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Blow-up boundary solutions for a class of nonhomogeneous logistic equations

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Abstract. In this paper we will be concerned with the equations $\Delta_{p(x)}u = g(x)f(u)$, where Ω is a bounded domain, g is a non-negative continuous function on Ω which is allowed to be unbounded on Ω and non-linearity f is a non-negative non-decreasing functions. We show that the equation $\Delta_{p(x)}u = g(x)f(u)$ admits a non-negative local weak solution $u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ such that $u(x) \to \infty$ as $x \to \partial\Omega$ if $\Delta_{p(x)}w = -g(x)$ in the weak sense for some $w \in W^{1,p(x)}_0(\Omega)$ and f satisfies a generalized Keller-Osserman condition.

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1. Introduction

Differential equations and variational problems with nonstandard $p(x)$-growth conditions have been studied intensively in the recent years. The results this paper have been obtained by Mohammed [24] in the case $p > 1$ is a real number.

In this paper, we will be concerned with local weak solutions to equations of the form

$$\Delta_{p(x)} = H(x,u), \quad x \in \Omega. \tag{1}$$

where $\Omega \subseteq \mathbf{R}^N$ is a bounded domain and $\Delta_{p(x)}v := div(|\nabla v|^{p(x)-2}\nabla v)$ is the $p(x)$-Laplacian, a function defined on $\mathbf{R}^n$ with $1 < p(x) < \infty$ and $H : \Omega \times \mathbf{R} \to \mathbf{R}$ is a continuous function with $H(x,t) = g(x)f(t)$.

By weak solution to (1) in the domain Ω we mean a function $u \in W^{1,p(x)}(\Omega)$ which satisfies

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi dx = -\int_\Omega H(x,u)\varphi dx \tag{2}$$

for all $\varphi \in W^{1,p(x)}_0(\Omega)$.

By local weak solution to (1) in the domain Ω we mean a function $u \in W^{1,p(x)}_{loc}(\Omega)$ which is a weak solution of (1) on D for every sub-domain D with $\overline{D} \subset \Omega$.

By local weak solution u of (1) we mean a (local weak) blow-up solution u which is continuous on Ω and

$$u(x) \to \infty \qquad as \quad d(x,\partial\Omega) \to 0.$$

We study in this paper the solutions $u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ to the problem

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) = g(x)f(u) & in \quad \Omega, \\ u(x) \to \infty & as \quad d(x,\partial\Omega) \to 0. \end{cases} \tag{3}$$

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The function g is supposed that is non-negative, which satisfies the following condition:

for any $x_0 \in \Omega$ satisfying $g(x_0) = 0$, there exists a sub-domain

$$O \text{ with } \overline{O} \subset \Omega \text{ containing } x_0 \text{ such that } g(x) > 0 \text{ for all } x \in \partial O. \tag{4}$$

Suppose that the non-linearity f satisfies

$(F1)$ $f : [0, \infty) \to [0, \infty)$ is a non-decreasing C^1 function such that $f(0) = 0$, and

$(F2)$ $f(s) > 0$ for $s > 0$.

The growth condition on f at infinity,

$$\int_1^\infty \frac{1}{(F(t))^{1/p(x)}} dt < \infty, \quad \textit{where} \quad F(t) := \int_0^t f(s) ds, \tag{5}$$

first introduced by Keller [18] and Osserman [25] and is crucial in the investigation of existence of blow-up solutions.

We will refer to the condition (5) as the generalized Keller-Osserman, or simply the Keller-Osserman condition.

Keller [18] and Osserman [25] gave the condition (5) that a necessary and sufficient for the equation $\Delta u = f(u)$ to admit a blow-up solution on a bounded domain Ω (with $p > 1$ a real number).

The Keller-Osserman type condition around the origin we have, also, in [26].

The important results clung of blow-up solutions have been obtained in the papers [1, 2, 3, 5, 7, 8, 9, 12, 15, 20, 21, 23] and references therein. Cîrstea and Rădulescu [6, 10, 11] prove the uniqueness and asymptotic behavior of solutions for problem

$$\triangle u = g(x) f(u), \; x \in \Omega, \; u(x) \to \infty, \text{ as dist } (x, \partial\Omega) \to \infty, \tag{6}$$

when $g \in C^{0,\alpha}(\Omega)$ is a nonnegative function and f is regularly varying.

We recall some definitions and basic properties of the variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbf{R}^N$.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x \in \Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real–valued function such that}$$

$$\int_\Omega |u(x)|^{p(x)} \, dx < \infty\},$$

endowed with the so–called *Luxemburg norm*

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} \, dx \le 1 \right\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [19]. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous, [19, Theorem 2.8].

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$, obtained by conjugating the exponent pointwise that is, $1/p(x) + 1/p'(x) = 1$, [19, Corollary 2.7]. For any $u \in$

$L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(x)}|v|_{p'(x)} \tag{7}$$

is valid.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbf{R}$ given by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ then the following relations hold

$$|u|_{p(x)} < 1\ (=1; >1) \quad \Leftrightarrow \quad \rho_{p(x)}(u) < 1\ (=1; >1) \tag{8}$$

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{9}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{10}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0, \tag{11}$$

since $p^+ < \infty$. For a proof of these facts see [19].

We will need the following comparison principle for weak solutions to equations.

Theorem 1.1. *(Weak comparison principle). Let $G : \mathbf{R} \to \mathbf{R}$ be continuous and further assume that it is non-increasing in the second variable. Let $u, v \in W^{1,p(x)}(\Omega)$ satisfy the inequalities*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \le \int_\Omega G(x,u)\varphi$$

and

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\varphi \ge \int_\Omega G(x,v)\varphi$$

for all non-negative $\varphi \in W_0^{1,p(x)}(\Omega)$. Then the inequality $u \le v$ on $\partial\Omega$ implies $u \le v$ in Ω.

Proof. See Lemma 2.3 in [28].

The other result which using is the following interior regularity result for weak solutions to equations. It is due to DiBenedetto [13] and Tolksdorf [27].

Theorem 1.2. *(DiBenedetto-Tolksdorf $C^{1,\alpha}$ interior regularity). Suppose $h(x,t)$ is measurable in $x \in \Omega$ and continuous in $t \in \mathbf{R}$ such that $|h(x,t)| \le \Gamma$ on $\Omega \times \mathbf{R}$. Let $u \in W^{1,p(x)} \cap L^\infty(\Omega)$ be a weak solution of $\Delta_{p(x)}u = h(x,u)$. Given a sub-domain D with $\overline{D} \subset \Omega$, there is $\alpha > 0$ and a positive constant C, depending on n, p, Γ, $||u||_\infty$ and D such that*

$$|\nabla u(x)| \le C \quad \text{and} \quad |\nabla u(x) - \nabla u(y)| \le C|x-y|^\alpha, \quad x, y \in D. \tag{12}$$

The paper is organized as follows. In Section 2 we present a sufficient condition on the weight g for problem (3) to admit a local weak blow-up solution. In Section 3 we investigated asymptotic boundary behavior of blow-up solutions.

2. Existence of blow-up solutions

In this section we assume that $H(x,t)$ satisfies the assumptions in Theorem 1.2.

We start with the following lemma that extends a result of Lair (see Theorem 1 of [20], see also [24]) to the $p(x)$-Laplacian case.

Lemma 2.1. *Let $D \subseteq \mathbf{R}^N$ be a bounded domain. Suppose that $g \in C(\overline{D})$ satisfied (4) on D. Let f satisfy the Keller-Osserman condition. Then the problem*

$$\begin{cases} div(|\nabla u|^{p(x)-2}\nabla u) = g(x)f(u) & in \quad D, \\ u(x) \to \infty & as \quad d(x,\partial D) \to 0, \end{cases} \tag{13}$$

admits a non-negative solution $u \in W^{1,p(x)}_{loc}(D) \cap C^{1,\alpha}(D)$, $0 < \alpha < 1$.

Proof. We follow the method used by Mohammed [24]. Let $u_k \in W^{1,p(x)}(D)$ be a weak solutions of

$$\begin{cases} div(|\nabla u|^{p(x)-2}\nabla u) = g(x)f(u) & x \in D, \\ u(x) = k & x \in \partial D, \end{cases} \tag{14}$$

for each $k = 1, 2, ...$, (see [12, Theorem 4.2]). Using the fact that $u \equiv 0$ is a solution of the above Dirichlet problem with $k = 0$, by the comparison principle we see that

$0 \leq u_k(x) \leq u_{k+1}(x), \quad x \in D,$

for all $k = 1, 2, \ldots$. By proceeding as in [20] we find that $\{u_k\}$ is uniformly bounded on sub-domains that are compactly contained in D. Let us consider U with $\overline{U} \subset D$ a sub-domain and take $x_0 \in U$. We have the following alternative: either $g(x_0) > 0$ or $g(x_0) = 0$. Suppose that $g(x_0) > 0$. Then there is a ball B containing x_0 such that $g > 0$ on $2B$. Let w be a blow-up solution of $\Delta_{p(x)}u = mf(u)$, $u = \infty$ on $\partial(2B)$, where $m > 0$ is the minimum of g on $2B$. The existence of such a blow-up solutions follows from [14, 22, 23]. Using again the comparison principle we deduce that $u_k \leq w$ on $2B$. But w is locally bounded. Therefore $u_k \leq C$ on B for all $k = 1, 2, ...$, and some $C > 0$. Now, suppose that $g(x_0) = 0$. Since condition (4) it follows that there exists a sub-domain O with $\overline{O} \subset D$ such that $g(x) > 0$ for all $x \in \partial O$. Now, arguing as in [20] we deduce that $u_k \leq C$ on ∂O for some C and all $k = 1, 2, \ldots$. Using again the comparison principle we see that $u_k \leq C$ on O for all $k = 1, 2, \ldots$. Therefore in any case we have that given $x_0 \in U$ there is a ball $B \subset U$ containing x_0 and a positive constant C_B such that $0 \leq u_k \leq C_B$ on B for all $k = 1, 2, \ldots$. By covering U by such balls we obtain that $\{u_k\}$ is indeed uniformly bounded on U.

From the Theorem 1.2 we see that sequences $\{u_k\}$ and $\{\nabla u_k\}$ are equicontinuous in subdomains compactly contained in Ω, and thus we can find a subsequence, which we still denote by $\{u_k\}$, such that $u_k \to u$ and $\nabla u_k \to v$ uniformly on compact subsets of D for some $u \in C(D)$ and $v \in (C(D))^n$. We immediately see that $v = \nabla u$ on D, and it follows from the interior $C^{1,\alpha}$ estimate (12) that $\nabla u \in C^{\alpha}(D)$ for some $0 < \alpha < 1$. Therefore $u \in W^{1,p(x)}_{loc}(D) \cap C^{1,\alpha}(D)$. Let U with $\overline{U} \subset D$ and $\varphi \in W^{1,p(x)}_0(U)$. Using again (12) we easily get that $|\nabla u_k|^{p(x)-1}|\nabla\varphi| \leq C|\nabla\varphi|$ on U and since the function $\xi \to |\xi|^{p(x)-2}\xi$ is continuous on $\mathbf{R}^n$, we deduce that

$$|\nabla u_k(x)|^{p(x)-2}\nabla u_k(x) \cdot \nabla\varphi(x) \to |\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla\varphi(x) \text{ for } x \in U.$$

Then by the dominated convergence theorem we obtain that

$$\int_U |\nabla u_k(x)|^{p(x)-2}\nabla u_k(x) \cdot \nabla\varphi(x) \to \int_U |\nabla u(x)|^{p(x)-2}\nabla u(x) \cdot \nabla\varphi(x).$$

Taking into account that $0 \le f(u_k) \le f(u_{k+1})$ and $f(u_k(x)) \to f(u(x))$ for each $x \in U$, with the monotone convergence theorem we get

$$\int_U gf(u_k)\varphi \to \int_U gf(u)\varphi.$$

Thus it follows that

$$\int_U |\nabla u|^{p(x)-2}\nabla u \cdot \nabla \varphi = -\int_U gf(u)\varphi, \quad \varphi \in W_0^{1,p(x)}(U),$$

and we see that u is a local weak solution of $\Delta_{p(x)}u = gf(u)$ on D. Using the fact that $u_k = k$ on ∂D we obtain that $u(x) \to \infty$ as $x \to \partial D$. ■ □

Now, we assume that f satisfies the Keller Osserman condition (5). Then it follows that (see Lemma 2.1 of [17])

$$\lim_{t\to\infty} \frac{(F(t))^{(p(x)-1)/p(x)}}{f(t)} = 0 \tag{15}$$

We obtain that for $t > 0$,

$$\int_t^\infty \frac{1}{f(s)^{1/(p(x)-1)}} ds < \infty.$$

Next, we define $\gamma : (0,\infty) \to (0, \gamma(0+))$ given by

$$\gamma(t) := \int_t^\infty \frac{1}{f(s)^{1/(p(x)-1)}} ds,$$

which is a decreasing function.

Next, we assume the following condition on $g \in C(\Omega)$ (introduced in [24]), which we will tell off to as the G-condition.

There exist a sequence $\{D_k\}$ of domains such that

(1) $\overline{D}_k \subseteq D_{k+1}$; $k = 1, 2, ...$

(2) $\Omega = \bigcup_{k=1}^\infty D_k$.

(3) g satisfied condition (4) on each D_k.

We consider the following Dirichlet problem:

$$\begin{cases} div(|\nabla w|^{p(x)-2}\nabla u) = -g(x), & x \in \Omega, \\ \quad w(x) = 0 \quad x \in \partial\Omega). \end{cases} \tag{16}$$

Next, we prove the following result

Theorem 2.1. *Let f be a function satisfying the Keller-Osserman condition, and suppose that $g \in C(\Omega)$ satisfy the G-condition. Then* (3) *admits a non-negative blow-up solution, if the Dirichlet problem* (16) *has a weak solution.*

Proof. We follow the method used by Mohammed [24]. Using the G-condition it follows that there exist domains D_j with $\overline{D}_j \subseteq D_{j+1} \subseteq \Omega$ such that $\bigcup_{j=1}^\infty D_j = \Omega$, and g satisfying the condition (4) on each D_j. Since $g \in C(\overline{D}_j)$ and g verifies condition (4) on D_j, by the Lemma 2.1 obtain that for each j there exists u_j a local weak blow-up solution of (3) with D_j replacing Ω. Using the comparison principle we get that $u_{j+1} \le u_j$ on D_j.

Now let $\varepsilon > 0$ be fixed, and we denote $v_j(x) := \gamma(u_j(x) + \varepsilon)$, $x \in D_j$. Then, it follows that

$$|\nabla v_j|^{p^+-2}\nabla v_j = |\gamma'(u_j+\varepsilon)|^{p^+-2}\gamma'(u_j+\varepsilon)|\nabla u_j|^{p^+-2}\nabla u_j.$$

and

$$\nabla\left(\left|\gamma'(u_j+\varepsilon)\right|^{p^+-2}\gamma'(u_j+\varepsilon)\right)=(p^+-1)\left|\gamma'(u_j+\varepsilon)\right|^{p^+-2}\gamma''(u_j+\varepsilon)\nabla u_j.$$

We also have that

$$\begin{aligned}\int_{D_j}|\nabla v_j|^{p(x)-2}\nabla v_j\cdot\nabla\varphi &\le \int_{D_j}|\nabla v_j|^{p^+-2}\nabla v_j\cdot\nabla\varphi\\ =\int_{D_j}|\nabla u_j|^{p^+-2}\nabla u_j\cdot\nabla\left(\left|\gamma'(u_j+\varepsilon)\right|^{p^+-2}\gamma'(u_j+\varepsilon)\varphi\right)\\ -\int_{D_j}|\nabla u_j|^{p^+-2}\nabla u_j\cdot\nabla\left(\left|\gamma'(u_j+\varepsilon)\right|^{p^+-2}\gamma'(u_j+\varepsilon)\right)\varphi\\ =-\int_{D_j}gf(u_j)\left|\gamma'(u_j+\varepsilon)\right|^{p^+-2}\gamma'(u_j+\varepsilon)\varphi\\ -(p^+-1)\int_{D_j}|\nabla u_j|^{p^+}|\gamma'(u_j)+\varepsilon|^{p^+-2}\gamma''(u_j+\varepsilon)\varphi.\end{aligned}$$

where $\varphi\in C_0^\infty(D_j)$ is a non-negative test function.

If we denote that

$$\left|\gamma'(t)\right|^{p^+-2}\gamma'(t)=-\frac{1}{f(t)}\quad and\quad \gamma''(t)=\frac{1}{p^+-1}\frac{f'(t)}{f(t)^{p^+/(p^+-1)}},$$

then we have the equation

$$\int_{D_j}|\nabla v_j|^{p^+-2}\nabla v_j\cdot\nabla\varphi=\int_{D_j}g\frac{f(u_j)}{f(u_j+\varepsilon)}\varphi-\int_{D_j}|\nabla u_j|^{p^+}\frac{f'(u_j+\varepsilon)}{f^2(u_j+\varepsilon)}\varphi.$$

Therefore, we obtain that

$$\int_{D_j}|\nabla v_j|^{p(x)-2}\nabla v_j\cdot\nabla\varphi\le\int_{D_j}g\varphi,\quad 0\le\varphi\in C_0^\infty(D_j).$$

Using the density argument we get that the above inequality is still valid for all $0\le\varphi\in W_0^{1,p(x)}(D_j)$.

Using again the comparison principle, we obtain that

$$v_j(x)\le w(x)\quad for\quad all\quad x\in D_j, \tag{17}$$

where w is a local weak solution to the Dirichlet problem (16). Let D a domain in Ω with $\overline{D}\subset\Omega$. We choose m such that $D\subseteq D_m$. We observe that the sequence $\{u_j(x)\}_{j=m+1}^\infty$, with $x\in D_m$, is a monotone non-increasing sequence bounded below by $\gamma^{-1}(w)$. Using the regularity theorem we also obtain that $\{\nabla u_j\}_{j=k}^\infty$ is equicontinuous on D_k. Hence by diagonal extraction we find a subsequence $\{u_j\}$ such that $u_j(x)\to u(x)$ and $\nabla u_j(x)\to\nabla u(x)$ for $x\in D$. We observe that for all $k\ge m+1$ the following inequalities holds:

$$|\nabla u_k|^{p(x)-1}|\nabla\varphi|\le C_m|\nabla\varphi|,\quad f(u_k)\le f(u_{m+1})\quad on\quad D,$$

where $\varphi\in C_0^\infty(D)$.

Taking into account these inequalities and the pointwise convergence we obtain

$$|\nabla u_k(x)|^{p(x)-2}\nabla u_k\cdot\nabla\varphi\to|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi,\quad f(u_k)\to f(u)\quad on\quad D.$$

By the Lebesgue convergence theorem, we see that

$$\int_D|\nabla u_k(x)|^{p(x)-2}\nabla u_k(x)\cdot\nabla\varphi(x)\to\int_D|\nabla u(x)|^{p(x)-2}\nabla u(x)\cdot\nabla\varphi(x)$$

and

$$\int_D gf(u_k)\varphi \to \int_D gf(u)\varphi.$$

Thus, we have that

$$\int_D |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi = -\int_D gf(u)\varphi, \quad \varphi \in C_0^\infty(D).$$

Using the usual density argument, we obtain that the above equation continues to hold for any $\varphi \in W_0^{1,p(x)}(D)$, and it follows that u is a local weak solution of $\triangle_{p(x)}u = gf(u)$ on Ω. But, by (17) we observe, since $\varepsilon > 0$ is arbitrary, that $\gamma(u) \le w$ on Ω. Taking $U \subseteq \Omega$ a neighborhood of the boundary $\partial\Omega$ such that $0 \le w \le \gamma(0+)$ it follows that $u(x) \ge \gamma^{-1}(w(x))$ on U and so $u(x) \to \infty$ as $x \to \partial\Omega$. ■ □

Corollary 2.1. *We assume that $g \in C(\Omega)$ for which the Dirichlet problem* (16) *admits a weak solution w and let f be a function which satisfies the Keller-Osserman condition. Then we have*

$$u(x) \ge \gamma^{-1}(w(x))$$

for any non-negative blow-up solution u of (3) *and for x near $\partial\Omega$.*

Proof. Since the proof of the above theorem we observe that the function

$$v(x) = \gamma(u+\varepsilon), \quad \varepsilon > 0,$$

where u is a non-negative blow-up solution of (3), which satisfies the inequality

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\varphi \le \int_\Omega g\varphi, \quad 0 \le \varphi \in W_0^{1,p(x)}(\Omega).$$

Using again the comparison principle we get

$$\gamma(u+\varepsilon)(x) \le w(x), \quad x \in \Omega.$$

Thus, we have

$$u(x) + \varepsilon \ge \gamma^{-1}(w(x)) \quad for \quad x \quad near \quad \partial\Omega.$$

Using the fact that $\varepsilon > 0$ is arbitrary, we conclude that $u(x) \ge \gamma^{-1}(w(x))$. ■ □

3. Asymptotic boundary behavior of blow-up solutions

In this section we investigated asymptotic boundary behavior of blow-up solutions of (3) near the boundary $\partial\Omega$. We denote by $d(x)$ the distance of $x \in \Omega$ to the boundary $\partial\Omega$ and with $d(x, \partial D)$ the distance of $x \in D$ to ∂D for other domains D. For the next result we assume that Ω is a bounded domain with C^2 boundary $\partial\Omega$. We know that (see [16]) these exists a positive number $\mu = \mu(\Omega)$, depending only on Ω, such that the distance function $d(x)$, $x \in \Omega$ is in $C^2(\overline{\Gamma}_\mu)$, where Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$ of class C^2 and

$$\Gamma_\mu : \{x \in \Omega : d(x) < \mu\}.$$

Moreover, we have that

$$|\nabla d(x)| = 1, \quad x \in \overline{\Gamma}_\mu.$$

Next, we prove the following result.

Theorem 3.1. *Let $\partial\Omega$ be of class C^2, let f a function which satisfies the Keller-Osserman condition, let $g \in C(\Omega)$ be such that the Dirichlet problem (16) has a solution. If* $\sup_\Omega g(x)d(x)^{(1-\beta)(p(x)-1)+1} \leq N < \infty$ *for some $0<\beta<1$, then there is a neighborhood O of the boundary $\partial\Omega$ and a positive constant α, depending only on Ω and the weight g, such that for any non-negative blow-up solution u of (3),*

$$u(x) \geq \gamma^{-1}(\alpha d(x)^\beta), \quad x \in O.$$

Proof. We follow the method used by Mohamed [24]. Let β be a real number such that $0<\beta<1$, and let $v(x) = \alpha d(x)^\beta$ with $\alpha > 0$ to be determined later. Using the fact that $v \in C^2(\Gamma_\mu)$, and that $|\nabla d(x)| = 1$ on Γ_μ we get

$$\nabla v = \alpha\beta d(x)^{\beta-1}\nabla d(x)$$

and

$$|\nabla v|^{p(x)-2}\nabla v = (\alpha\beta)^{p(x)-1} d(x)^{(p(x)-1)(\beta-1)}\nabla d(x).$$

Set $M := \max\{|\triangle d(x)| : x \in \overline{\Gamma}_\mu\}$.

On $\Gamma_\eta = \{x \in \Omega : d(x) < \eta\}$, where $0 < \eta \leq \mu$, we have:

$$\begin{aligned} &-div(|\nabla v|^{p(x)-2}\nabla v) - g(x) \\ &= d(x)^{(p(x)-1)(\beta-1)-1}\{-(\alpha\beta)^{p(x)-1}[ln(\alpha\beta)\cdot\nabla(p(x)-1)\nabla d(x)\cdot d(x) \\ &\quad + d(x)\cdot(\beta-1)\cdot\nabla(p(x)-1)\cdot d(x) lnd(x) + 1 + d(x)\cdot\triangle d(x)] \\ &\quad - g(x)d(x)^{(p(x)-1)(1-\beta)+1}\} \\ &\geq d(x)^{(p(x)-1)(\beta-1)-1}\{-(\alpha\beta)^{p(x)-1}[ln(\alpha\beta)\cdot\nabla(p(x)-1)\nabla d(x)\cdot d(x) \\ &\quad + d(x)\cdot(\beta-1)\cdot\nabla(p(x)-1)\cdot d(x) lnd(x) + 1 + d(x)\cdot M] - N\} \end{aligned}$$

We now choose $\eta > 0$ small enough and $\alpha > 0$ big enough such that

$$-div(|\nabla v|^{p(x)-2}\nabla v) - g(x) \geq 0.$$

Taking into account this choices of α and η it follows that the function v satisfies

$$-div(|\nabla v|^{p(x)-2}\nabla v) - g(x) \geq 0 \quad on \quad \Gamma_\eta.$$

Now, since w is the solution to the Dirichlet problem (16) it follows from Corrollary 2.1 that for any blow-up solution u of (3), we have $u(x) \geq \gamma^{-1}(w(x))$ in some neighborhood U of $\partial\Omega$. We choose $\eta > 0$ such that $\Gamma_\eta \subseteq U$ and α large enough such that $\alpha\eta^\beta \geq w$ on the set $\{x \in \Omega : d(x) = \eta\}$ so that $v \geq w$ on the boundary $\partial\Gamma_\eta$. Using the weak comparison principle we get that $w \leq v$ on Γ_η. Hence we see that $u \geq \gamma^{-1}(v)$ on Γ_η. With this the proof is complete. ■ □

We assume that f satisfies the Keller-Osserman condition and we consider the function

$$\psi(t) = \int_t^\infty \frac{1}{q(s)F(s)^{1/p(s)}}ds, \quad t > 0. \tag{18}$$

where $1/(q(s)) + 1/(p(s)) = 1$

The function ψ is decreasing and let ϕ be the inverse of the function ψ. Next, we use the following condition, introduced in [3]. At this condition we will as the Bandle-Marcus condition on f. Suppose that the function ψ defined in (18) satisfy

$$\lim_{t\to\infty}\inf \frac{\psi(\beta t)}{\psi(t)} > 1 \quad for \quad any \quad 0<\beta<1. \tag{19}$$

We recall a result from [3](see also [24])

Lemma 3.1. *Assume that $\psi \in C[t_0, \infty)$ is strictly monotone decreasing and satisfies* (19) *and let $\phi := \psi^{-1}$. If is given a positive number γ there exist positive numbers η_γ, ρ_γ with the following property:*

$$\text{If } \gamma > 1, \quad \text{then } \phi\left((1-\eta)\rho\right) \leq \gamma\phi(\rho) \quad \text{for all} \quad \eta \in [0, \eta_\gamma], \quad \rho \in [0, \rho_\gamma]; \tag{20}$$

$$\text{If } \gamma < 1, \quad \text{then } \phi\left((1+\eta)\rho\right) \geq \gamma\phi(\rho) \quad \text{for all } \eta \in [0, \eta_\gamma], \quad \rho \in [0, \rho_\gamma]. \tag{21}$$

Given a bounded domain D with C^2 boundary ∂D and a function f with satisfies the Keller-Osserman condition, we note that if $u \in W^{1,p(x)}_{loc}(D) \cap C(D)$ is a solution of

$$\triangle_{p(x)} u = f(u), \quad u(x) \to \infty, \quad x \to \partial D,$$

then by [23] it is known that

$$\lim_{d(x,\partial D)} \frac{u(x)}{\phi(d(x, \partial D))} = 1. \tag{22}$$

A similar result for weak blow-up solutions of (3) when $g \in C(\overline{\Omega})$ we prove in the next theorem.

Let Ω be a bounded domain in $\mathbf{R}^n$. We say that Ω satisfies a uniform interior(exterior) sphere condition if there is $R > 0$ such that for any $y \in \partial\Omega$ and any $0 < r < R$ there is a ball $B := B_r(x)$ contained in Ω (contained in the complement Ω^c) such that $\partial B \cap \partial\Omega = \{y\}$. (see [24])

Theorem 3.2. *We assume that Ω is a bounded domain that satisfies both the uniform interior and exterior sphere conditions. Let f be a function which satisfies both the Keller-Osserman and the Bandle-Marcus conditions. If $g \in C(\overline{\Omega})$ such that $g > 0$ on the boundary $\partial\Omega$ then for any non-negative solution u of* (3) *we have*

$$\lim_{d(x) \to 0} \frac{u(x)}{\phi(g(x)^{1/p^+} d(x))} = 1. \tag{23}$$

Proof. We follow the method introduced by Bandle and Marcus in [3] (see also [23], [24]). Let $z \in \partial\Omega$. First we prove that

$$\lim_{x \to z} \sup \frac{u(x)}{\phi(g^{1/p^+}(x) d(x))} \leq 1.$$

Let $\varepsilon > 0$. Taking $\gamma = 1 + \varepsilon$ in Lemma 3.1 we find $\eta_\varepsilon < 1$ and $\rho_\varepsilon > 0$ such that (20) holds for all $\eta \in [0, \eta_\varepsilon]$, $\rho \in [0, \rho_\varepsilon]$. We choose $\eta > 0$ such that $2\eta - \eta^2 < \eta_\varepsilon$. We observe that $g^{1/p^+}(x) d(x) \to 0$ as $x \to z$ and we see that g^{1/p^+} is continuous on $\overline{\Omega}$. Hence there is $r_1 = r_1(z, \eta)$ such that $|g^{1/p^+}(x) - g^{1/p^+}(z)| < g^{1/p^+}(z)\eta$ and $g^{1/p^+}(x) d(x, \partial B) < \rho_\varepsilon$ when $|x - z| < r_1$, $x \in \overline{\Omega}$.

We observe that $g^{1/p^+}(z) > g^{1/p^+}(x)(1 - \eta)$ for any $x \in \Omega$, $|x - z| < r_1$. Hence, for such x we see that

$$\alpha^{1/p^+} > g^{1/p^+}(z)(1 - \eta) > g^{1/p^+}(x)(1 - \eta)^2.$$

Taking into account the fact that ϕ is decreasing and the above inequality, we have that

$$\phi\left(\alpha^{1/p+} d(x)\right) \leq \phi\left(g^{1/p^+}(x) d(x)\right)\left(1 - (2\eta - \eta^2)\right), \quad x \in \Omega, |x - z| < r_1.$$

It follows from (20) of Lemma 3.1 and the above inequality that

$$\phi\left(\alpha^{1/p^+} d(x)\right) \leq (1 + \varepsilon)\phi\left(g^{1/p^+}(x) d(x)\right), \quad x \in \Omega, \ |x - z| < r_1. \tag{24}$$

Using the fact that Ω satisfies the interior sphere condition we choose a ball $B := B(x_0, r) \subseteq \Omega$ such that $\partial B \cap \partial\Omega = \{z\}$. We assume that the radius r is small enough such that $r < r_1/2$ and $g > 0$ on B.

Set $\alpha = \inf\{g(x) : x \in B\} > 0$ and let $w \in W^{1,p(x)}_{loc}(B) \cap C(B)$ be a blow-up solution of

$$\triangle_{p(x)} w = \alpha f(w).$$

It follows from the comparison principle that

$$u(x) \le w(x), \quad x \in B \tag{25}$$

Letting αf instead of f in (18) and taking ϕ^α for the corresponding inverse, we observe that

$$\phi^\alpha(s) = \phi(\alpha^{1/p^+} s).$$

It follows from (22) that

$$\lim_{d(x,\partial B)\to 0} \frac{w(x)}{\phi(\alpha^{1/p^+} d(x, \partial B))} = 1.$$

Also for the each $\varepsilon > 0$ given there exists a $\rho > 0$ such that

$$w(x) \le (1+\varepsilon)\phi\left(\alpha^{1/p^+} d(x,\partial B)\right), \quad \text{with } x \in B \text{ and } d(x, \partial B) < \rho. \tag{26}$$

Next, we choose $x \in B$ on segment $\overline{x_0 z}$ such that $d(x, \partial B) = d(x)$. We observe that for this x we have from (24) and (26) that

$$\phi\left(\alpha^{1/p^+} d(x,\partial B)\right) \le (1+\varepsilon)^2 \phi\left(g^{1/p^+}(x) d(x)\right).$$

From the above inequality and (25) we obtain that

$$u(x) \le (1+\varepsilon)^2 \phi\left(g^{1/p^+}(x) d(x)\right)$$

for $x \in B$ which lies on the segment $\overline{x_0 z}$. Hence

$$\lim_{d(x)\to 0} \sup \frac{u(x)}{\phi(g^{1/p^+}(x) d(x))} \le 1. \tag{27}$$

Taking into account the fact that Ω satisfies the exterior sphere condition and using a similar argument as in the above we can show that

$$\lim_{d(x)\to 0} \inf \frac{u(x)}{\phi(g^{1/p^+}(x) d(x))} \ge 1.$$

Let $z \in \partial\Omega$ and let $B = B(x_0, r) \subseteq \Omega^c$ such that $\partial B \cap \partial\Omega = \{z\}$. We denote $D(r) = B(x_0, 2r) \setminus B(x_0, r)$ and set $\beta := \sup\{g(x) : x \in \overline{\Omega \cap D(r)}\}$. From [23] it follows that there exists $w \in W^{1,p(x)}_{loc}(D(r))$ a solution of

$$\triangle_{p(x)} w = \beta f(w), \quad w(x) \to \infty \text{ on } \partial B(x_0, r) \text{ and } w(x) = 0 \text{ on } \partial B(x_0, 2r).$$

Using the comparison principle we obtain that

$$w(x) \le u(x), \quad x \in \Omega \cap D(r). \tag{28}$$

By [23] we see that w satisfies

$$\lim_{D(r)\ni x \to \partial B(x_0,r)} \frac{w(x)}{\phi(\beta^{1/p^+} d(x, \partial B(x_0, r)))} \ge 1.$$

In a similar manner it follows from (21) of Lemma 3.1 that

$$\lim_{x\to\partial\Omega} \inf \frac{u(x)}{\phi(g(x)^{1/p^+} d(x, \partial\Omega))} \ge 1. \tag{29}$$

Using (27) and (29) it follows that the proof is complete. ■ □

Note that, in general, a solutions u of (3) does not satisfy the limit in (23). From [4] we see that the limit superior in (3.10) could be zero, while the limit inferior in (29) could be infinity.

Therefore we assume that

$$\inf_{\Gamma_\eta} g(x) > 0 \text{ for some } \eta > 0,$$

$$g(x)^{1/p^+} d(x, \partial\Omega) \to 0 \text{ as } d(x) \to 0. \tag{30}$$

Theorem 3.3. *We assume that $g \in C(\Omega)$ satisfy (30) and let f be a function which satisfies both the Keller-Osserman and the Bandle-Marcus conditions. If u is a blow-up solution of (3) then we have*

$$\lim_{d(x)\to 0} \sup \frac{u(x)}{\phi(g^{1/p^+}(x)d(x))} \leq 1.$$

Proof. We follow the method used by Mohamed [24]. We prove arguing by contradiction. We assume that there is a sequence of points $x_i \in \Omega$ with $d(x_i) \to 0$ as $i \to \infty$ such that

$$\frac{u(x_i)}{\phi(g^{1/p^+}(x_i)d(x_i))} > 1 + \alpha$$

for all $i = 1, 2, ...$ and for some $\alpha > 0$. It follows from continuity that for each $i = 1, 2, ...$, there is a ball $B_i(x_i) \subseteq \Omega$ centered at x_i such that

$$u(x) \geq (1+\alpha)\phi\left(g^{1/p^+}(x)d(x)\right), \quad x \in B_i(x_i). \tag{31}$$

For $\gamma := 1 + \alpha/2$ we find η_α and ρ_α such that inequality (20) of Lemma 3.1 holds for all $\eta \in [0, \eta_\alpha]$ and all $\rho \in [0, \rho_\alpha]$. We denote

$$\Omega_i := \{x \in \Omega : d(x) > d(x_i)\}.$$

We choose $\nu > 0$ such that for all $x \in \Omega$ with $d(x) < \nu$ we have

$$g^{1/p^+}(x)d(x) \in [0, \rho_\alpha], \qquad \frac{d(x,\partial\Omega) - d(x,\partial\Omega_i)}{d(x,\partial\Omega)} \in [0, \eta_\alpha],$$

for all i sufficiently large.

We observe that

$$\phi\left(g(x)^{1/p^+} d(x,\partial\Omega_i)\right) = \phi\left(g(x)^{1/p^+} d(x,\partial\Omega)\left(1 - \frac{d(x,\partial\Omega) - d(x,\partial\Omega_i)}{d(x,\partial\Omega)}\right)\right)$$
$$\leq (1+\alpha/2)\phi\left(g(x)^{1/p^+} d(x,\partial\Omega)\right). \tag{32}$$

Using the comparison principle, we obtain that

$$w_i(x) \geq u(x), \quad x \in \Omega_i,$$

where w_i is a blow-up solution of (3) on the set Ω_i.

By this and (31) it follows that

$$w_i(x) \geq (1 + \alpha/(\alpha+2))\,\phi\left(g(x)^{1/p^+} d(x,\partial\Omega_i)\right), \quad x \in B_i(x_i) \cap (\Omega_i \setminus \Omega_\nu),$$

which is in contradiction with the fact that

$$\lim_{d(x,\partial\Omega_i)\to 0} \sup \frac{w_i(x)}{\phi(g^{1/p^+}(x)d(x,\partial\Omega_i))} \leq 1,$$

and thus the proof is complete. ■ □

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