$= g \odot [(g \odot y) \rightsquigarrow (g \odot t)] = g \odot (y \rightsquigarrow t) = (y \rightsquigarrow t) \wedge g,$$

hence $(x \rightsquigarrow z, y \rightsquigarrow t) \in \theta_S$. ■

For $x \in A$ we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \rightarrow A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A, x/S \wedge y/S = (x \wedge y)/S, x/S \vee y/S = (x \vee y)/S, x/S \odot y/S = (x \odot y)/S$, $x/S \rightarrow y/S = (x \rightarrow y)/S, x/S \rightsquigarrow y/S = (x \rightsquigarrow y)/S$.

So, p_S is an onto morphism of pseudo-MTL algebras.

Remark 3.1. *Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.*

Proposition 3.1. *If $a \in A$, then $a/S \in B(A[S])$ iff there is $e \in S \cap B(A)$ such that $a \vee a^-, a \vee a^\sim \geq e$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof. For $a \in A$, we have by Proposition 2.3, $a/S \in B(A[S]) \Leftrightarrow a/S \vee (a/S)^- = a/S \vee (a/S)^\sim = \mathbf{1} \Leftrightarrow (a \vee a^-)/S = (a \vee a^\sim)/S = 1/S$ iff there is $e_1, e_2 \in S \cap B(A)$ such that $(a \vee a^-) \wedge e_1 = 1 \wedge e_1 = e_1$ and $(a \vee a^\sim) \wedge e_2 = 1 \wedge e_2 = e_2$. If denote $e = e_1 \wedge e_2 \in S \cap B(A)$, then $a \vee a^-, a \vee a^\sim \geq e$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 = e \vee e^- = e \vee e^\sim \geq 1$, we deduce that $e/S \in B(A[S])$. ■

As in the case of MTL algebras we have the following result:

Theorem 3.1. *If A' is a pseudo-MTL algebra and $f : A \rightarrow A'$ is a morphism of pseudo-MTL algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there is an unique morphism of pseudo-MTL algebras $f' : A[S] \rightarrow A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ & {}_f\searrow \quad \swarrow_{f'} & \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Definition 3.2. *Theorem 3.1 allows us to call $A[S]$ the* pseudo-MTL algebra of fractions relative to the $\wedge-$closed system S.

Remark 3.2. *If pseudo-MTL algebra A is a $MTL-$ algebra, then $A[S]$ is a $MTL-$ algebra, called the MTL-algebra of fractions relative to the $\wedge-$closed system S.*

Example 3.1. *If A is a pseudo $MTL-$ algebra and $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A)\backslash\{1\}) = \varnothing$, then for $x, y \in A, (x, y) \in \theta_S \iff x \wedge 1 = y \wedge 1 \iff x = y$, hence in this case $A[S] = A$.*

Example 3.2. *If A is a pseudo $MTL-$ algebra and S is a $\wedge-$closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.*

4. Strong multipliers on a pseudo MTL - algebra

The concept of *maximal lattice of quotients* for a distributive lattice was defined by J. Schmid in [17] taking as a guide-line the construction of *complete ring of quotients* by partial morphisms introduced by G. Findlay and J. Lambek (see [13], p.36). The central role in the constructions of maximal lattice of quotients for a distributive lattice due to J. Schmid is played by the concept of *multiplier* (defined for a distributive lattice by W. H. Cornish in [5]).

In this section by A we denote the universe of a pseudo $MTL-$ algebra.

We denote by $C(A) = \{x \in A : x \odot (x \rightsquigarrow a) = (x \to a) \odot x$, for every $a \le x, a \in A\}$. We remark that if A is a $MTL-$ algebra or a pseudo $BL-$ algebra, then $C(A) = A$.

Lemma 4.1. *In a pseudo $MTL-$ algebra A if $e \in B(A)$ and $x \in C(A)$, then $e \odot x \in C(A)$.*

Proof. Let $a \in A$ such that $a \le e \odot x$. Then $(e \odot x) \odot [(e \odot x) \rightsquigarrow a] = x \odot (e \odot [(e \odot x) \rightsquigarrow a]) \overset{c_{57}}{=} x \odot e \odot [(e \odot e \odot x) \rightsquigarrow (e \odot a)]) =$

$x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot a)]) \overset{c_{57}}{=} x \odot e \odot (x \rightsquigarrow a) = e \odot x \odot (x \rightsquigarrow a) \overset{a \le x, x \in C(A)}{=}$ $e \odot (x \to a) \odot x \overset{c_{57}}{=} [(x \odot e) \to (a \odot e)] \odot e \odot x = [(x \odot e \odot e) \to (a \odot e)] \odot e \odot x \overset{c_{57}}{=}$ $[(x \odot e) \to a] \odot (x \odot e) = [(e \odot x) \to a] \odot (e \odot x).$■

Also, we denote by $\mathcal{I}(A) = \{I \subseteq A :$ if $x, y \in A, x \le y$ and $y \in I$, then $x \in I\}$ and by $\mathcal{I}'(A) = \{I = J \cap C(A), J \in \mathcal{I}(A)\}$. Clearly, if $I_1, I_2 \in \mathcal{I}'(A)$, then $I_1 \cap I_2 \in \mathcal{I}'(A)$. Also, if $I \in \mathcal{I}'(A)$, then $0 \in I$. If A is a $MTL-$ algebra or a pseudo $BL-$ algebra, then $\mathcal{I}'(A) = \mathcal{I}(A)$ is the set of all ordered ideals of A.

Definition 4.1. *By* partial strong multiplier *on A we mean a map $f : I \to A$, where $I \in \mathcal{I}'(A)$, which verifies the next axioms:*

(M_1) *$f(e \odot x) = e \odot f(x)$, for every $e \in B(A)$ and $x \in I$;*
(M_2) *$x \odot (x \rightsquigarrow f(x)) = f(x)$, for every $x \in I$;*
(M_3) *If $e \in I \cap B(A)$, then $f(e) \in B(A)$;*
(M_4) *$x \wedge f(e) = e \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$.*

Remark 4.1. *The axiom M_2, $x \odot (x \rightsquigarrow f(x)) = f(x)$ implies $f(x) \le x$, for every $x \in I$, and since $x \in I \subseteq C(A)$, this axiom become $x \odot (x \rightsquigarrow f(x)) = (x \to f(x)) \odot x = f(x)$, for every $x \in I$.*

Remark 4.2. *If pseudo $MTL-$algebra A is a $MTL-$algebra, the Definition 4.1 coincide with the definition for partial strong multipliers in a $MTL-$algebra, see* [15].

By $dom(f) \in \mathcal{I}'(A)$ we denote the domain of f; if $dom(f) = C(A)$, we call f *total*. To simplify the language, we will use *strong multiplier* instead *partial strong multiplier,* using *total* to indicate that the domain of a certain multiplier is $C(A)$.

Example 4.1. *The map $\mathbf{0} : C(A) \to A$ defined by $\mathbf{0}(x) = 0$, for every $x \in C(A)$ is a total strong multiplier on A.*

Example 4.2. *The map* $\mathbf{1} : C(A) \to A$ *defined by* $\mathbf{1}(x) = x$, *for every* $x \in C(A)$ *is also a total strong multiplier on* A.

Example 4.3. *For* $a \in B(A)$ *and* $I \in \mathcal{I}'(A)$, *the map* $f_a : I \to A$ *defined by* $f_a(x) = a \wedge x \overset{c_{52}}{=} a \odot x$, *for every* $x \in I$ *is a strong multiplier on* A *(called principal).*

Indeed, for $x \in I$ and $e \in B(A)$, we have $f_a(e \odot x) = a \wedge (e \odot x) = a \wedge (e \wedge x) = e \wedge (a \wedge x) = e \odot (a \wedge x) = e \odot f_a(x)$ and $x \odot (x \rightsquigarrow f_a(x)) = x \odot (x \rightsquigarrow (a \wedge x)) \overset{c_{30}}{=} x \odot [(x \rightsquigarrow a) \wedge (x \rightsquigarrow x)] = x \odot (x \rightsquigarrow a) = x \wedge a = f_a(x)$.

Also, if $e \in I \cap B(A)$, $f_a(e) = e \wedge a \in B(A)$ and $x \wedge (a \wedge e) = e \wedge (a \wedge x)$, for every $x \in I$.

If $dom(f_a) = C(A)$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$ and $\overline{f_1} = \mathbf{1}$.

For $I \in \mathcal{I}'(A)$, we denote $M(I, A) = \{f : I \to A \mid f$ is a strong multiplier on $A\}$ and $M(A) = \bigcup_{I \in \mathcal{I}'(A)} M(I, A)$.

Proposition 4.1. *If* $I_1, I_2 \in \mathcal{I}'(A)$ *and* $f_i \in M(I_i, A), i = 1, 2$, *then*
(c_{60}) $f_1(x) \odot [x \rightsquigarrow f_2(x)] = [x \to f_1(x)] \odot f_2(x)$, *for every* $x \in I_1 \cap I_2$.

Proof. For $x \in I_1 \cap I_2$ we have $f_1(x) \odot [x \rightsquigarrow f_2(x)] \overset{M_2}{=} [(x \to f_1(x)) \odot x] \odot (x \rightsquigarrow f_2(x)) = (x \to f_1(x)) \odot [x \odot (x \rightsquigarrow f_2(x))] \overset{M_2}{=} [x \to f_1(x)] \odot f_2(x)$. ■

Definition 4.2. *For* $I_1, I_2 \in \mathcal{I}'(A)$ *and* $f_i \in M(I_i, A), i = 1, 2$, *we define* $f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \otimes f_2$, $f_1 \leftrightarrow f_2$, $f_1 \leftrightsquigarrow f_2 : I_1 \cap I_2 \to A$ *by* $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$, $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$, $(f_1 \otimes f_2)(x) = f_1(x) \odot [x \rightsquigarrow f_2(x)] \overset{c_{60}}{=} [x \to f_1(x)] \odot f_2(x)$, $(f_1 \leftrightarrow f_2)(x) = [f_1(x) \to f_2(x)] \odot x$, $(f_1 \leftrightsquigarrow f_2)(x) = x \odot [f_1(x) \rightsquigarrow f_2(x)]$, *for every* $x \in I_1 \cap I_2$.

Lemma 4.2. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A)$.

Proof. It is sufficient to verify only M_2 (for M_1, M_3 and M_4 see [15]).

For every $x \in I_1 \cap I_2$ we have $x \odot [x \rightsquigarrow (f_1 \wedge f_2)(x)] = x \odot [x \rightsquigarrow (f_1(x) \wedge f_2(x))] \overset{c_{30}}{=} x \odot [(x \rightsquigarrow f_1(x)) \wedge (x \rightsquigarrow f_2(x))] \overset{c_{34}}{=} [x \odot (x \rightsquigarrow f_1(x))] \wedge [x \odot (x \rightsquigarrow f_2(x))] \overset{M_2}{=} f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x)$. ■

Lemma 4.3. $f_1 \vee f_2 \in M(I_1 \cap I_2, A)$.

Proof. The axioms M_1, M_3 and M_4 are verified as in the case of $MTL-$ algebras (see [15]). To verify M_2, let $x \in I_1 \cap I_2$. Then $x \odot [x \rightsquigarrow (f_1 \vee f_2)(x)] = x \odot [x \rightsquigarrow (f_1(x) \vee f_2(x))] \overset{c_{32}}{=} x \odot [(x \rightsquigarrow f_1(x)) \vee (x \rightsquigarrow f_2(x))] \overset{c_{30}}{=} [x \odot (x \rightsquigarrow f_1(x))] \vee [x \odot (x \rightsquigarrow f_2(x))] \overset{M_2}{=} f_1(x) \vee f_2(x) = (f_1 \vee f_2)(x)$. ■

Lemma 4.4. $f_1 \otimes f_2 \in M(I_1 \cap I_2, A)$.

Proof. By using c_{57} and c_{58} the axioms M_1, M_3 and M_4 are verified as in the case of $MTL-$ algebras (see [15]). For M_2, let $x \in I_1 \cap I_2$ and denote $f = f_1 \otimes f_2$.

To prove the equality $x \odot (x \rightsquigarrow f(x)) = f(x)$ since by c_1, $x \odot (x \rightsquigarrow f(x)) \leq f(x)$, it is sufficient to prove that $f(x) \leq x \odot (x \rightsquigarrow f(x))$.

We have $f(x) = f_1(x) \odot (x \rightsquigarrow f_2(x)) = x \odot (x \rightsquigarrow f_1(x)) \odot (x \rightsquigarrow f_2(x))$ and $x \odot (x \rightsquigarrow f(x)) = x \odot [x \rightsquigarrow (x \odot (x \rightsquigarrow f_1(x)) \odot (x \rightsquigarrow f_2(x)))]$. So, to prove that $f(x) \leq x \odot (x \rightsquigarrow f(x))$ it is sufficient to prove that $x \odot (x \rightsquigarrow f_1(x)) \odot (x \rightsquigarrow f_2(x)) \leq x \odot [x \rightsquigarrow (x \odot (x \rightsquigarrow f_1(x)) \odot (x \rightsquigarrow f_2(x)))]$, that is $\alpha \leq x \rightsquigarrow (x \odot \alpha)$ (with $\alpha \overset{not}{=} (x \rightsquigarrow f_1(x)) \odot (x \rightsquigarrow f_2(x))$), which is true using a_3. ■

Lemma 4.5. $f_1 \leftrightsquigarrow f_2 \in M(I_1 \cap I_2, A)$.

Proof. By using c_{57} and c_{58} the axioms M_1, M_3 and M_4 are verified as in the case of $MTL-$ algebras (see [15]). For M_2, let $x \in I_1 \cap I_2$ and denote $f = f_1 \leftrightsquigarrow f_2 : I_1 \cap I_2 \to A$; then $f(x) = x \odot [f_1(x) \rightsquigarrow f_2(x)]$. We have $f_1(x) \rightsquigarrow f_2(x) \leq x \rightsquigarrow [x \odot (f_1(x) \rightsquigarrow f_2(x))]$, hence $x \odot [f_1(x) \rightsquigarrow f_2(x)] \leq x \odot [x \rightsquigarrow (x \odot (f_1(x) \rightsquigarrow f_2(x)))]$ $\Leftrightarrow f(x) \leq x \odot [x \rightsquigarrow f(x)] \overset{c_1}{\Leftrightarrow} f(x) = x \odot [x \rightsquigarrow f(x)]$. ■

Lemma 4.6. $f_1 \leftrightarrow f_2 \in M(I_1 \cap I_2, A)$.

Proof. By using c_{57} and c_{58} the axioms M_1, M_3 and M_4 are verified as in the case of $MTL-$ algebras (see [15]). For M_2, let $x \in I_1 \cap I_2$ and denote $f = f_1 \leftrightarrow f_2 : I_1 \cap I_2 \to A$; then $f(x) = [f_1(x) \to f_2(x)] \odot x$. We have $f_1(x) \to f_2(x) \leq x \to [(f_1(x) \to f_2(x)) \odot x]$, hence $[f_1(x) \to f_2(x)] \odot x \leq [x \to ((f_1(x) \to f_2(x)) \odot x)] \odot x$ $\Leftrightarrow f(x) \leq [x \to f(x)] \odot x \overset{c_2}{\Leftrightarrow} f(x) = [x \to f(x)] \odot x$.

Using Remark 4.1 we deduce that $x \odot (x \rightsquigarrow f(x)) = (x \to f(x)) \odot x = f(x)$, for every $x \in I$. ■

Proposition 4.2. $(M(A), \wedge, \vee, \otimes, \leftrightarrow, \leftrightsquigarrow, \mathbf{0}, \mathbf{1})$ *is a pseudo* $MTL-$ *algebra.*

Proof. We verify the axioms of a pseudo $MTL-$ algebra.

(a_1). Obviously $(M(A), \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded (distributive) lattice.

(a_2). As in the case of $MTL-$ algebras (see [15]), using c_{60}.

(a_3). Let $f_i \in M(I_i, A)$, where $I_i \in \mathcal{I}'(A)$, $i = 1, 2, 3$.

¿From $f_1 \leq f_2 \leftrightarrow f_3$ for $x \in I_1 \cap I_2 \cap I_3$, we deduce that

$$f_1(x) \leq (f_2 \leftrightarrow f_3)(x) \Leftrightarrow f_1(x) \leq [f_2(x) \to f_3(x)] \odot x.$$

So, by c_3,we deduce that

$$f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq [f_2(x) \to f_3(x)] \odot x \odot [x \rightsquigarrow f_2(x)] \Leftrightarrow$$

$$f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq (f_2(x) \to f_3(x)) \odot f_2(x) \Leftrightarrow$$

Since $(f_2(x) \to f_3(x)) \odot f_2(x) \leq f_3(x)$ we deduce that $(f_1 \otimes f_2)(x) \leq f_3(x)$, for every $x \in I_1 \cap I_2 \cap I_3$, that is, $f_1 \otimes f_2 \leq f_3$.

Conversely, if $(f_1 \otimes f_2)(x) \leq f_3(x)$, then we have $[x \to f_1(x)] \odot f_2(x) \leq f_3(x)$, for every $x \in I_1 \cap I_2 \cap I_3$. Obviously,

$$[x \to f_1(x)] \leq f_2(x) \to f_3(x) \overset{c_3}{\Rightarrow} (x \to f_1(x)) \odot x \leq (f_2(x) \to f_3(x)) \odot x$$

$$\Rightarrow f_1(x) \leq (f_2(x) \to f_3(x)) \odot x \Rightarrow f_1(x) \leq (f_2 \leftrightarrow f_3)(x).$$

Hence, $f_1 \leq f_2 \leftrightarrow f_3$ iff $f_1 \otimes f_2 \leq f_3$, for all $f_1, f_2, f_3 \in M(A)$.

If $f_2 \leq f_1 \leftrightsquigarrow f_3$ for $x \in I_1 \cap I_2 \cap I_3$, then we have

$$f_2(x) \leq (f_1 \leftrightsquigarrow f_3)(x) \Leftrightarrow f_2(x) \leq x \odot [f_1(x) \rightsquigarrow f_3(x)].$$

So, by c_3, we have

$$[x \to f_1(x)] \odot f_2(x) \leq [x \to f_1(x)] \odot x \odot [f_1(x) \rightsquigarrow f_3(x)] \Leftrightarrow$$

$$(f_1 \otimes f_2)(x) \leq f_1(x) \odot (f_1(x) \rightsquigarrow f_3(x)).$$

Since $f_1(x) \odot (f_1(x) \rightsquigarrow f_3(x)) \leq f_3(x)$ we deduce that $(f_1 \otimes f_2)(x) \leq f_3(x)$, for every $x \in I_1 \cap I_2 \cap I_3$, that is, $f_1 \otimes f_2 \leq f_3$.

Conversely if $(f_1 \otimes f_2)(x) \leq f_3(x)$, then we have $f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq f_3(x)$, for every $x \in I_1 \cap I_2 \cap I_3$. It is obvious that

$$x \rightsquigarrow f_2(x) \leq f_1(x) \rightsquigarrow f_3(x) \overset{c_3}{\Rightarrow} x \odot (x \rightsquigarrow f_2(x)) \leq x \odot (f_1(x) \rightsquigarrow f_3(x))$$

$$\Rightarrow f_2(x) \leq x \odot (f_1(x) \rightsquigarrow f_3(x)) \Rightarrow f_2(x) \leq (f_1 \leftrightsquigarrow f_3)(x).$$

Hence, $f_2 \leq f_1 \rightsquigarrow f_3$ iff $f_1 \otimes f_2 \leq f_3$ for all $f_1, f_2, f_3 \in M(A)$.

(a_4). For the preliniarity equation we have

$$\begin{aligned}[(f_1 \leftrightarrow f_2) \vee (f_2 \leftrightarrow f_1)](x) &= [(f_1 \leftrightarrow f_2)(x)] \vee [(f_2 \leftrightarrow f_1)(x)] = \\ &= [(f_1(x) \rightarrow f_2(x)) \odot x] \vee [(f_2(x) \rightarrow f_1(x)) \odot x] = \\ &\stackrel{c_{30}}{=} [(f_1(x) \rightarrow f_2(x)) \vee (f_2(x) \rightarrow f_1(x))] \odot x \stackrel{a_4}{=} 1 \odot x = x = \mathbf{1}(x),\end{aligned}$$

and

$$\begin{aligned}[(f_1 \rightsquigarrow f_2) \vee (f_2 \rightsquigarrow f_1)](x) &= [(f_1 \rightsquigarrow f_2)(x)] \vee [(f_2 \rightsquigarrow f_1)(x)] = \\ &= [x \odot (f_1(x) \rightsquigarrow f_2(x))] \vee [x \odot (f_2(x) \rightsquigarrow f_1(x))] = \\ &\stackrel{c_{30}}{=} x \odot [(f_1(x) \rightsquigarrow f_2(x)) \vee (f_2(x) \rightsquigarrow f_1(x))] \stackrel{a_4}{=} x \odot 1 = x = \mathbf{1}(x),\end{aligned}$$

hence $(f_1 \leftrightarrow f_2) \vee (f_2 \leftrightarrow f_1) = (f_1 \rightsquigarrow f_2) \vee (f_2 \rightsquigarrow f_1) = \mathbf{1}$.

Finally, we deduce that $(M(A), \wedge, \vee, \otimes, \leftrightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a pseudo $MTL-$ algebra. ■

Remark 4.3. *To prove that* $(M(A), \wedge, \vee, \otimes, \leftrightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ *is a pseudo* MTL*-algebra it is sufficient to ask for strong multipliers only the axioms* M_1 *and* M_2.

Remark 4.4. *If pseudo* $MTL-$ *algebra* A *is a pseudo* $BL-$ *algebra (i.e.* $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ *, for all* $x, y \in A$*), then pseudo* $MTL-$ *algebra* $M(A)$ *is also a pseudo* $BL-$ *algebra. Indeed, let* $f_i \in M(I_i, A)$*, where* $I_i \in \mathcal{I}'(A)$*,* $i = 1, 2$*. Then*

$$\begin{aligned}(f_1 &\leftrightarrow f_2) \otimes f_1 = f_1 \wedge f_2 \Leftrightarrow [(f_1 \leftrightarrow f_2) \otimes f_1](x) = (f_1 \wedge f_2)(x) \Leftrightarrow \\ &\Leftrightarrow (f_1 \leftrightarrow f_2)(x) \odot [x \rightsquigarrow f_1(x)] = (f_1 \wedge f_2)(x) \Leftrightarrow \\ [(f_1(x) &\rightarrow f_2(x)) \odot x] \odot [x \rightsquigarrow f_1(x)] = (f_1 \wedge f_2)(x) \Leftrightarrow \\ &\Leftrightarrow [f_1(x) \rightarrow f_2(x)] \odot [x \odot (x \rightsquigarrow f_1(x))] = f_1(x) \wedge f_2(x) \Leftrightarrow \\ [f_1(x) &\rightarrow f_2(x)] \odot (x \wedge f_1(x)) = f_1(x) \wedge f_2(x) \Leftrightarrow \\ &\Leftrightarrow [f_1(x) \rightarrow f_2(x)] \odot f_1(x) = f_1(x) \wedge f_2(x),\end{aligned}$$

for every $x \in I_1 \cap I_2$, which is true because A is a pseudo $BL-$ algebra.

Also,

$$\begin{aligned}f_1 \otimes (f_1 &\rightsquigarrow f_2) = f_1 \wedge f_2 \Leftrightarrow [f_1 \otimes (f_1 \rightsquigarrow f_2)](x) = (f_1 \wedge f_2)(x) \Leftrightarrow \\ &\Leftrightarrow [x \rightarrow f_1(x)] \odot [x \odot (f_1(x) \rightsquigarrow f_2(x))] = (f_1 \wedge f_2)(x) \Leftrightarrow \\ [(x &\rightarrow f_1(x)) \odot x] \odot (f_1(x) \rightsquigarrow f_2(x)) = (f_1 \wedge f_2)(x) \Leftrightarrow \\ &\Leftrightarrow (x \wedge f_1(x)) \odot (f_1(x) \rightsquigarrow f_2(x)) = (f_1 \wedge f_2)(x) \Leftrightarrow \\ &\Leftrightarrow f_1(x) \odot (f_1(x) \rightsquigarrow f_2(x)) = (f_1 \wedge f_2)(x),\end{aligned}$$

for every $x \in I_1 \cap I_2$, which is true because A is a pseudo $BL-$ algebra. ■

Remark 4.5. *If pseudo* MTL *-algebra* A *is a* MTL *-algebra then pseudo* MTL *-algebra* $M(A)$ *is also a* MTL *-algebra. Indeed if* $I_1, I_2 \in \mathcal{I}'(A)$ *and* $f_i \in M(I_i, A)$*,* $i = 1, 2$ *we have*

$$(f_1 \leftrightarrow f_2)(x) = [f_1(x) \rightarrow f_2(x)] \odot x = x \odot [f_1(x) \rightsquigarrow f_2(x)] = (f_1 \rightsquigarrow f_2)(x),$$

for all $x \in I_1 \cap I_2$*, then* $f_1 \leftrightarrow f_2 = f_1 \rightsquigarrow f_2$*, and pseudo* MTL *-algebra* $M(A)$ *is commutative, so is a* MTL *-algebra.*

Definition 4.3. *(*[12]*) A pseudo* MTL *algebra* A *is called*

(i) *A* pseudo $IMTL$ algebra *(*pseudo involutive algebra*) if it satisfies the equation* (pDN) $(x^-)^\sim = (x^\sim)^- = x$;

(ii) *a* pseudo WNM algebra *(*pseudo weak nilpotent minimum*) if it satisfies the equation*
(W) $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = (x \odot y)^\sim \vee [(x \wedge y) \rightsquigarrow (x \odot y)] = 1;$

(iii) *a* pseudo NM algebra *(*pseudo nilpotent minimum*) if it is a WNM algebra satisfying the axiom* (pDN).

Theorem 4.1. *If A is a pseudo $IMTL$ algebra (resp. a pseudo WNM algebra, a pseudo NM algebra), then $M(A)$ is also a pseudo $IMTL$ algebra (resp. a pseudo WNM algebra, a pseudo NM algebra).*

Proof. Suppose A is a pseudo $IMTL$ algebra. For $f \in M(I, A)$, with $I \in \mathcal{I}'(A)$ and $x \in I$, we have $(f^-)^\sim = (f \leftrightarrow \mathbf{0}) \leftrightsquigarrow \mathbf{0}$ and $(f^\sim)^- = (f \leftrightsquigarrow \mathbf{0}) \leftrightarrow \mathbf{0}$, so $(f^-)^\sim = x \odot [(f(x))^- \odot x]^\sim \overset{c_{47}}{=} x \odot [x \rightsquigarrow ((f(x))^-)^\sim] \overset{pDN}{=} x \odot [x \rightsquigarrow f(x)] \overset{M_2}{=} f(x)$, and $(f^\sim)^-(x) = [x \odot f^\sim(x)]^- \odot x \overset{c_{47}}{=} [x \rightarrow ((f(x))^\sim)^-] \odot x \overset{pDN}{=} [x \rightarrow f(x)] \odot x \overset{M_2}{=} f(x)$, hence $(f^-)^\sim = (f^\sim)^- = f$, that is, $M(A)$ is a pseudo $IMTL$ algebra.

Suppose that A is a pseudo WNM algebra. Let $f \in M(I, A), g \in M(J, A)$ with $I, J \in \mathcal{I}'(A), x \in I \cap J$ and denote $a = f(x), b = g(x)$. We have $((f \otimes g)^\sim \vee ((f \wedge g) \leftrightsquigarrow (f \otimes g)))(x) = ((f \otimes g)^\sim(x)) \vee (x \odot ((f \wedge g)(x) \rightsquigarrow (f \otimes g)(x))) = (x \odot (a \odot (x \rightsquigarrow b))^\sim) \vee (x \odot ((a \wedge b) \rightsquigarrow (a \odot (x \rightsquigarrow b)))) \overset{c_{30}}{=} x \odot ((a \odot (x \rightsquigarrow b))^\sim \vee ((a \wedge b) \rightsquigarrow (a \odot (x \rightsquigarrow b))))$.

Since $b \leq x \rightsquigarrow b$ we deduce that $a \wedge b \leq a \wedge (x \rightsquigarrow b)$, hence, using c_5, $(a \wedge (x \rightsquigarrow b)) \rightsquigarrow (a \odot (x \rightsquigarrow b)) \leq (a \wedge b) \rightsquigarrow (a \odot (x \rightsquigarrow b))$.

Since A is supposed a pseudo $WNM-$algebra we obtain $1 = (a \odot (x \rightsquigarrow b))^\sim \vee ((a \wedge (x \rightsquigarrow b)) \rightsquigarrow (a \odot (x \rightsquigarrow b))) \leq (a \odot (x \rightsquigarrow b))^\sim \vee ((a \wedge b) \rightsquigarrow (a \odot (x \rightsquigarrow b)))$, hence $(a \odot (x \rightsquigarrow b))^\sim \vee ((a \wedge b) \rightsquigarrow (a \odot (x \rightsquigarrow b))) = 1$. Then $((f \otimes g)^\sim \vee ((f \wedge g) \leftrightsquigarrow (f \otimes g)))(x) = x \odot 1 = x = \mathbf{1}(x) \Leftrightarrow (f \otimes g)^\sim \vee ((f \wedge g) \leftrightsquigarrow (f \otimes g)) = \mathbf{1}$.

Also we have $((f \otimes g)^- \vee ((f \wedge g) \leftrightarrow (f \otimes g)))(x) = ((f \otimes g)^-(x)) \vee (((f \wedge g)(x) \rightarrow (f \otimes g)(x)) \odot x) = (((x \rightarrow b) \odot a)^- \odot x) \vee (((a \wedge b) \rightarrow ((x \rightarrow b) \odot a)) \odot x) \overset{c_{30}}{=} (((x \rightarrow b) \odot a)^- \vee ((a \wedge b) \rightarrow ((x \rightarrow b) \odot a))) \odot x$.

Since $b \leq x \rightarrow b$ we deduce that $a \wedge b \leq a \wedge (x \rightarrow b)$, hence using c_5, $(a \wedge (x \rightarrow b)) \rightarrow ((x \rightarrow b) \odot a) \leq (a \wedge b) \rightarrow ((x \rightarrow b) \odot a)$.

Since A is supposed a pseudo $WNM-$algebra we obtain $1 = ((x \rightarrow b) \odot a)^- \vee ((a \wedge (x \rightarrow b)) \rightarrow ((x \rightarrow b) \odot a)) \leq ((x \rightarrow b) \odot a)^- \vee ((a \wedge b) \rightarrow ((x \rightarrow b) \odot a))$, hence $((x \rightarrow b) \odot a)^- \vee ((a \wedge b) \rightarrow ((x \rightarrow b) \odot a)) = 1$. Then $((f \otimes g)^- \vee ((f \wedge g) \leftrightarrow (f \otimes g)))(x) = x \odot 1 = x = \mathbf{1}(x) \Leftrightarrow (f \otimes g)^- \vee ((f \wedge g) \leftrightarrow (f \otimes g)) = \mathbf{1}$, that is $M(A)$ is a pseudo WNM algebra.

Suppose now A is a pseudo NM algebra. Then A is a pseudo WNM algebra and a pseudo $IMTL$ algebra, so $M(A)$ is a pseudo WNM algebra and a pseudo $IMTL$ algebra, hence $M(A)$ is a pseudo NM algebra.■

Lemma 4.7. *Let the map $v_A : B(A) \rightarrow M(A)$ defined by $v_A(a) = \overline{f_a}$ for every $a \in B(A)$. Then v_A is a monomorphism of pseudo $MTL-$ algebras.*

Proof. Clearly, $v_A(0) = \overline{f_0} = \mathbf{0}$. Let $a, b \in B(A)$ and $x \in C(A)$. We have:

$$v_A(a \vee b) = v_A(a) \vee v_A(b), v_A(a \wedge b) = v_A(a) \wedge v_A(b),$$

$$(v_A(a) \otimes v_A(b))(x) = v_A(a)(x) \odot (x \rightsquigarrow v_A(b)(x)) = (a \wedge x) \odot (x \rightsquigarrow (b \wedge x))$$
$$= (a \odot x) \odot (x \rightsquigarrow (b \wedge x)) = a \odot [x \odot (x \rightsquigarrow (b \wedge x))] = a \odot (b \wedge x)$$
$$= a \wedge (b \wedge x) = (a \wedge b) \wedge x = (v_A(a \wedge b))(x) = (v_A(a \odot b))(x),$$

hence $v_A(a \odot b) = v_A(a) \otimes v_A(b)$.

Also, since $a \to b, a \rightsquigarrow b \in B(A)$, we have

$$(v_A(a) \leftrightarrow v_A(b))(x) = [v_A(a)(x) \to v_A(b)(x)] \odot x = [(a \wedge x) \to (b \wedge x)] \odot x$$
$$= [(a \odot x) \to (b \odot x)] \odot x \overset{c_{58}}{=} (a \to b) \odot x = x \wedge (a \to b) = v_A(a \to b)(x),$$
$$(v_A(a) \leftrightsquigarrow v_A(b))(x) = x \odot [v_A(a)(x) \rightsquigarrow v_A(b)(x)] = x \odot [(a \wedge x) \rightsquigarrow (b \wedge x)]$$
$$= x \odot [(x \odot a) \rightsquigarrow (x \odot b)] \overset{c_{58}}{=} x \odot (a \rightsquigarrow b) = x \wedge (a \rightsquigarrow b) = v_A(a \rightsquigarrow b)(x).$$

Consequently, we have $v_A(a) \leftrightarrow v_A(b) = v_A(a \to b)$, $v_A(a) \leftrightsquigarrow v_A(b) = v_A(a \rightsquigarrow b)$. This proves that v_A is a morphism of pseudo MTL-algebras.

To prove the injectivity of v_A, we let $a, b \in B(A)$ such that $v_A(a) = v_A(b)$. Then $a \wedge x = b \wedge x$, for every $x \in C(A)$, hence for $x = 1 \in C(A)$ we obtain that $a \wedge 1 = b \wedge 1 \Rightarrow a = b$.■

We have for pseudo $MTL-$ algebras the next analogous definitions, results and remarks as in [15] for $MTL-$ algebras:

Definition 4.4. *A nonempty set $I \subseteq A$ is called* regular *if for every $x, y \in A$ such that $x \wedge e = y \wedge e$ for every $e \in I \cap B(A)$, then $x = y$.*

For example $A, C(A)$ are regular subsets of A (since if $x, y \in A$ (or, $C(A)$) and $x \wedge e = y \wedge e$ for every $e \in B(A)$, then for $e = 1$ we obtain $x \wedge 1 = y \wedge 1 \Leftrightarrow x = y$).

More generally, every subset of A which contains 1 is regular.

We denote $R(A) = \{I \subseteq A : I \text{ is a regular subset of } A\}$.

Lemma 4.8. *If $I_1, I_2 \in \mathcal{I}'(A) \cap R(A)$, then $I_1 \cap I_2 \in \mathcal{I}'(A) \cap R(A)$.*

Remark 4.6. *By Lemmas 4.2-4.6, 4.8 and Proposition 4.2 we deduce that $M_r(A) = \{f \in M(A) : dom(f) \in \mathcal{I}'(A) \cap R(A)\}$ is a pseudo $MTL-$ subalgebra of $M(A)$.*

Proposition 4.3. *$M_r(A)$ is a Boolean subalgebra of $M(A)$.*

Proof. Let $f : I \to A$ be a strong multiplier on A with $I \in \mathcal{I}'(A) \cap \mathcal{R}(A)$. To prove that $M_r(A)$ is a Boolean algebra, using Proposition 2.5 it is suffice to prove that $f = (f^-)^\sim = (f^\sim)^-$ and $f \otimes g = f \wedge g$, for all $g \in M_r(A)$. Let $g \in M_r(A), g : J \to A$.

Then for all $x \in I \cap J$ and $e \in I \cap J \cap B(A)$,

$$e \wedge [f \otimes g](x) = e \wedge [(x \to f(x)) \odot g(x)] = e \odot [x \to f(x)] \odot g(x) = [x \to f(x)] \odot e \odot g(x) =$$
$$\overset{c_{57}}{=} [(x \odot e) \to (f(x) \odot e)] \odot e \odot g(x) = [(e \odot x) \to (f(x) \odot e)] \odot e \odot g(x) =$$
$$= [(e \odot x) \to (f(e) \odot x)] \odot x \odot g(e) =$$
$$\overset{c_{58}}{=} [e \to f(e)] \odot x \odot g(e) = [e \to f(e)] \odot e \odot g(x) = [e \wedge f(e)] \odot g(x) =$$
$$= e \wedge f(e) \wedge g(x) = e \wedge f(e) \wedge (g(x) \wedge x) = e \wedge g(x) \wedge [f(e) \wedge x] = e \wedge g(x) \wedge [e \wedge f(x)] =$$
$$= e \wedge [f(x) \wedge g(x)] = e \wedge [f \wedge g](x),$$

hence $[f \otimes g](x) = f(x) \wedge g(x)$, (since $I \cap J \in \mathcal{R}(A)$), so, $f \otimes g = f \wedge g$.

For all $x \in I$ we have

$(f^-)^\sim(x) = x \odot (f^-(x))^\sim = x \odot [(f(x))^- \odot x]^\sim \overset{c_{47}}{=} x \odot [x \rightsquigarrow ((f(x))^-)^\sim]$ and $(f^\sim)^-(x) = (f^\sim(x))^- \odot x = [x \odot (f(x))^\sim]^- \odot x \overset{c_{47}}{=} [x \to ((f(x))^\sim)^-] \odot x$,

so, for all $e \in I \cap B(A)$ we obtain

$$e \wedge (f^-)^\sim(x) = e \wedge (x \odot [x \rightsquigarrow ((f(x))^-)^\sim]) = e \odot x \odot [x \rightsquigarrow ((f(x))^-)^\sim] =$$

$$= \quad x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot ((f(x))^-)^\sim)] =$$
$$= \quad x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [(f(x))^- \rightsquigarrow 0])] =$$
$$\overset{c_{57}}{=} x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [(e \odot (f(x))^-) \rightsquigarrow 0])] =$$

$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [e \odot (f(x))^{-}]^{\sim})] =$$

$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [e \odot [f(x) \to 0]]^{\sim})] =$$

$$\stackrel{c_{57}}{=} x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot e \odot ([e \odot f(x)]^{-})^{\sim})] =$$

$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot ([x \odot f(e)]^{-})^{\sim})] =$$

$$\stackrel{c_{57}}{=} x \odot e \odot [x \rightsquigarrow ([x \odot f(e)]^{-})^{\sim}] = x \odot e \odot [x \rightsquigarrow ([x \wedge f(e)]^{-})^{\sim}] =$$

$$\stackrel{c_{49}}{=} x \odot e \odot [x \rightsquigarrow [x^{-} \vee (f(e))^{-}]^{\sim}] \stackrel{c_{48}}{=} x \odot e \odot [x \rightsquigarrow [(x^{-})^{\sim} \wedge f(e)]] =$$

$$\stackrel{c_{30}}{=} x \odot e \odot ([x \rightsquigarrow (x^{-})^{\sim}] \wedge [x \rightsquigarrow f(e)]) =$$

$$\stackrel{c_{41}}{=} x \odot e \odot (1 \wedge [x \rightsquigarrow f(e)]) = x \odot e \odot [x \rightsquigarrow f(e)] =$$

$$= e \odot x \odot [x \rightsquigarrow f(e)] = e \odot [x \wedge f(e)] = e \wedge x \wedge f(e) = x \wedge f(e) = e \wedge f(x),$$

and

$$e \wedge (f^{\sim})^{-}(x) = e \wedge [x \to ((f(x))^{\sim})^{-}] \odot x = [x \to ((f(x))^{\sim})^{-}] \odot e \odot x =$$

$$\stackrel{c_{57}}{=} [(x \odot e) \to (((f(x))^{\sim})^{-} \odot e)] \odot e \odot x =$$

$$\stackrel{c_{57}}{=} [(x \odot e) \to (([e \odot f(x)]^{\sim})^{-} \odot e)] \odot e \odot x =$$

$$= [(x \odot e) \to (([x \odot f(e)]^{\sim})^{-} \odot e)] \odot e \odot x =$$

$$\stackrel{c_{57}}{=} [x \to ([x \odot f(e)]^{\sim})^{-}] \odot e \odot x = [x \to [(x^{\sim})^{-} \wedge f(e)]] \odot x \odot e =$$

$$\stackrel{c_{30}}{=} ([x \to (x^{\sim})^{-}] \wedge [x \to f(e)]) \odot x \odot e =$$

$$= (1 \wedge [x \to f(e)]) \odot x \odot e = [x \to f(e)] \odot x \odot e =$$

$$= [x \wedge f(e)] \odot e = e \wedge f(x) \wedge e = e \wedge f(x).$$

So, $f \otimes g = f \wedge g$ and $f = (f^{-})^{\sim} = (f^{\sim})^{-}$, that is, $M_r(A)$ is a Boolean algebra. ■

Remark 4.7. *The axioms M_3, M_4 are necessary in the proof of Proposition 4.3.*

Definition 4.5. *Given two strong multipliers f_1, f_2 on A, we say that f_2* extends *f_1 if $dom(f_1) \subseteq dom(f_2)$ and $f_{2|dom(f_1)} = f_1$; we write $f_1 \leq f_2$ if f_2 extends f_1. A strong multiplier f is called* maximal *if f can not be extended to a strictly larger domain.*

Lemma 4.9. (*i*) *If $f_1, f_2 \in M(A)$, $f \in M_r(A)$ and $f \leq f_1, f \leq f_2$, then f_1 and f_2 coincide on the $dom(f_1) \cap dom(f_2)$,*

(*ii*) *Every strong multiplier $f \in M_r(A)$ can be extended to a strong maximal multiplier. More precisely, each principal strong multiplier f_a with $a \in B(A)$ and $dom(f_a) \in \mathcal{I}'(A) \cap R(A)$ can be uniquely extended to a total strong multiplier $\overline{f_a}$ and each non-principal strong multiplier can be extended to a strong maximal non-principal one.*

Proof. As in the case of $MTL-$ algebras (see [15]), using Lemma 4.1.

On the Boolean algebra $M_r(A)$ we consider the relation ρ_A defined by $(f_1, f_2) \in \rho_A$ iff f_1 and f_2 coincide on the intersection of their domains.

Lemma 4.10. *ρ_A is a congruence on Boolean algebra $M_r(A)$.*

Proof. The reflexivity and the symmetry of ρ_A are immediately; to prove the transitivity of ρ_A let $(f_1, f_2), (f_2, f_3) \in \rho_A$. Therefore f_1, f_2 and respectively f_2, f_3 coincide on the intersection of their domains. If by contrary, there exists $x_0 \in dom(f_1) \cap dom(f_3)$ such that $f_1(x_0) \neq f_3(x_0)$, since $dom(f_2) \in \mathcal{R}(A)$, there exists $e \in dom(f_2) \cap B(A)$ such that $e \wedge f_1(x_0) \neq e \wedge f_3(x_0) \Leftrightarrow f_1(e \odot x_0) \neq f_3(e \odot x_0)$ which is contradictory, since by Lemma 4.1, $e \odot x_0 = e \wedge x_0 \in dom(f_1) \cap dom(f_2) \cap dom(f_3)$.

To prove the compatibility of ρ_A with the operations $\wedge, \vee$ and $^{\sim}$ on $M_r(A)$, let $(f_1, f_2), (g_1, g_2) \in \rho_A$. So, we have f_1, f_2 and respectively g_1, g_2 coincide on the intersection of their domains. Let $x \in dom(f_1) \cap dom(f_2) \cap dom(g_1) \cap dom(g_2)$. Then $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, hence

$$(f_1 \wedge g_1)(x) = f_1(x) \wedge g_1(x) = f_2(x) \wedge g_2(x) = (f_2 \wedge g_2)(x),$$

$$(f_1 \vee g_1)(x) = f_1(x) \vee g_1(x) = f_2(x) \vee g_2(x) = (f_2 \vee g_2)(x).$$

For $x \in dom(f_1) \cap dom(f_2)$ we have

$$f_1^{\sim}(x) = (f_1 \leftrightsquigarrow \mathbf{0})(x) = x \odot [f_1(x) \rightsquigarrow \mathbf{0}(x)] = x \odot [f_2(x) \rightsquigarrow \mathbf{0}(x)] = (f_2 \leftrightsquigarrow \mathbf{0})(x) = f_2^{\sim}(x),$$

that is the pairs $(f_1 \wedge g_1, f_2 \wedge g_2), (f_1 \vee g_1, f_2 \vee g_2), (f_1^{\sim}, f_2^{\sim})$ coincide on the intersection of their domains, hence ρ_A is compatible with the operations $\wedge, \vee$ and $^{\sim}$. ■

For $f \in M_r(A)$ with $I = dom(f) \in \mathcal{I}'(A) \cap R(A)$, we denote by $[f, I]$ the congruence class of f modulo ρ_A and $A'' = M_r(A)/\rho_A$.

Since the class of Boolean algebras is equational, from Proposition 4.2, Remark 4.6 and Lemma 4.10 we deduce:

Theorem 4.2. *A'' is a Boolean algebra, where for $[f, I], [g, J] \in A'', [f, I] \wedge [g, J] = [f \wedge g, I \cap J]$, $[f, I] \vee [g, J] = [f \vee g, I \cap J]$, $[f, I] \otimes [g, J] = [f \otimes g, I \cap J], [f, I] \leftrightarrow [g, J] = [f \leftrightarrow g, I \cap J]$, $[f, I] \leftrightsquigarrow [g, J] = [f \leftrightsquigarrow g, I \cap J], \mathbf{0} = [\mathbf{0}, C(A)]$ and $\mathbf{1} = [\mathbf{1}, C(A)]$.*

Remark 4.8. *If we denote by $\mathcal{F} = \mathcal{I}'(A) \cap R(A)$ and consider the partially ordered systems $\{\delta_{I,J}\}_{I,J \in \mathcal{F}, I \subseteq J}$ (where for $I, J \in \mathcal{F}$, $I \subseteq J, \delta_{I,J} : M(J, A) \to M(I, A)$ is defined by $\delta_{I,J}(f) = f_{|I}$), then by above construction of A'' we deduce that A'' is the inductive limit $A'' = \varinjlim_{I \in \mathcal{F}} M(I, A)$.*

Lemma 4.11. *Let the map $\overline{v_A} : B(A) \to A''$ defined by $\overline{v_A}(a) = [\overline{f_a}, C(A)]$ for every $a \in B(A)$. Then:*
(i) $\overline{v_A}$ is a monomorphism of Boolean algebras;
(ii) $\overline{v_A}(B(A)) \in R(A'')$.

Proof. (i). Follows from Lemma 7.1.
(ii). As in the case of MTL algebras (see [15]). ■

Remark 4.9. *Since for every $a \in B(A)$, $\overline{f_a}$ is the unique strong maximal multiplier on $[\overline{f_a}, C(A)]$ (by Lemma 7.7) we can identify $[\overline{f_a}, C(A)]$ with $\overline{f_a}$. So, since $\overline{v_A}$ is injective map, the elements of $B(A)$ can be identified with the elements of the set $\{ \overline{f_a} : a \in B(A)\}$.*

Lemma 4.12. *In view of the identifications made above, if $[f, dom(f)] \in A''$ (with $f \in M_r(A)$ and $I = dom(f) \in \mathcal{I}'(A) \cap R(A)$), then $I \cap B(A) \subseteq \{a \in B(A) : \overline{f_a} \wedge [f, dom(f)] \in B(A)\}$.*

Proof. As in the case of MTL algebras (see [15]).■

5. Maximal pseudo MTL-algebra of quotients

The scope of this section is to define the notions of pseudo MTL *-algebra of fractions* and *maximal* pseudo MTL - *algebra of quotients* for a pseudo MTL - algebra.

Definition 5.1. *Let A be a pseudo MTL - algebra. A pseudo MTL - algebra F is called* pseudo MTL - algebra of fractions of A *if:*

(Fr_1) *$B(A)$ is a pseudo MTL - subalgebra of F;*

(Fr_2) *For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B(A)$.*

So, pseudo MTL - algebra $B(A)$ is a pseudo MTL - algebra of fractions of itself (since $1 \in B(A)$).

As a notational convenience, we write $A \preceq F$ to indicate that F is a pseudo MTL - algebra of fractions of A.

Definition 5.2. *$Q(A)$ is the* maximal pseudo MTL - algebra of quotients of A *if $A \preceq Q(A)$ and for every pseudo MTL - algebra F with $A \preceq F$ there exists a monomorphism of pseudo MTL - algebras $i : F \to Q(A)$.*

Remark 5.1. *If $A \preceq F$, then F is a Boolean algebra. Indeed, if $a' \in F$ such that $((a')^-)^\sim \neq a'$ or $((a')^\sim)^- \neq a'$ or $a' \wedge x \neq a' \odot x$ for some $x \in F$ then there exists $e, f, g \in B(A)$ such that $e \wedge a', f \wedge a', g \wedge a' \in B(A)$ and*

$$e \wedge a' \neq e \wedge ((a')^-)^\sim = ((e \wedge a')^-)^\sim \text{ or}$$

$$f \wedge a' \neq f \wedge ((a')^\sim)^- = ((f \wedge a')^\sim)^- \text{ or}$$

$$\begin{aligned} g \wedge a' \wedge x &\neq g \wedge (a' \odot x) \Leftrightarrow g \odot (a' \wedge x) \neq g \odot (a' \odot x) \Leftrightarrow \\ (g \odot a') \wedge (g \odot x) &\neq (g \odot a') \odot (g \odot x) \Leftrightarrow (g \wedge a') \wedge (g \odot x) \neq (g \wedge a') \odot (g \odot x), \end{aligned}$$

a contradiction !.

We also have for pseudo MTL - algebras the next analogous definitions, results and remarks as in [15] for MTL - algebras:

Lemma 5.1. *Let $A \preceq F$; then for every $a', b' \in F, a' \neq b'$, and any finite sequence $c'_1, ..., c'_n \in F$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c'_i \in B(A)$ for $i = 1, 2, ..., n$ $(n \geq 2)$.*

Lemma 5.2. *Let $A \prec F$ and $a' \in F$. Then $I_{a'} = \{e \in B(A) : e \wedge a' \in B(A)\} \in \mathcal{I}(B(A)) \cap R(A) = \mathcal{I}'(B(A)) \cap R(A)$.*

Theorem 5.1. *A'' is the maximal pseudo MTL - algebra $Q(A)$ of quotients of A.*

Remark 5.2. 1. *If pseudo MTL - algebra A is a MTL - algebra or a pseudo BL - algebra, then $Q(A)$ is the maximal MTL - algebra of quotients or the maximal pseudo BL - algebra of quotients of A.*

2. *If A is a pseudo MTL - algebra with $B(A) = \{0, 1\} = L_2$ and $A \preceq F$ then $F = \{0, 1\}$, hence $Q(A) = A'' \approx L_2$.*
3. *More general, if A is a pseudo $MTL-$ algebra such that $B(A)$ is finite and $A \preceq F$ then $F = B(A)$, hence in this case $Q(A) = B(A)$.*

6. Topologies on a pseudo MTL-algebra

Definition 6.1. *A non-empty set $\mathcal{F}$ of elements $I \in \mathcal{I}(A)$ will be called a* topology *on A if the following axioms hold:*

(top_1) *If $I_1 \in \mathcal{F}, I_2 \in \mathcal{I}(A)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $A \in \mathcal{F}$);*

(top_2) *If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.*

Remark 6.1. 1. *$\mathcal{F}$ is a topology on A iff $\mathcal{F}$ is a filter of the lattice of power set of A; for this reason a topology on $\mathcal{I}(A)$ is usually called a Gabriel filter on $\mathcal{I}(A)$.*

2. *Clearly, if $\mathcal{F}$ is a topology on A, then $(A, \mathcal{F} \cup \{\emptyset\})$ is a topological space.*

Any intersection of topologies on A is a topology; so, the set $T(A)$ of all topologies of A is a complete lattice with respect to inclusion.

Example 6.1. *If $I \in \mathcal{I}(A)$, then the set $\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$ is a topology on A.*

Example 6.2. *If we denote $R(A) = \{I \subseteq A : I$ is a regular subset of $A\}$, then $\mathcal{F} = \mathcal{I}(A) \cap R(A)$ is a topology on A.*

Example 6.3. *A nonempty set $I \subseteq A$ will be called* dense *(see [10]) if for $x \in A$ such that $e \wedge x = 0$ for every $e \in I \cap B(A)$, then $x = 0$. If we denote by $D(A)$ the set of all dense subsets of A, then $R(A) \subseteq D(A)$ and $\mathcal{F} = \mathcal{I}(A) \cap D(A)$ is a topology on A.*

Example 6.4. *For any $\wedge-$ closed subset S of A, the set $\mathcal{F}_S = \{I \in \mathcal{I}(A) : I \cap S \cap B(A) \neq \oslash\}$ is a topology on A.*

7. Localization of pseudo MTL-algebras

In [10], G. Georgescu exhibited the localization lattice $L_{\mathcal{F}}$ of a distributive lattice L with respect to a topology $\mathcal{F}$ on L in a similar way as for rings or monoids.

The concept of localization MTL algebras was studied in [16] for *commutative* case (taking as a guide-line the case of distributive lattices).

The aim of this section is to define the notion of *localization pseudo MTL - algebra* of a pseudo MTL - algebra. In the least part it is proved that the maximal pseudo $MTL-$ algebra of fractions and the pseudo MTL - algebra of fractions relative to a $\wedge$ -closed system are pseudo MTL - algebras of localization.

In this section by A we consider a pseudo MTL - algebra.

Let $\mathcal{F}$ be a topology on A and we consider the relation $\theta_{\mathcal{F}}$ on A defined in the following way: $(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(A)$.

Lemma 7.1. *$\theta_{\mathcal{F}}$ is a congruence on A.*

Proof. See [16] for the case of $MTL-$ algebras. ■

We shall denote by $a/\theta_{\mathcal{F}}$ the congruence class of an element $a \in A$ and by $p_{\mathcal{F}} : A \to A/\theta_{\mathcal{F}}$ the canonical morphism of pseudo MTL-algebras.

Proposition 7.1. *For $a \in A$, $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$ iff there exists $I \in \mathcal{F}$ such that $a \vee a^-, a \vee a^\sim \geq e$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Proof. Using Proposition 2.3, for $a \in A$, we have $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}) \Leftrightarrow a/\theta_{\mathcal{F}} \vee (a/\theta_{\mathcal{F}})^- = a/\theta_{\mathcal{F}} \vee (a/\theta_{\mathcal{F}})^\sim = 1/\theta_{\mathcal{F}} \Leftrightarrow (a \vee a^-)/\theta_{\mathcal{F}} = (a \vee a^\sim)/\theta_{\mathcal{F}} = 1/\theta_{\mathcal{F}} \Leftrightarrow$ there exist $K, J \in \mathcal{F}$ such that $(a \vee a^-) \wedge e = 1 \wedge e = e$, for every $e \in K \cap B(A) \Leftrightarrow a \vee a^- \geq e$, for every $e \in K \cap B(A)$ and $(a \vee a^{\sim}) \wedge e = 1 \wedge e = e$, for every $e \in J \cap B(A) \Leftrightarrow a \vee a^{\sim} \geq e$, for every $e \in J \cap B(A)$.

If we denote $I = K \cap J$, then $I \in \mathcal{F}$ and for every $e \in I \cap B(A), a \vee a^{-}, a \vee a^{\sim} \geq e$.

If $a \in B(A)$, then $1 = a \vee a^{-} = a \vee a^{\sim} \geq e$, for every $e \in I \cap B(A), I \in \mathcal{F}$, hence $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.■

Corollary 7.1. *If $\mathcal{F} = \mathcal{I}(A) \cap R(A)$, then for $a \in A$, $a \in B(A)$ iff $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

We recall that for a pseudo $MTL-$ algebra A, we denote by $C(A) = \{x \in A : x \odot (x \rightsquigarrow a) = (x \to a) \odot x$, for every $a \leq x, a \in A\}$.

For a topology $\mathcal{F}$ on a pseudo $MTL-$algebra A and we denote by $\mathcal{F}' = \{I = J \cap C(A) : J \in \mathcal{F}\}$.

Definition 7.1. *Let $\mathcal{F}$ be a topology on A. A $\mathcal{F}-$* multiplier *is a mapping $f : I \to A/\theta_{\mathcal{F}}$ where $I \in \mathcal{F}'$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:*

(M_5) $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$;

(M_6) $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f(x)) = f(x)$.

Remark 7.1. *The axiom M_6, $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f(x)) = f(x)$, for every $x \in I$, implies $f(x) \leq x/\theta_{\mathcal{F}}$, so, since $x/\theta_{\mathcal{F}} \in C(A/\theta_{\mathcal{F}})$ this axiom become $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f(x)) = (x/\theta_{\mathcal{F}} \to f(x)) \odot x/\theta_{\mathcal{F}} = f(x)$, for every $x \in I$.*

By $dom(f) \in \mathcal{F}'$ we denote the domain of f; if $dom(f) = C(A)$, we called f *total.*

To simplify language, we will use $\mathcal{F}-$ *multiplier* instead *partial* $\mathcal{F}-$ *multiplier,* using *total* to indicate that the domain of a certain $\mathcal{F}-$ multiplier is $C(A)$.

If $\mathcal{F} = \{A\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of A so a $\mathcal{F}-$ multiplier is a total strong multiplier in sense of Definition 4.1, which verify the conditions M_1 and M_2.

The maps $\mathbf{0}, \mathbf{1} : C(A) \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in C(A)$ are $\mathcal{F}-$ multipliers in the sense of Definition 7.1.

Also, for $a \in B(A)$ and $I \in \mathcal{F}'$, $f_a : I \to A/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in I$, is a $\mathcal{F}-$ multiplier. If $dom(f_a) = C(A)$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}'$ and $M(A/\theta_{\mathcal{F}}) = \bigcup_{I \in \mathcal{F}'} M(I, A/\theta_{\mathcal{F}})$. If $I_1, I_2 \in \mathcal{F}'$, $I_1 \subseteq I_2$ we have a canonical mapping $\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}})$ defined by $\varphi_{I_1,I_2}(f) = f_{|I_1}$ for $f \in M(I_2, A/\theta_{\mathcal{F}})$. Let us consider the directed system of sets

$\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}'}, \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}', I_1 \subseteq I_2} \rangle$ and denote by $A_{\mathcal{F}}$ the inductive limit (in the category of sets) $A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}'} M(I, A/\theta_{\mathcal{F}})$. For any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ with $I \in \mathcal{F}'$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

Remark 7.2. *If $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$ multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}'$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Proposition 7.2. *If $I_1, I_2 \in \mathcal{F}'$ and $f_i \in M(I_i, A/\theta_{\mathcal{F}}), i = 1, 2$, then*

(c_{61}) $f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] = [x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x)$, *for every $x \in I_1 \cap I_2$.*

Proof. For $x \in I_1 \cap I_2$ we have $f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] = [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot x/\theta_{\mathcal{F}}] \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)) = (x/\theta_{\mathcal{F}} \to f_1(x)) \odot [x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x))] = [x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x)$.■

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}'$, $i = 1, 2$), $\mathcal{F}-$multipliers. Let us consider the mappings $f_1 \curlywedge f_2, f_1 \curlyvee f_2, f_1 \otimes f_2, f_1 \leftrightarrow f_2, f_1 \leftrightsquigarrow f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by

$$(f_1 \curlywedge f_2)(x) = f_1(x) \wedge f_2(x), (f_1 \curlyvee f_2)(x) = f_1(x) \vee f_2(x),$$

$$(f_1 \otimes f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \stackrel{c_{61}}{=} [x/\theta_{\mathcal{F}} \rightarrow f_1(x)] \odot f_2(x),$$
$$(f_1 \leftrightarrow f_2)(x) = [f_1(x) \rightarrow f_2(x)] \odot x/\theta_{\mathcal{F}},$$
$$(f_1 \leftrightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)],$$

for any $x \in I_1 \cap I_2$, and let

$$\begin{aligned} \widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} &= (I_1 \cap \widehat{I_2, f_1} \wedge f_2), \widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2), \\ \widehat{(I_1, f_1)} \otimes \widehat{(I_2, f_2)} &= (I_1 \cap \widehat{I_2, f_1} \otimes f_2), \widehat{(I_1, f_1)} \leftrightarrow \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \leftrightarrow f_2), \\ \text{and } \widehat{(I_1, f_1)} &\leftrightsquigarrow \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \leftrightsquigarrow f_2). \end{aligned}$$

Clearly, the definitions of the operations $\curlywedge, \curlyvee, \otimes, \leftrightsquigarrow$ and $\leftrightarrow$ on $A_{\mathcal{F}}$ are correct.

Lemma 7.2. $f_1 \curlywedge f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See [16] and Lemma 4.2. ■

Lemma 7.3. $f_1 \curlyvee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See [16] and Lemma 4.3.■

Lemma 7.4. $f_1 \otimes f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See [16] and Lemma 4.4. ■

Lemma 7.5. $f_1 \leftrightsquigarrow f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See [16] and Lemma 4.5. ■

Lemma 7.6. $f_1 \leftrightarrow f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See [16] and Lemma 4.6.■

Proposition 7.3. $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \leftrightarrow, \leftrightsquigarrow, \mathbf{0} = (\widehat{C(A), \mathbf{0}}), \mathbf{1} = (\widehat{C(A), \mathbf{1}}))$ *is a pseudo* MTL*-algebra.*

Proof. See the proof of Proposition 4.2.■

Remark 7.3. $(M(A/\theta_{\mathcal{F}}), \curlywedge, \curlyvee, \otimes, \leftrightarrow, \leftrightsquigarrow, \mathbf{0}, \mathbf{1})$ *is also a pseudo* MTL*-algebra.*

Definition 7.2. *The pseudo* MTL*-algebra* $A_{\mathcal{F}}$ *will be called the* localization MTL-algebra *of* A *with respect to the topology* $\mathcal{F}$ *.*

Remark 7.4. *If pseudo* $MTL-$ *algebra* A *is a* $MTL-$ *algebra in* [16] *will be called* $A_{\mathcal{F}}$ *the* localization MTL-algebra of A *with respect to the topology* $\mathcal{F}$ *.*

Theorem 7.1. (i)**:** *If pseudo* MTL*-algebra* A *is a* $MTL-$*algebra (resp. a pseudo* BL*-algebra) then* $A_{\mathcal{F}}$ *is also a* $MTL-$*algebra (resp. a pseudo* BL*-algebra);*
(ii)**:** *If pseudo* MTL*-algebra* A *is a pseudo* $IMTL$*-algebra (resp. a pseudo* WNM*-algebra or a pseudo* NM*-algebra) then* $A_{\mathcal{F}}$ *is also a pseudo* $IMTL$*-algebra (resp. a pseudo* WNM*-algebra or a pseudo* NM*-algebra).*

Proof. (i). See Remarks 4.4 and 4.5.
(ii). See the proof of Theorem 4.1. ■

Remark 7.5. *If pseudo* $MTL-$ *algebra* A *is a* $MTL-$*algebra (resp. a pseudo* BL*-algebra, a pseudo* $IMTL$*-algebra, a pseudo* WNM*-algebra, a pseudo* NM*-algebra), then pseudo* $MTL-$ *algebra* $M(A/\theta_{\mathcal{F}})$ *is a* $MTL-$*algebra (resp. a pseudo* BL*-algebra, a pseudo* $IMTL$*-algebra, a pseudo* WNM*-algebra, a pseudo* NM*-algebra).*

Lemma 7.7. *Let the map* $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ *defined by* $v_{\mathcal{F}}(a) = (\widehat{C(A), \overline{f_a}})$ *for every* $a \in B(A)$. *Then:*
(i) $v_{\mathcal{F}}$ *is a morphism of pseudo* MTL*-algebras;*
(ii) *For* $a \in B(A)$, $(\widehat{C(A), \overline{f_a}}) \in B(A_{\mathcal{F}})$;
(iii) $v_{\mathcal{F}}(B(A)) \in R(A_{\mathcal{F}})$.

Proof. $(i), (iii)$. As in the case of $MTL-$ algebras (see [16]).

(ii). For $a \in B(A)$ we have $a \vee a^{\sim} = a \vee a^{-} = 1$, hence $(a \wedge x) \vee [x \odot (a \wedge x)^{\sim}] \stackrel{c_{48}}{=} (a \wedge x) \vee [x \odot (a^{\sim} \vee x^{\sim})] \stackrel{c_{30}}{=} (a \wedge x) \vee [(x \odot a^{\sim}) \vee (x \odot x^{\sim})] \stackrel{c_{37}}{=} (a \wedge x) \vee [(x \odot a^{\sim}) \vee 0) = (a \wedge x) \vee (x \wedge a^{\sim}) \stackrel{c_{35}}{=} x \wedge (a \vee a^{\sim}) = x \wedge 1 = x$, and $(a \wedge x) \vee [(a \wedge x)^{-} \odot x] \stackrel{c_{49}}{=} (a \wedge x) \vee [(a^{-} \vee x^{-}) \odot x] \stackrel{c_{30}}{=} (a \wedge x) \vee [(a^{-} \odot x) \vee (x^{-} \odot x)] \stackrel{c_{37}}{=} (a \wedge x) \vee [(a^{-} \odot x) \vee 0) = (a \wedge x) \vee (a^{-} \wedge x) \stackrel{c_{35}}{=} (a \vee a^{-}) \wedge x = x \wedge 1 = x$, for every $x \in C(A)$. We deduce that $(a \wedge x)/\theta_{\mathcal{F}} \vee [x/\theta_{\mathcal{F}} \odot ((a \wedge x)/\theta_{\mathcal{F}})^{\sim}] = (a \wedge x)/\theta_{\mathcal{F}} \vee [((a \wedge x)/\theta_{\mathcal{F}})^{-} \odot x/\theta_{\mathcal{F}}] = x/\theta_{\mathcal{F}}$ hence $\overline{f_a} \vee (\overline{f_a})^{\sim} = \overline{f_a} \vee (\overline{f_a})^{-} = \mathbf{1}$, that is, $(\widehat{C(A), \overline{f_a}}) \curlyvee (\widehat{C(A), \overline{f_a}})^{\sim} = (\widehat{C(A), \overline{f_a}}) \curlyvee (\widehat{C(A), \overline{f_a}})^{-} = (\widehat{C(A), \mathbf{1}})$, so by Proposition 2.3, $(\widehat{C(A), \overline{f_a}}) \in B(A_{\mathcal{F}})$.■

8. Applications

In the following we describe the localization pseudo MTL-algebra $A_{\mathcal{F}}$ in some special instances.

1. If $I \in \mathcal{I}(A)$, and $\mathcal{F}$ is the topology $\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$ (see Example 6.1), then $A_{\mathcal{F}}$ is isomorphic with $M(I \cap C(A), A/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(A)$.

If I is a regular subset of A, then $\theta_{\mathcal{F}}$ is the identity, hence $A_{\mathcal{F}}$ is isomorphic with $M(I \cap C(A), A)$ (see [15]), which in generally is not a Boolean algebra.

2. **Main remark.** To obtain the maximal pseudo MTL -algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we have to develope another theory of multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

Definition 8.1. *Let* $\mathcal{F}$ *be a topology on* A. *A* strong - $\mathcal{F}-$ multiplier *is a mapping* $f : I \to A/\theta_{\mathcal{F}}$ *(where* $I \in \mathcal{F}' = \{J \cap C(A) : J \in \mathcal{F})$ *which verifies the axioms* M_5 *and* M_6 *(see Definition 7.1) and*
(M_7) *If* $e \in I \cap B(A)$, *then* $f(e) \in B(A/\theta_{\mathcal{F}})$;
(M_8) $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, *for every* $e \in I \cap B(A)$ *and* $x \in I$.

Remark 8.1. *If* $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ *is a pseudo* $MTL-$ *algebra, the maps* $\mathbf{0}, \mathbf{1} : C(A) \to A/\theta_{\mathcal{F}}$ *defined by* $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ *and* $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ *for every* $x \in C(A)$ *are strong -* $\mathcal{F}-$ *multipliers. We recall that if* $f_i : I_i \to A/\theta_{\mathcal{F}}$ *, (with* $I_i \in \mathcal{F}'$, $i = 1, 2)$ *are* $\mathcal{F}-$*multipliers* $f_1 \curlywedge f_2, f_1 \curlyvee f_2, f_1 \otimes f_2, f_1 \leftrightarrow f_2, f_1 \leftrightsquigarrow f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ *defined by* $(f_1 \curlywedge f_2)(x) = f_1(x) \wedge f_2(x), (f_1 \curlyvee f_2)(x) = f_1(x) \vee f_2(x), (f_1 \otimes f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \stackrel{c_{61}}{=} [x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x), (f_1 \leftrightarrow f_2)(x) = [f_1(x) \to f_2(x)] \odot x/\theta_{\mathcal{F}}$, $(f_1 \leftrightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)]$, *for any* $x \in I_1 \cap I_2$ *are* $\mathcal{F}-$*multipliers. If* f_1, f_2 *are strong -* $\mathcal{F}-$ *multipliers then* $f_1 \curlywedge f_2, f_1 \curlyvee f_2, f_1 \otimes f_2, f_1 \leftrightarrow f_2, f_1 \leftrightsquigarrow f_2$ *are also strong -* $\mathcal{F}-$ *multipliers. Indeed, if* $e \in I_1 \cap I_2 \cap B(A)$, *then*

$$(f_1 \curlywedge f_2)(e) = f_1(e) \wedge f_2(e) \in B(A/\theta_{\mathcal{F}}),$$

$$(f_1 \curlyvee f_2)(e) = f_1(e) \vee f_2(e) \in B(A/\theta_{\mathcal{F}}),$$

$$(f_1 \otimes f_2)(e) = [e/\theta_{\mathcal{F}} \to f_1(e)] \odot f_2(e) = [(e^{-})/\theta_{\mathcal{F}} \vee f_1(e)] \odot f_2(e) \in B(A/\theta_{\mathcal{F}}),$$

$$(f_1 \leftrightarrow f_2)(e) = [f_1(e) \to f_2(e)] \odot e/\theta_{\mathcal{F}} = [(f_1(e))^{-} \vee f_2(e)] \odot e/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}),$$

$$(f_1 \rightsquigarrow\!\!\!\!\leftarrow f_2)(e) = e/\theta_{\mathcal{F}} \odot [f_1(e) \rightsquigarrow f_2(e)] = e/\theta_{\mathcal{F}} \odot [(f_1(e))^{\sim} \vee f_2(e)] \in B(A/\theta_{\mathcal{F}}).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x/\theta_{\mathcal{F}} \wedge (f_1 \curlywedge f_2)(e) = x/\theta_{\mathcal{F}} \wedge f_1(e) \wedge f_2(e) = [x/\theta_{\mathcal{F}} \wedge f_1(e)] \wedge [x/\theta_{\mathcal{F}} \wedge f_2(e)] =$$

$$= [e/\theta_{\mathcal{F}} \wedge f_1(x)] \wedge [e/\theta_{\mathcal{F}} \wedge f_2(x)] = e/\theta_{\mathcal{F}} \wedge (f_1 \curlywedge f_2)(x)$$

and

$$x/\theta_{\mathcal{F}} \wedge (f_1 \curlyvee f_2)(e) = x/\theta_{\mathcal{F}} \wedge [f_1(e) \vee f_2(e)] =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(e)] \vee [x/\theta_{\mathcal{F}} \wedge f_2(e)] =$$
$$= [e/\theta_{\mathcal{F}} \wedge f_1(x)] \vee [e/\theta_{\mathcal{F}} \wedge f_2(x)] =$$
$$= e/\theta_{\mathcal{F}} \wedge [f_1(x) \vee f_2(x)] = e/\theta_{\mathcal{F}} \wedge (f_1 \curlyvee f_2)(x)$$

and

$$x/\theta_{\mathcal{F}} \wedge (f_1 \otimes f_2)(e) = x/\theta_{\mathcal{F}} \wedge [(e/\theta_{\mathcal{F}} \to f_1(e)) \odot f_2(e)]$$
$$= [(e/\theta_{\mathcal{F}} \to f_1(e)) \odot f_2(e)] \odot x/\theta_{\mathcal{F}} = [(e/\theta_{\mathcal{F}} \to f_1(e)) \odot x/\theta_{\mathcal{F}}] \odot f_2(e)$$
$$\stackrel{c_{58}}{=} [((e \odot x)/\theta_{\mathcal{F}} \to (f_1(e) \odot x/\theta_{\mathcal{F}})) \odot x/\theta_{\mathcal{F}}] \odot f_2(e)$$
$$= [(e \odot x)/\theta_{\mathcal{F}} \to (f_1(e) \odot x/\theta_{\mathcal{F}})] \odot [x/\theta_{\mathcal{F}} \odot f_2(e)]$$
$$= [(e \odot x)/\theta_{\mathcal{F}} \to (e/\theta_{\mathcal{F}} \odot f_1(x))] \odot [e/\theta_{\mathcal{F}} \odot f_2(x)]$$
$$= [((e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}}) \to (e/\theta_{\mathcal{F}} \odot f_1(x))) \odot e/\theta_{\mathcal{F}}] \odot f_2(x)$$
$$\stackrel{c_{57}}{=} [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot e/\theta_{\mathcal{F}}] \odot f_2(x) = [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot f_2(x)] \odot e/\theta_{\mathcal{F}}$$
$$= [(f_1 \otimes f_2)(x)] \odot e/\theta_{\mathcal{F}} = e/\theta_{\mathcal{F}} \wedge (f_1 \otimes f_2)(x)$$

and

$$e/\theta_{\mathcal{F}} \wedge (f_1 \leftrightarrow f_2)(x) = [(f_1(x) \to f_2(x)) \odot x/\theta_{\mathcal{F}}] \wedge e/\theta_{\mathcal{F}}$$
$$= [(f_1(x) \to f_2(x)) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} = [(f_1(x) \to f_2(x)) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}}$$
$$\stackrel{c_{57}}{=} [((f_1(x) \odot e/\theta_{\mathcal{F}}) \to (f_2(x) \odot e/\theta_{\mathcal{F}})) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}}$$
$$= [((x/\theta_{\mathcal{F}} \odot f_1(e)) \to (x/\theta_{\mathcal{F}} \odot f_2(e))) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} =$$
$$= [((x/\theta_{\mathcal{F}} \odot f_1(e)) \to (x/\theta_{\mathcal{F}} \odot f_2(e))) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} \stackrel{c_{58}}{=} [(f_1(e) \to f_2(e)) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} =$$
$$= [(f_1(e) \to f_2(e)) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} = [(f_1 \leftrightarrow f_2)(e)] \odot x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} \wedge (f_1 \leftrightarrow f_2)(e)$$

and

$$e/\theta_{\mathcal{F}} \wedge (f_1 \leftrightsquigarrow f_2)(x) = e/\theta_{\mathcal{F}} \wedge [x/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))]$$
$$= (e \odot x)/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)] = x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))]$$
$$\stackrel{c_{57}}{=} x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}} \odot f_1(x)) \rightsquigarrow (e/\theta_{\mathcal{F}} \odot f_2(x)))]$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}} \odot f_1(e)) \rightsquigarrow (x/\theta_{\mathcal{F}} \odot f_2(e)))] =$$
$$= e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}} \odot f_1(e)) \rightsquigarrow (x/\theta_{\mathcal{F}} \odot f_2(e)))] \stackrel{c_{58}}{=} e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(e) \rightsquigarrow f_2(e))] =$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(e) \rightsquigarrow f_2(e))] = x/\theta_{\mathcal{F}} \odot (f_1 \leftrightsquigarrow f_2)(e) = x/\theta_{\mathcal{F}} \wedge (f_1 \leftrightsquigarrow f_2)(e).$$

Remark 8.2. *Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a pseudo $MTL-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s - A_{\mathcal{F}}$ which will be called the strong-localization pseudo $MTL-$ algebra of A with respect to the topology $\mathcal{F}$.*

So, if $\mathcal{F} = \mathcal{I}(A) \cap R(A)$ is the topology of regular ideals, then $\theta_{\mathcal{F}}$ is the identity congruence of A and we obtain the definition for multipliers on A, so

$$s - A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}'} (\, s - M(I, A)),$$

where $s - M(I, A)$ is the set of strong multipliers of A having the domain I (see Definition 4.1, $M_1 - M_4$).

In this situation we obtain:

Proposition 8.1. *In the case $\mathcal{F} = \mathcal{I}(A) \cap R(A)$, $A_{\mathcal{F}}$ is exactly the maximal pseudo MTL-algebra $Q(A)$ of quotients of A which is a Boolean algebra. If pseudo $MTL-$ algebra A is a $MTL-$ algebra, $A_{\mathcal{F}}$ is exactly the maximal MTL-algebra $Q(A)$ of quotients of A.*

3. Denoting by $\mathcal{D}$ the topology of dense subsets of A, then (since $R(A) \subseteq D(A)$) there exists a morphism of pseudo MTL -algebras $\alpha : Q(A) \to s - A_{\mathcal{D}}$ such that the diagrame

$$\begin{array}{ccc} B(A) & \xrightarrow{\overline{v_A}} & Q(A) \\ \searrow_{v_D} & & \swarrow_{\alpha} \\ & s - A_{\mathcal{D}} & \end{array}$$

is commutative (i.e. $\alpha \circ \overline{v_A} = v_{\mathcal{D}}$). Indeed, if $[f, I] \in Q(A)$ (with $I \in \mathcal{I}'(A) \cap R(A)$ and $f : I \to A$ a strong multiplier in the sense of Definition 4.1) we denote by $f_{\mathcal{D}}$ the strong - $\mathcal{D}-$multiplier $f_{\mathcal{D}} : I \to A/\theta_{\mathcal{D}}$ defined by $f_{\mathcal{D}}(x) = f(x)/\theta_{\mathcal{D}}$ for every $x \in I$. Thus, α is defined by $\alpha([f, I]) = [f_{\mathcal{D}}, I]$.

4. Let $S \subseteq A$ a $\wedge-$closed system of pseudo MTL- algebra A.

As in the case of $MTL-$algebras we obtain the following result:

Proposition 8.2. *If $\mathcal{F}_S$ is the topology associated with a $\wedge-$closed system $S \subseteq A$, then the pseudo MTL-algebra $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

Remark 8.3. *In the proof of Proposition 8.2 the axiom M_8 is not necessarily.*

Concluding remarks

Since in particular a $MTL-$ algebra is a pseudo $MTL-$ algebra we obtain in this paper a part of the results about localization of $MTL-$ algebras, so we deduce that the main results of this paper are generalization of the analogous results relative to $MTL-$ algebras in [15], [16].

We use in the construction of localization pseudo $MTL-$ algebra $A_{\mathcal{F}}$ the Boolean center $B(A)$ of a pseudo $MTL-$ algebra A; as a consequence of this fact, $s - A_{\mathcal{F}}$ is a Boolean algebra in some particular cases.

A very interesting subject for future research would be a treatment of the localization for pseudo $MTL-$ algebras or residuated lattices without use the Boolean center.

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