

Hölder's and Hardy's Two Dimensional Diamond-alpha Inequalities on Time Scales

MOULAY RCHID SIDI AMMI AND DELFIM F. M. TORRES

ABSTRACT. We prove a two dimensional Hölder and reverse-Hölder inequality on time scales via the diamond-alpha integral. Other integral inequalities are established as well, which have as corollaries some recent proved Hardy-type inequalities on time scales.

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1. Introduction

The theory and applications of dynamic derivatives on time scales is receiving an increase of interest and attention. This relative new area was created in order to unify and generalize discrete and continuous analysis. It was introduced by Stefan Hilger [5, 6], then used as a tool in several computational and numerical applications [1, 3, 4]. One important and very active subject being developed within the theory of time scales is the study of inequalities [2, 8, 9, 13, 14, 16, 17]. The primary purpose of this paper is to prove more general two dimensional reverse-Hölder's and Hölder's inequalities on time scales, using the recent theory of combined dynamic derivatives and the more general notion of diamond- α integral [10, 11, 12]. As particular cases, we get Hardy's inequalities [7, 16].

Hölder's inequalities and their extensions have received considerable attention in the theory of differential and difference equations, as well as other areas of mathematics [8, 13, 16, 17]. Recently, authors in [16] proved a time scale version of Hölder's inequality in the two dimensional case, by using the Δ -integral. Here we extend this result to more general diamond- α integral inequalities. The results in [16] are obtained choosing $\alpha = 1$; different inequalities on time scales follow by choosing $0 \leq \alpha < 1$ (e.g., for $\alpha = 0$ one gets new ∇ -integral inequalities).

2. Preliminaries

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers. Let \mathbb{T} be a time scale with the topology that it inherits from the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$.

If $\sigma(t) > t$ we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are simultaneously right-scattered and left-scattered are said to be isolated. If $\sigma(t) = t$, then t is called right-dense; if $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

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The mappings $\mu, \nu : \mathbb{T} \rightarrow [0, +\infty)$ defined by $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$ are called, respectively, the forward and backward graininess function.

Given a time scale \mathbb{T} , we introduce the sets \mathbb{T}^κ , \mathbb{T}_κ , and \mathbb{T}_κ^κ as follows. If \mathbb{T} has a left-scattered maximum t_1 , then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum t_2 , then $\mathbb{T}_\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, $\mathbb{T}_\kappa^\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a real valued function on a time scale \mathbb{T} . Then, for $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number, if one exists, such that for all $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|.$$

We say that f is delta differentiable on \mathbb{T}^κ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Similarly, for $t \in \mathbb{T}_\kappa$ we define $f^\nabla(t)$ to be the number, if one exists, such that for all $\epsilon > 0$, there is a neighborhood V of t such that for all $s \in V$

$$|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|.$$

We say that f is nabla differentiable on \mathbb{T}_κ , provided that $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

For $f : \mathbb{T} \rightarrow \mathbb{R}$ we define the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$. Similarly, we define the function $f^\rho : \mathbb{T} \rightarrow \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, that is, $f^\rho = f \circ \rho$.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous, provided it is continuous at all right-dense points in \mathbb{T} and its left-sided limits finite at all left-dense points in \mathbb{T} . A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called ld-continuous, provided it is continuous at all left-dense points in \mathbb{T} and its right-sided limits finite at all right-dense points in \mathbb{T} .

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$, provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the delta integral of f is defined by

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$

A function $G : \mathbb{T} \rightarrow \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \rightarrow \mathbb{R}$, provided $G^\nabla(t) = g(t)$ holds for all $t \in \mathbb{T}_\kappa$. Then the nabla integral of g is defined by $\int_a^b g(t) \nabla t = G(b) - G(a)$. For more on the delta and nabla calculus on time scales, we refer the reader to [1, 3, 4]. We review now the recent diamond- α derivative and integral [10, 11, 12].

Let \mathbb{T} be a time scale and f differentiable on \mathbb{T} in the Δ and ∇ senses. For $t \in \mathbb{T}$, we define the diamond- α dynamic derivative $f^{\diamond\alpha}(t)$ by

$$f^{\diamond\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$

Thus, f is diamond- α differentiable if and only if f is Δ and ∇ differentiable. The diamond- α derivative reduces to the standard Δ derivative for $\alpha = 1$, or the standard ∇ derivative for $\alpha = 0$. On the other hand, it represents a ‘‘weighted derivative’’ for $\alpha \in (0, 1)$. Diamond- α derivatives have shown in computational experiments to provide efficient and balanced approximation formulas, leading to the design of more reliable numerical methods [11, 12].

Let $f, g : \mathbb{T} \rightarrow \mathbb{R}$ be diamond- α differentiable at $t \in \mathbb{T}$. Then,

- (i) $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$ with

$$(f + g)^{\diamond\alpha}(t) = (f)^{\diamond\alpha}(t) + (g)^{\diamond\alpha}(t).$$

- (ii) For any constant c , $cf : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$ with

$$(cf)^{\diamond\alpha}(t) = c(f)^{\diamond\alpha}(t).$$

(ii) $fg : \mathbb{T} \rightarrow \mathbb{R}$ is diamond- α differentiable at $t \in \mathbb{T}$ with

$$(fg)^{\diamond_\alpha}(t) = (f)^{\diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)(g)^\Delta(t) + (1 - \alpha)f^\rho(t)(g)^\nabla(t).$$

Let $a, t \in \mathbb{T}$, and $h : \mathbb{T} \rightarrow \mathbb{R}$. Then, the diamond- α integral from a to t of h is defined by

$$\int_a^t h(\tau) \diamond_\alpha \tau = \alpha \int_a^t h(\tau) \Delta \tau + (1 - \alpha) \int_a^t h(\tau) \nabla \tau, \quad 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on \mathbb{T} . It is clear that the diamond- α integral of h exists when h is a continuous function. Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$, and f and g be continuous functions on $[a, b] \cap \mathbb{T}$. Then (cf. [12, Theorem 3.7] and [13, Lemma 2.2]), the following properties hold:

- (a) $\int_a^t (f(\tau) + g(\tau)) \diamond_\alpha \tau = \int_a^t f(\tau) \diamond_\alpha \tau + \int_a^t g(\tau) \diamond_\alpha \tau$;
- (b) $\int_a^t cf(\tau) \diamond_\alpha \tau = c \int_a^t f(\tau) \diamond_\alpha \tau$;
- (c) $\int_a^t f(\tau) \diamond_\alpha \tau = \int_a^b f(\tau) \diamond_\alpha \tau + \int_b^t f(\tau) \diamond_\alpha \tau$.
- (d) If $f(t) \geq 0$ for all $t \in [a, b]_{\mathbb{T}}$, then $\int_a^b f(t) \diamond_\alpha t \geq 0$.
- (e) If $f(t) \leq g(t)$ for all $t \in [a, b]_{\mathbb{T}}$, then $\int_a^b f(t) \diamond_\alpha t \leq \int_a^b g(t) \diamond_\alpha t$.
- (f) If $f(t) \geq 0$ for all $t \in [a, b]_{\mathbb{T}}$, then $\int_a^b f(t) \diamond_\alpha t = 0$ if and only if $\int_a^b f(t) \diamond_\alpha t = 0$.

3. Main Results

We prove new diamond- α inequalities. As particular cases we get Δ -inequalities on time scales for $\alpha = 1$, and ∇ -inequalities on time scales when $\alpha = 0$. In the sequel we use $[a, b]$ to denote $[a, b] \cap \mathbb{T}$. We also suppose that all integrals converge.

Theorem 3.1 (reverse diamond- α Hölder's inequality). *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and f and g be two positive functions satisfying $0 < m \leq \frac{f^p}{g^q} \leq M < +\infty$ on the set $[a, b]$. If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\left(\int_a^b f^p(t) \diamond_\alpha t \right)^{\frac{1}{p}} \left(\int_a^b g^q(t) \diamond_\alpha t \right)^{\frac{1}{q}} \leq \left(\frac{M}{m} \right)^{\frac{1}{pq}} \int_a^b f(t)g(t) \diamond_\alpha t. \quad (1)$$

Proof. We have $\frac{f^p}{g^q} \leq M$. Then, $f^{\frac{p}{q}} \leq M^{\frac{1}{q}}g$. Multiplying by $f > 0$, it follows that

$$f^p = f^{1+\frac{p}{q}} \leq M^{\frac{1}{q}}fg.$$

Using properties (e) and (b), we can write that

$$\left(\int_a^b f^p(t) \diamond_\alpha t \right)^{\frac{1}{p}} \leq M^{\frac{1}{pq}} \left(\int_a^b f(t)g(t) \diamond_\alpha t \right)^{\frac{1}{p}}. \quad (2)$$

In the same manner, we have $m^{\frac{1}{p}}g^{\frac{q}{p}} \leq f$. Then,

$$\int_a^b m^{\frac{1}{p}}g^q(t) \diamond_\alpha t = m^{\frac{1}{p}} \int_a^b g^{1+\frac{q}{p}}(t) \diamond_\alpha t \leq \int_a^b f(t)g(t) \diamond_\alpha t.$$

We obtain that

$$m^{\frac{1}{pq}} \left(\int_a^b g^q(t) \diamond_\alpha t \right)^{\frac{1}{q}} \leq \left(\int_a^b f(t)g(t) \diamond_\alpha t \right)^{\frac{1}{q}}. \quad (3)$$

Gathering (2) and (3), the intended inequality (1) is proved. \square

Remark 3.1. For the particular case $\mathbb{T} = \mathbb{R}$, Theorem 3.1 gives [7, Theorem 2.1]. For $\alpha = 1$, Theorem 3.1 coincides with [16, Lemma 1].

We now define the diamond- α integral for a function of two variables. The double integral is defined as an iterated integral. Let \mathbb{T} be a time scale with $a, b \in \mathbb{T}$, $a < b$, and f be a real-valued function on $\mathbb{T} \times \mathbb{T}$. Because we need notation for partial derivatives with respect to time scale variables x and y we denote the time scale partial derivative of $f(x, y)$ with respect to x by $f^{\diamond_\alpha^1}(x, y)$ and let $f^{\diamond_\alpha^2}(x, y)$ denote the time scale partial derivative with respect to y . Definition of these partial derivatives are now given. Fix an arbitrary $y \in \mathbb{T}$. Then the diamond- α derivative of function

$$\begin{aligned} \mathbb{T} &\rightarrow \mathbb{R} \\ x &\mapsto f(x, y) \end{aligned}$$

is denoted by $f^{\diamond_\alpha^1}$. Let now $x \in \mathbb{T}$. The diamond- α derivative of function

$$\begin{aligned} \mathbb{T} &\rightarrow \mathbb{R} \\ y &\mapsto f(x, y) \end{aligned}$$

is denoted by $f^{\diamond_\alpha^2}$. If function f has a \diamond_α^1 antiderivative A , i.e., $A^{\diamond_\alpha^1} = f$, and A has a \diamond_α^2 antiderivative B , i.e., $B^{\diamond_\alpha^2} = A$, then

$$\begin{aligned} \int_a^b \int_a^b f(x, y) \diamond_\alpha x \diamond_\alpha y &:= \int_a^b (A(b, y) - A(a, y)) \diamond_\alpha y \\ &= B(b, b) - B(b, a) - B(a, b) + B(a, a). \end{aligned}$$

Note that $(B^{\diamond_\alpha^2})^{\diamond_\alpha^1} = f$.

Theorem 3.2 (two dimensional diamond- α Hölder's inequality). *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $f, g, h : [a, b] \times [a, b] \rightarrow \mathbb{R}$ be \diamond_α -integrable functions, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then,*

$$\begin{aligned} &\int_a^b \int_a^b |h(x, y) f(x, y) g(x, y)| \diamond_\alpha x \diamond_\alpha y \\ &\leq \left(\int_a^b \int_a^b |h(x, y) f(x, y)|^p \diamond_\alpha x \diamond_\alpha y \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |h(x, y) g(x, y)|^q \diamond_\alpha x \diamond_\alpha y \right)^{\frac{1}{q}}. \quad (4) \end{aligned}$$

Proof. Inequality (4) is trivially true in the case when f or g or h is identically zero. Suppose that

$$\left(\int_a^b \int_a^b |h(x, y)| |f(x, y)|^{\frac{1}{p}} \diamond_\alpha x \diamond_\alpha y \right) \left(\int_a^b \int_a^b |h(x, y)| |g(x, y)|^{\frac{1}{q}} \diamond_\alpha x \diamond_\alpha y \right) \neq 0,$$

and let

$$A(x, y) = \frac{|h^{\frac{1}{p}}(x, y)| |f(x, y)|}{\int_a^b \int_a^b |h(x, y)| |f(x, y)|^{\frac{1}{p}} \diamond_\alpha x \diamond_\alpha y},$$

and

$$B(x, y) = \frac{|h^{\frac{1}{q}}(x, y)| |g(x, y)|}{\int_a^b \int_a^b |h(x, y)| |g(x, y)|^{\frac{1}{q}} \diamond_\alpha x \diamond_\alpha y}.$$

From the well-known Young's inequality $\xi\lambda \leq \frac{\xi^p}{p} + \frac{\lambda^q}{q}$, valid for nonnegative real numbers ξ and λ , we have that

$$\begin{aligned} & \int_a^b \int_a^b A(x, y)B(x, y) \diamond_{\alpha} x \diamond_{\alpha} y \\ & \leq \int_a^b \int_a^b \left[\frac{A^p(x, y)}{p} + \frac{B^q(x, y)}{q} \right] \diamond_{\alpha} x \diamond_{\alpha} y \\ & \leq \frac{1}{p} \int_a^b \int_a^b \frac{|h||f|^p \diamond_{\alpha} x \diamond_{\alpha} y}{\int_a^b \int_a^b |h||f|^p} + \frac{1}{q} \int_a^b \int_a^b \frac{|h||g|^q \diamond_{\alpha} x \diamond_{\alpha} y}{\int_a^b \int_a^b |h||g|^q} \\ & \leq \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

and the desired result follows. \square

Remark 3.2. For the particular case $\alpha = 1$, Theorem 3.2 coincides with [16, Theorem 4].

Theorem 3.3 (two dimensional diamond- α Cauchy-Schwartz's inequality). *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$. For \diamond_{α} -integrable functions $f, g, h : [a, b] \times [a, b] \rightarrow \mathbb{R}$, we have:*

$$\begin{aligned} & \int_a^b \int_a^b |h(x, y)f(x, y)g(x, y)| \diamond_{\alpha} x \diamond_{\alpha} y \\ & \leq \sqrt{\left(\int_a^b \int_a^b |h(x, y)||f(x, y)|^2 \diamond_{\alpha} x \diamond_{\alpha} y \right) \left(\int_a^b \int_a^b |h(x, y)||g(x, y)|^2 \diamond_{\alpha} x \diamond_{\alpha} y \right)}. \quad (5) \end{aligned}$$

Proof. The Cauchy-Schwartz inequality (5) is the particular case $p = q = 2$ of (4). \square

We now obtain some general results for estimating the diamond-alpha double integral $\int_a^b \int_a^b K(x, y)f(x)g(y) \diamond_{\alpha} x \diamond_{\alpha} y$.

Theorem 3.4 (diamond- α Hardy-type inequalities). *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $K(x, y), f(x), g(y), \varphi(x)$, and $\psi(y)$ be nonnegative functions. Let*

$$F(x) = \int_a^b K(x, y)\psi^{-p}(y) \diamond_{\alpha} y$$

and

$$G(y) = \int_a^b K(x, y)\varphi^{-q}(x) \diamond_{\alpha} x,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Then, the two inequalities

$$\begin{aligned} & \int_a^b \int_a^b K(x, y)f(x)g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ & \leq \left(\int_a^b \varphi^p(x)F(x)f^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b \psi^q(y)G(y)g^q(y) \diamond_{\alpha} y \right)^{\frac{1}{q}} \quad (6) \end{aligned}$$

and

$$\int_a^b G^{1-p}(y)\psi^{-p}(y) \left(\int_a^b K(x, y)f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \leq \int_a^b \varphi^p(x)F(x)f^p(x) \diamond_{\alpha} x \quad (7)$$

hold and are equivalent.

Equation (7) is the diamond- α Hardy's inequality.

Proof. First, we prove that (6) hold. Write

$$\int_a^b \int_a^b K(x, y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y = \int_a^b \int_a^b K(x, y) f(x) \frac{\varphi(x)}{\psi(y)} g(y) \frac{\psi(y)}{\varphi(x)} \diamond_{\alpha} x \diamond_{\alpha} y.$$

Applying Hölder's inequality on time scale, we have

$$\begin{aligned} \int_a^b \int_a^b K(x, y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ \leq \left(\int_a^b \varphi^p(x) F(x) f^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b \psi^q(y) G(y) g^q(y) \diamond_{\alpha} y \right)^{\frac{1}{q}}. \end{aligned}$$

Now we show that (6) is equivalent to (7). Suppose that inequality (6) is verified. Set

$$g(y) = G^{1-p}(y) \psi^{-p}(y) \left(\int_a^b K(x, y) f(x) \diamond_{\alpha} x \right)^{p-1}.$$

Using (6) and the fact that $\frac{1}{p} + \frac{1}{q} = 1$, we obtain:

$$\begin{aligned} & \int_a^b G^{1-p}(y) \psi^{-p}(y) \left(\int_a^b K(x, y) f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \\ &= \int_a^b \int_a^b K(x, y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ &\leq \left(\int_a^b \varphi^p(x) F(x) f^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b \psi^q(y) G(y) g^q(y) \diamond_{\alpha} y \right)^{\frac{1}{q}} \\ &= \left(\int_a^b \varphi^p(x) F(x) f^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_a^b G^{1-p}(y) \psi^{-p}(y) \left(\int_a^b K(x, y) f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \right)^{\frac{1}{q}}. \end{aligned}$$

Inequality (7) is obtained by dividing both sides of the previous inequality by

$$\left(\int_a^b G^{1-p}(y) \psi^{-p}(y) \left(\int_a^b K(x, y) f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \right)^{\frac{1}{q}}.$$

Reciprocally, suppose that (7) is valid. From Hölder's inequality we can write that

$$\begin{aligned} & \int_a^b \int_a^b K(x, y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ &= \int_a^b \left(\psi^{-1}(y) G^{-\frac{1}{q}}(y) \int_a^b K(x, y) f(x) \diamond_{\alpha} x \right) \psi(y) G^{\frac{1}{q}}(y) g(y) \diamond_{\alpha} y \\ &\leq \left(\int_a^b G^{1-p}(y) \psi^{-p}(y) \left(\int_a^b K(x, y) f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_a^b \psi^q(y) G(y) g^q(y) \diamond_{\alpha} y \right)^{\frac{1}{q}}. \end{aligned}$$

Using (7), we get that

$$\begin{aligned} & \int_a^b \int_a^b K(x, y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ &\leq \left(\int_a^b \varphi^p(x) F(x) f^p(x) \diamond_{\alpha} x \right)^{\frac{1}{q}} \left(\int_a^b \psi^q(y) G(y) g^q(y) \diamond_{\alpha} y \right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. \square

Remark 3.3. Choose $\mathbb{T} = \mathbb{R}$. In this particular case the inequalities (6) and (7) give the Hardy type inequalities proved in [7]. If

$$\left(f(x) \frac{\varphi(x)}{\psi(y)} \right)^p = K \left(g(y) \frac{\psi(y)}{\varphi(x)} \right)^q, \quad (8)$$

then (6) takes the form of equality. In this case there exist arbitrary constants A and B , not both zero, such that

$$f^p(x) = A \varphi^{-(p+q)}(x) \text{ and } g^q(y) = B \psi^{-(p+q)}(y).$$

This is possible only if

$$\int_a^b F(x) \varphi^{-q}(x) \diamond_{\alpha} x < \infty \text{ and } \int_a^b G(y) \psi^{-p}(y) \diamond_{\alpha} y < \infty.$$

If (8) does not hold, inequalities in Theorem 3.4 are strict.

As corollaries of Theorem 3.4 we have the following results.

Corollary 3.1. Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $h(y)$, $f(x)$, $g(y)$, $\varphi(x)$, and $\psi(y)$ be nonnegative functions, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Setting $H(y) = h(y) \psi^{-p}(y)$, then the two inequalities

$$\begin{aligned} & \int_a^b \int_a^y h(y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ &\leq \left(\int_a^b \varphi^p(x) f^p(x) \left(\int_x^b H(y) \diamond_{\alpha} y \right) \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ &\quad \left(\int_a^b \psi^q(y) g^q(y) h(y) \left(\int_a^y \varphi^{-q}(x) \diamond_{\alpha} x \right) \diamond_{\alpha} y \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} \int_a^b H(y) \left(\int_a^y \varphi^{-q} \diamond_{\alpha} x \right)^{1-p} \left(\int_a^y f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \\ \leq \left(\int_a^b \varphi^p(x) f^p(x) \left(\int_x^b H(y) \diamond_{\alpha} y \right) \diamond_{\alpha} x \right)^{\frac{1}{p}} \end{aligned}$$

hold and are equivalent.

Proof. Use Theorem 3.4 with $K(x, y) = \begin{cases} h(y), & \text{if } x \leq y \\ 0, & \text{if } x > y. \end{cases}$ \square

Corollary 3.2. *Let \mathbb{T} be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $h(y)$, $f(x)$, $g(y)$, $\varphi(x)$, and $\psi(y)$ be nonnegative, and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then, the two inequalities*

$$\begin{aligned} \int_a^b \int_y^b h(y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ \leq \left(\int_a^b \varphi^p(x) f^p(x) \left(\int_a^x H(y) \diamond_{\alpha} y \right) \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ \left(\int_a^b \psi^q(y) g^q(y) h(y) \left(\int_y^b \varphi^{-q}(x) \diamond_{\alpha} x \right) \diamond_{\alpha} y \right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned} \int_a^b H(y) \left(\int_y^b \varphi^{-q} \diamond_{\alpha} x \right)^{1-p} \left(\int_y^b f(x) \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \\ \leq \left(\int_a^b \varphi^p(x) f^p(x) \left(\int_a^x H(y) \diamond_{\alpha} y \right) \diamond_{\alpha} x \right)^{\frac{1}{p}} \end{aligned}$$

hold and are equivalent.

Proof. Use Theorem 3.4 with $K(x, y) = \begin{cases} 0, & \text{if } x \leq y \\ h(y), & \text{if } x > y. \end{cases}$ \square

Remark 3.4. *When $\alpha = 1$, Corollaries 3.1 and 3.2 coincide, respectively, with Theorems 7 and 8 in [16]. In the particular case $\mathbb{T} = \mathbb{R}$, they give Theorem 3 and Theorem 4 of [7].*

It is interesting to consider the case when functions $F(x)$ and $G(y)$ of Theorem 3.4 are bounded. We then obtain the following:

Theorem 3.5. *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, $K(x, y)$, $f(x)$, $g(y)$, $\varphi(x)$, $\psi(x)$ be nonnegative functions and $F(x) = \int_a^b \frac{K(x, y)}{\psi^p(y)} \diamond_{\alpha} y \leq F_1(x)$, $G(y) = \int_a^b \frac{K(x, y)}{\varphi^q(x)} \diamond_{\alpha} x \leq G_1(y)$. Then, the inequalities*

$$\begin{aligned} \int_a^b \int_a^b K(x, y) f(x) g(y) \diamond_{\alpha} x \diamond_{\alpha} y \\ \leq \left(\int_a^b \varphi^p(x) F_1(x) f^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \left(\int_a^b \psi^q(y) G_1(y) g^q(y) \diamond_{\alpha} y \right)^{\frac{1}{q}} \quad (9) \end{aligned}$$

and

$$\int_a^b G_1^{1-p}(y)\psi^{-p}(y) \left(\int_a^b K(x,y)f(x)\diamond_{\alpha}x \right)^p \diamond_{\alpha}y \leq \int_a^b \varphi^p(x)F_1(x)f^p(x)\diamond_{\alpha}x \quad (10)$$

hold and are equivalent.

The following result extends the one found in [15].

Theorem 3.6. *Let $F, G, L(f, g), M(f)$, and $N(g)$ be positive functions, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, such that*

$$0 < \int_a^b M^p(f(t))F^p(t)\diamond_{\alpha}t < \infty, \quad 0 < \int_c^d N^q(g(t))G^q(t)\diamond_{\alpha}t < \infty.$$

Then, the inequalities

$$\begin{aligned} \int_a^b \int_c^d \frac{F(x)G(y)}{L(f(x), g(y))} \diamond_{\alpha}x \diamond_{\alpha}y \\ \leq C \left(\int_a^b M^p(f(t))F^p(t)\diamond_{\alpha}t \right)^{\frac{1}{p}} \left(\int_c^d N^q(g(t))G^q(t)\diamond_{\alpha}t \right)^{\frac{1}{q}} \end{aligned} \quad (11)$$

and

$$\int_c^d N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha}x \right)^p \diamond_{\alpha}y \leq C^p \int_a^b M^p(f(t))F^p(t)\diamond_{\alpha}t, \quad (12)$$

where C is a constant, are equivalent.

Proof. Suppose that the inequality (12) is valid. Then,

$$\begin{aligned} & \int_a^b \int_c^d \frac{F(x)G(y)}{L(f(x), g(y))} \diamond_{\alpha}x \diamond_{\alpha}y \\ &= \int_c^d N(g(y))G(y) \left(N^{-1}(g(y)) \int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha}x \right) \diamond_{\alpha}y \\ &\leq \left(\int_c^d N^q(g(y))G^q(y)\diamond_{\alpha}y \right)^{\frac{1}{q}} \left(\int_c^d N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha}x \right)^p \diamond_{\alpha}y \right)^{\frac{1}{p}} \\ &\leq C^p \left(\int_a^b M^p(f(t))F^p(t)\diamond_{\alpha}t \right)^{\frac{1}{p}} \left(\int_c^d N^q(g(t))G^q(t)\diamond_{\alpha}t \right)^{\frac{1}{q}}. \end{aligned}$$

We just proved inequality (11). Let us now suppose that the inequality (11) is valid. By setting $G(y) = N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right)^{\frac{p}{q}} \diamond_{\alpha} y$ and applying (11), we obtain that

$$\begin{aligned} & \int_c^d N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \\ &= \int_c^d \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right) N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right)^{\frac{p}{q}} \diamond_{\alpha} y \\ &\leq C \left(\int_a^b M^p(f(x)) F^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_c^d N^q(g(y)) N^{-pq}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \right)^{\frac{1}{q}} \\ &= C \left(\int_a^b M^p(f(x)) F^p(x) \diamond_{\alpha} x \right)^{\frac{1}{p}} \\ &\quad \left(\int_c^d N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \right)^{\frac{1}{q}}. \end{aligned}$$

It follows (12):

$$\int_c^d N^{-p}(g(y)) \left(\int_a^b \frac{F(x)}{L(f(x), g(y))} \diamond_{\alpha} x \right)^p \diamond_{\alpha} y \leq C^p \int_a^b M^p(f(t)) F^p(t) \diamond_{\alpha} t.$$

□

4. Conclusion

The study of integral inequalities on time scales via the diamond- α integral, which is defined as a linear combination of the delta and nabla integrals, plays an important role in the development of the theory of time scales [8, 9, 13, 14]. In this paper we generalize some delta-integral inequalities on time scales to diamond- α integrals. As special cases, one obtains previous Hölder's and Hardy's inequalities.

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(Moulay Rchid Sidi Ammi) DEPARTMENT OF MATHEMATICS, FAC. DES SCI. ET TECH.
(FST-ERRACHIDIA)
UNIVERSITY MY ISMAIL, BP: 509 BOUTALAMINE, ERRACHIDIA, 52000, MOROCCO
E-mail address: sidiammi@ua.pt

(Delfim F. M. Torres) AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES, 6-8 MELROSE ROAD
MUIZENBERG 7945, CAPE TOWN, SOUTH AFRICA
ON LEAVE OF ABSENCE FROM:
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AVEIRO, 3810-193 AVEIRO, PORTUGAL
E-mail address: delfim@aims.ac.za, delfim@ua.pt