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Approximate reasoning using Yager implication

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Abstract. Using Generalized Modus Ponens reasoning we examine the values of the inferred conclusion working with the Yager implication in order to interpret a fuzzy if-then rule with a single input single output and with the t-norm $t(x,y) = \max((1+\lambda)(x+y-1) - \lambda xy, 0)$, $\lambda \geq -1$ for composition operation. In this way we complete some of our previous results.

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1. Introduction

The database of a rule-based system may contains imprecisions which are inherent in the description of the rules by the expert. A difficulty appears in the utilization of these rules when the observed facts do not match the condition expressed in the premise of the rule, but are not too different from them. Another difficulty is given by representation of knowledge expressed, in most cases, by means of natural language statements. These problems led Zadeh to outline the theory of approximate reasoning [24], that is the deduction of imprecise conclusion from a set of imprecise premises based on fuzzy logic.

Starting from Modus Ponens rule, Zadeh [25] obtains the Generalized Modus Ponens (GMP) reasoning. Stated in the form of a syllogism the Generalized Modus Ponens looks like

$$\begin{array}{l} if\ X\ is\ A\ then\ Y\ is\ B \\ \quad X\ is\ A' \\ \hline \qquad\qquad\qquad Y\ is\ B' \end{array}$$

where A, B, A' and B' are modeled by fuzzy sets.

An investigation of inference processes in the fuzzy if-then rules is a subject of many papers in literature [1-25].

2. Knowledge representation

In an expert system an elementary piece of information can be represented in canonical form "$X\ is\ A$", where X is a variable representing the attribute of the entity and A is its value.
The proposition

$$X\ is\ A$$

can be understood as

$$the\ quantity\ X\ satisfies\ the\ predicate\ A$$

or

$$\textit{the variable } X \textit{ takes its values in the set } A.$$

As pointed out Zadeh [22, 23], the semantic content of the proposition

$$X \ is \ A$$

can be represented by

$$\pi_X = \mu_A,$$

where π_X is the possibility distribution restricting the possible value of X and μ_A is the membership function of the set A.

Because the majority of practical applications work with trapezoidal or triangular distributions and these representations are still a subject of various recent papers ([21], for instance) we will work with membership functions represented by trapezoidal fuzzy numbers. Such a number $N = (a, b, \alpha, \beta)$ is defined as

$$\mu_N(x) = \begin{cases} 0 \ \ for \ \ x < a - \alpha \\ \dfrac{x - a + \alpha}{\alpha} \ \ \ for \ \ \ x \in [a - \alpha, a] \\ 1 \ \ \ for \ \ \ x \in [a, b] \\ \dfrac{b + \beta - x}{\beta} \ \ \ for \ \ \ x \in [b, b + \beta] \\ 0 \ \ for \ \ x > b + \beta \end{cases}$$

In order to represent a rule, the notion of fuzzy implication is used. We recall an axiomatic approach (formulated by Fodor in [7]) for the definition of fuzzy implication.

Definition 2.1. *An implication is a function* $I : [0,1]^2 \to [0,1]$ *satisfying the following conditions:*

I1: If $x \leq z$ *then* $I(x,y) \geq I(z,y)$ *for all* $x, y, z \in [0,1]$

I2: If $y \leq z$ *then* $I(x,y) \leq I(x,z)$ *for all* $x, y, z \in [0,1]$

I3: $I(0,y) = 1$ *(falsity implies anything) for all* $y \in [0,1]$

I4: $I(x,1) = 1$ *(anything implies tautology) for all* $x \in [0,1]$

I5: $I(1,0) = 0$ *(Booleanity)*

Another basic notion very important in Generalized Modus Ponens reasoning is given by the next definition.

Definition 2.2. *A function* $T : [0,1]^2 \to [0,1]$ *is a t-norm iff it is commutative, associative, non-decreasing and* $T(x,1) = x \ \forall x \in [0,1]$.

Let X and Y be two variables whose domains are U and V, respectively. A causal link from X to Y is represented as a conditional possibility distribution [24, 25] $\pi_{Y/X}$ which restricts the possible values of Y for a given value of X. For the rule

$$if \ X \ is \ A \ then \ Y \ is \ B$$

we have

$$\forall \ u \ \in U, \ \forall \ v \ \in V, \ \pi_{Y/X}(v,u) = \mu_A(u) \to \mu_B(v)$$

where $\to$ is an implication operator and μ_A and μ_B are the possibility distributions of the propositions "*X is A*" and "*Y is B*", respectively.

If $\mu_{A'}$ is the possibility distribution of the proposition

$$X \ is \ A'$$

then from the rule

$$if\ X\ is\ A\ then\ Y\ is\ B$$

and the fact

$$X\ is\ A'$$

Generalized Modus Ponens rule computes the possibility distribution $\mu_{B'}$ of the conclusion

$$Y\ is\ B'$$

as

$$\mu_{B'}(v) = \sup_{u \in U} T\left(\mu_{A'}(u), \pi_{Y/X}(v, u)\right),$$

where T is a t-norm.

3. Main results

In some of our papers [10, 11, 12, 13] we analyzed the Generalized Modus Ponens reasoning with t-norm

$$t(x, y) = \max\left((1 + \lambda)(x + y - 1) - \lambda xy, 0\right), \quad \lambda \geq -1$$

and various implication operators: Reichenbach, Willmott, Rescher-Gaines, Kleene-Dienes, Brouwer-Gödel, Goguen, Lukasiewicz, Fodor. In this paper we will continue the researches above mentioned using the same t-norm and Yager implication:

$$I_Y(x, y) = \begin{cases} 1 & if \quad x = 0 \\ y & if \quad x > 0 \end{cases}$$

Theorem 3.1. *From the rule "$if\ X\ is\ A\ then\ Y\ is\ B$" and the fact "$X\ is\ A'$" the Generalized Modus Ponens rule with Yager implication gives the following results:*

1. *$B' = B$ if $A' \subset A$ or $A' = A$*
2. *$B' \supseteq B$ if $A' \supset A$*
3. *if A and A' have a partial overlapping then*
   - *3.1. $B' = V$ if $core(A') \nsubseteq support(A)$*
   - *3.2. $B' \supseteq B$ if $core(A') \subseteq support(A)$*

Proof. 1.a. The premise contains the observation, i. e. $\mu_{A'}(u) \leq \mu_A(u)\ \forall u \in U$

$$\begin{aligned}\mu_{B'}(v) = sup_{u \in U} max((1 + \lambda)(\mu_{A'}(u) + I_Y(\mu_A(u), \mu_B(v)) - 1)\\ -\lambda \mu_{A'}(u) I_Y(\mu_A(u), \mu_B(v)), 0)\end{aligned}$$

- value on the set $U_1 = \{u \in U / \mu_A(u) = 0\}$

$$\mu_{B'}(v) = sup_{u \in U_1} max(\mu_{A'}(u), 0) = 0$$

- value on the set $U_2 = \{u \in U / \mu_A(u) > 0\}$

$$\begin{aligned}\mu_{B'}(v) &= sup_{u \in U_2} max((1 + \lambda)(\mu_{A'}(u) + \mu_B(v) - 1) - \lambda \mu_{A'}(u) \mu_B(v), 0)\\ &= sup_{u \in U_2} max(\mu_{A'}(u)(1 + \lambda - \lambda \mu_B(v)) - (1 + \lambda)(1 - \mu_B(v)), 0)\\ &= \mu_B(v)\end{aligned}$$

and this value is obtained for $u_0 \in U_2$ such that

$$\mu_{A'}(u_0) = \mu_A(u_0) = 1.$$

1.b. The premise and the observation coincide, i. e. $\mu_A(u) = \mu_{A'}(u)\ \forall u \in U$

In this case one repeat the proof from the previous case taking account the equality

$$\mu_A(u) = \mu_{A'}(u)\ \ \forall u \in U;$$

it results

$$\mu_{B'}(v) = \mu_B(v).$$

2. The observation contains the premise, i. e. $\mu_A(u) \le \mu_{A'}(u)\ \forall u \in U$

• value on the set $U_1 = \{u \in U/\mu_A(u) = 0\}$

$$\begin{aligned}\mu_{B'}(v) &= sup_{u\in U_1} max(\mu_{A'}(u), 0)\\ &= \mu_B(v) \text{ on the set } U_2 = \{u \in U_1/\mu_{A'}(u) \le \mu_B(v)\}\\ &> \mu_B(v) \text{ on the set } U_3 = \{u \in U_1/\mu_{A'}(u) > \mu_B(v)\}\end{aligned}$$

• value on the set $U_4 = \{u \in U/\mu_A(u) > 0\}$

$$\begin{aligned}\mu_{B'}(v) &= sup_{u\in U_4} max((1+\lambda)(\mu_{A'}(u) + \mu_B(v) - 1) - \lambda\mu_{A'}(u)\mu_B(v), 0)\\ &= sup_{u\in U_4} max(\mu_{A'}(u)(1+\lambda-\lambda\mu_B(v)) + (1+\lambda)(\mu_B(v)-1), 0) = \mu_B(v)\end{aligned}$$

and this value is obtained for $\mu_{A'}(u_0) = 1$.

Finally, it results $\mu_{B'}(v) \ge \mu_B(v)$.

3. There is a partial overlapping between the sets A and A'

• the case $core(A') \nsubseteq support(A)$

On the set $U_1 = \{u \in U/\mu_A(u) = 0\}$ it results

$$\mu_{B'}(v) = sup_{u\in U_1} \mu_{A'}(u) = 1.$$

• the case $core(A') \subseteq support(A)$

Let $u_0 \in U$ such that $\mu_{A'}(u_0) = 1$. It results

$$\mu_{B'}(v) \ge T(\mu_{A'}(u_0), I_Y(\mu_A(u_0), \mu_B(v))) = I_Y(\mu_A(u_0), \mu_B(v)) = \mu_B(v).$$

The final result is

$$\begin{aligned}&\mu_{B'}(v) = 1 \text{ if } core(A') \nsubseteq support(A)\\ &\mu_{B'}(v) \ge \mu_B(v) \text{ if } core(A') \subseteq support(A).\end{aligned}$$

□

Theorem 3.2. *If the premise and the observation are contradictory (i. e. $\mu_{A'}(u) = 1 - \mu_A(u)\ \ \forall u \in U$) then $\mu_{B'}(v) = 1\ \forall v \in V$.*

Proof. We have

$$\begin{aligned}\mu_{B'}(v) = sup_{u\in U} max(&(1+\lambda)(1 - \mu_A(u) + I_Y(\mu_A(u), \mu_B(v)) - 1)\\ &-\lambda(1 - \mu_A(u))I_Y(\mu_A(u), \mu_B(v))).\end{aligned}$$

For $\mu_A(u) = 0$ it results

$$\mu_{B'}(v) = 1$$

and for $\mu_A(u) > 0$ we obtain

$$\mu_{B'}(v) = sup_{u\in U} max(\mu_B(v) - \mu_A(u)(1 + \lambda - \lambda\mu_B(v)), 0) \in [0, \mu_B(v)).$$ □

4. Comments about the results

If the observation is more precise than the premise of the rule then it gives more information than the premise. However, it does not seem reasonable to think that the Generalized Modus Ponens allows to obtain a conclusion more precise than that of the rule. The result of the inference is valid if $\mu_{B'}(v) = \mu_B(v)$, $\forall v \in V$. The result from the Theorem 3.1 satisfies this request that represents the *subset property* of GMP reasoning.

When the observation and the premise of the rule coincide the convenient behavior of the fuzzy deduction is to obtain an identical conclusion. Our result satisfies this request that is the *basic property* of GMP inference.

The other results from Theorem 3.1 show that the GMP inference with Yager implication and t-norm $t(x, y) = \max((1 + \lambda)(x + y - 1) - \lambda xy, 0)$, $\lambda \geq -1$ satisfies the *superset property.*

The result obtained in the case when the observation contains the premise is very general and it does not offer enough information about the conclusion inferred. The result of inference depends on compatibility between the observation and the premise.

To express this compatibility, the following quantities [18, 4] are frequently used:

(a) $D.I = sup_{\{u \in U/\mu_A(u)=0\}} \mu_{A'}(u)$, named uniform degree of non-determination; it appears when the support of the premise does not contain the support of the observation;

(b) $I = sup_{\{u \in U/\mu_{A'}(u) \geq \mu_A(u)\}} (\mu_{A'}(u) - \mu_A(u))$.

The uncertainty propagated is expressed with the help of $D.I$ and I and it corresponds to value $\mu_{B'}$ on the set $\{v \in V/\mu_B(v) = 0\}$.

Theorem 4.1. *If $\mu_{A'}(u) \geq \mu_A(u)$ $\forall u \in U$ then the uncertainty propagated during the inference with Yager implication is*

$$\mu_{B'}(v) = D.I.$$

Proof. The result is obtained from the expression of $\mu_{B'}(v)$ for $\mu_B(v) = 0$; we have

$$\mu_{B'}(v) = sup_{\{u \in U, \mu_A(u)>0\}} max((1 + \lambda)(\mu_{A'}(u) - 1), 0) = 0$$

and

$$\mu_{B'}(v) = sup_{\{u \in U, \mu_A(u)=0\}} max(\mu_{A'}(u), 0) = sup_{\{u \in U, \mu_A(u)=0\}} \mu_{A'}(u) = D.I. \qquad \square$$

5. Conclusion

The results explained in the previous sections show that the Generalized Modus Ponens rule works with the parametric t-norm $t(x, y) = max((1+\lambda)(x+y-1)-\lambda xy, 0)$, $\lambda \geq -1$ and the Yager implication. The results given by theorems 3.1 and 3.2 prove that this type of reasoning satisfies some rational properties (see [8], pp. 54-55) of GMP inference.

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