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On pseudo-BCK algebras with pseudo-double negation

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Abstract. A special class of pseudo-BCK algebras is that of the pseudo-BCK algebras with pseudo-double negation which generalize some particular structures, such as pseudo-MV algebras. The aim of this paper is to present some new properties of the pseudo-BCK algebras with pseudo-double negation. As main results, we prove some conditions for a pseudo-BCK lattice with pseudo-double negation to be distributive and we present equivalent definitions for pseudo-BCK algebras with pseudo-double negation.

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1. Introduction

Pseudo-BCK algebras were introduced in [4] by G. Georgescu and A. Iorgulescu as a generalization of BCK algebras in order to give a corresponding structure to pseudo-MV algebras, since the bounded commutative BCK algebras correspond to MV algebras. More properties of pseudo-BCK algebras and their connection with other fuzzy structures were established by A. Iorgulescu in [7], [8], [9], [10]. A special class of pseudo-BCK algebras is that of pseudo-BCK algebras with pseudo-double negation which generalize some particular structures, such as pseudo-MV algebras. For this reason, the investigation of properties of this class of pseudo-BCK algebras seems to be interesting and useful as well. In this paper we present some new properties of the pseudo-BCK algebras with pseudo-double negation, we prove equivalent definitions for these structures and we present some conditions for a pseudo-BCK lattice with pseudo-double negation to be distributive. We also prove that every bounded locally finite pseudo-hoop is a pseudo-BCK algebra with double-negation.

2. Preliminaries

Definition 2.1. ([4]) *A pseudo-BCK algebra (more precisely, reversed left-pseudo-BCK algebra) is a structure* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ *where* $\leq$ *is a binary relation on* A, $\rightarrow$ *and* $\rightsquigarrow$ *are binary operations on* A *and* 1 *is an element of* A *satisfying, for all* $x, y, z \in A$, *the axioms:*
(A_1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;
(A_2) $x \leq (x \rightarrow y) \rightsquigarrow y, \quad x \leq (x \rightsquigarrow y) \rightarrow y$;
(A_3) $x \leq x$;
(A_4) $x \leq 1$;
(A_5) *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$;
(A_6) $x \leq y$ *iff* $x \rightarrow y = 1$ *iff* $x \rightsquigarrow y = 1$.

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A pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ is *commutative* if $\rightarrow = \rightsquigarrow$. Every commutative pseudo-BCK algebra is a BCK algebra.

Example 2.1. *([3]) Let's consider $A = \{o_1, a_1, b_1, c_1, o_2, a_2, b_2, c_2, 1\}$ with $o_1 < a_1, b_1 < c_1 < 1$ and a_1, b_1 incomparable, $o_2 < a_2, b_2 < c_2 < 1$ and a_2, b_2 incomparable. Let's also assume that any element of the set $\{o_1, a_1, b_1, c_1\}$ is incomparable with any element of the set $\{o_2, a_2, b_2, c_2\}$. Consider the operations $\rightarrow, \rightsquigarrow$ given by the following tables:*

$\rightarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	o_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	a_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	o_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	c_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	o_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

$\rightsquigarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	b_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	o_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	b_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	b_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	b_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

Then, $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK algebra.

Proposition 2.1. *([9], [10]) In every pseudo-BCK algebra the following properties hold:*
(c_1) *$x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$;*
(c_2) *$x \leq y, y \leq z$ implies $x \leq z$;*
(c_3) *$x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$ and $x \rightsquigarrow (y \rightarrow z) = y \rightarrow (x \rightsquigarrow z)$;*
(c_4) *$z \leq y \rightarrow x$ iff $y \leq z \rightsquigarrow x$;*
(c_5) *$z \rightarrow x \leq (y \rightarrow z) \rightarrow (y \rightarrow x)$ $\quad z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$;*
(c_6) *$x \leq y \rightarrow x$, $\quad x \leq y \rightsquigarrow x$;*
(c_7) *$1 \rightarrow x = x = 1 \rightsquigarrow x$;*
(c_8) *$x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$;*
(c_9) *$[(y \rightarrow x) \rightsquigarrow x] \rightarrow x = y \rightarrow x$, $\quad [(y \rightsquigarrow x) \rightarrow x] \rightsquigarrow x = y \rightsquigarrow x$.*

Proposition 2.2. *([11]) Let $(A, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. If $\bigvee_{i \in I} x_i$ exists, then so does $\bigwedge_{i \in I}(x_i \rightarrow y)$ and $\bigwedge_{i \in I}(x_i \rightsquigarrow y)$ and we have:*
(c_{10}) *$(\bigvee_{i \in I} x_i) \rightarrow y = \bigwedge_{i \in I}(x_i \rightarrow y)$, $\quad (\bigvee_{i \in I} x_i) \rightsquigarrow y = \bigwedge_{i \in I}(x_i \rightsquigarrow y)$.*

Definition 2.2. *([7]) If there is an element 0 of a pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$, such that $0 \leq x$ (i.e. $0 \rightarrow x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then 0 is called the* zero *of $\mathcal{A}$. A pseudo-BCK algebra with zero is called* bounded pseudo-BCK algebra *and it is denoted by $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$.*

Example 2.2. *([3]) Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$ and a, b incomparable. Consider the operations $\rightarrow, \rightsquigarrow$ given by the following tables:*

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then, $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra.

Definition 2.3. *([7]) A pseudo-BCK algebra with* (pP) condition *(i.e. with* pseudo-product *condition) or a* pseudo-BCK(pP) algebra *for short, is a pseudo-BCK algebra* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ *satisfying (pP) condition:*
(pP) there exists, for all $x, y \in A$, $x \odot y = min\{z \mid x \leq y \rightarrow z\} = min\{z \mid y \leq x \rightsquigarrow z\}$.

Remark 2.1. *Any bounded linearly ordered pseudo-BCK algebra is with (pP) condition (see [7]). If the pseudo-BCK algebra is not bounded this result is not always valid, as we can see in the following example communicated by J.* Kühr.
Let $(Q, +, 0, \leq)$ *be the additive group of rationals with the usual linear order and take* $A = \{x \in Q : -\sqrt{2} < x \leq 0\}$. *Then* $(A, \rightarrow, 0)$ *is a linear BCK algebra with* $x \rightarrow y = min\{0, y - x\}$. *We have* $\{z \in A : (-1) \leq (-1) \rightarrow z = min\{0, z + 1\}\} = A$, *thus* $(-1) \odot (-1) = minA$ *doesn't exist in* $(A, \rightarrow, 0)$.

Example 2.3. (1) *If* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ *is the bounded pseudo-BCK lattice from Example 2.2, then* $min\{z \mid b \leq a \rightarrow z\} = min\{a, b, c, 1\}$ *and* $min\{z \mid a \leq b \rightsquigarrow z\} = min\{a, b, c, 1\}$ *do not exist. Thus,* $b \odot a$ *does not exist, so* $\mathcal{A}$ *is not a pseudo-BCK(pP) algebra. Moreover, since* $(A, \leq)$ *is a lattice, it follows that* $\mathcal{A}$ *is a pseudo-BCK lattice.*
(2) *If* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ *is a reduct of a residuated lattice, then it is obvious that* $\mathcal{A}$ *is a bounded pseudo-BCK(pP) algebra.*

Let $(A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ is the bounded pseudo-BCK(pP) algebra. For any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. The *order* of $x \in A$, denoted $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$. A pseudo-BCK(pP) algebra A is *locally finite* if for any $x \in A$, $x \neq 1$ implies $ord(x) < \infty$.

We recall the definition and some properties of pseudo-hoops which supply some examples of structures studied in this paper. Pseudo-hoops were originally introduced by Bosbach in [1] and [2] under the name of *complementary semigroups* and their properties were recently studied in [5].

Definition 2.4. *([5]) A* pseudo-hoop *is an algebra* $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ *of the type* $(2, 2, 2, 0)$ *such that, for all* $x, y, z \in A$:
(H_1) $x \odot 1 = 1 \odot x = x$;
(H_2) $x \rightarrow x = x \rightsquigarrow x = 1$;
(H_3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
(H_4) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
(H_5) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

If the operation $\odot$ is commutative, or equivalently $\rightarrow = \rightsquigarrow$, then the pseudo-hoop is said to be *hoop*. On the pseudo-hoop A we define $x \leq y$ iff $x \rightarrow y = 1$ (equivalent to $x \rightsquigarrow y = 1$) and $\leq$ is a partial order on A. A pseudo-hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$ for all $x \in A$.

Proposition 2.3. *([5]) In every pseudo-hoop* $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ *the following hold:*
(h_1) $(A, \leq)$ *is a meet-semillatice with* $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$;
(h_2) $x \odot y \leq z$ *iff* $x \leq y \rightarrow z$ *iff* $y \leq x \rightsquigarrow z$;
(h_3) $x \rightarrow x = x \rightsquigarrow x = 1$;
(h_4) $1 \rightarrow x = 1 \rightsquigarrow x = x$;
(h_5) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;
(h_6) $x \leq (x \rightarrow y) \rightsquigarrow y$;
(h_7) $x \leq (x \rightsquigarrow y) \rightarrow y$;
(h_8) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$;
(h_9) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$.

Proposition 2.4. *Every pseudo-hoop is a pseudo-BCK(pP) algebra.*

Proof. Suppose that $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop. We will prove that it is a pseudo-BCK(pP) algebra.
(A_1) follows from (h_8) and (h_9);
(A_2) follows from (h_6) and (h_7);
(A_3) follows from (h_3);
(A_4) follows from (h_5);
(A_5) and (A_6) follow by the definition of $\leq$ and from the fact that $\leq$ is a partial order on A.
The (pP) condition is a consequence of (h_2). Thus, $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK(pP) algebra. □

Definition 2.5. *(*[7], [12]*)* (1) *Let $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. If the poset $(A, \leq)$ is a lattice, then we say that $\mathcal{A}$ is a* pseudo-BCK lattice.
(2) *An algebra $(A, \vee, \rightarrow, \rightsquigarrow, 1)$ is called* pseudo-BCK join-semilattice *if $(A, \vee)$ is a join-semilattice, $(A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK algebra and $x \rightarrow y = 1$ iff $x \vee y = y$.*
(3) *An algebra $(A, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a* pseudo-BCK meet-semilattice *if$(A, \wedge)$ is a meet-semilattice, $(A, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK algebra and $x \rightarrow y = 1$ iff $x \wedge y = x$.*

Example 2.4. (1) *In the case of the pseudo-BCK algebra from Example 2.2, since $(A, \leq)$ is a lattice, it follows that $\mathcal{A}$ is a pseudo-BCK lattice;*
(2) *One can easily check that the pseudo-BCK algebra from Example 2.1 is a pseudo-BCK join-semilattice;*
(3) *Given a pseudo-hoop $(A, \odot, \rightarrow, \rightsquigarrow, 1)$, applying the property (h_1) it follows that $(A, \wedge, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK meet-semilattice, where $x \wedge y = x \odot (x \rightsquigarrow y) = (x \rightarrow y) \odot x$.*

Proposition 2.5. *(*[10]*) In a bounded pseudo-BCK algebra the following hold:*
(c_{11}) $1^{-} = 0 = 1^{\sim}, \quad 0^{-} = 1 = 0^{\sim}$;
(c_{12}) $x \leq (x^{-})^{\sim}, \quad x \leq (x^{\sim})^{-}$;
(c_{13}) $x \rightarrow y \leq y^{-} \rightsquigarrow x^{-}, \quad x \rightsquigarrow y \leq y^{\sim} \rightarrow x^{\sim}$;
(c_{14}) $x \leq y$ *implies* $y^{-} \leq x^{-}$ *and* $y^{\sim} \leq x^{\sim}$;
(c_{15}) $x \rightarrow y^{\sim} = y \rightsquigarrow x^{-}$ *and* $x \rightsquigarrow y^{-} = y \rightarrow x^{\sim}$;
(c_{16}) $((x^{-})^{\sim})^{-} = x^{-}, \quad ((x^{\sim})^{-})^{\sim} = x^{\sim}$.

Proposition 2.6. *In a bounded pseudo-BCK algebra the following hold:*
(c_{17}) $x \rightarrow y^{-\sim} = y^{-} \rightsquigarrow x^{-} = x^{-\sim} \rightarrow y^{-\sim}$ *and* $x \rightsquigarrow y^{\sim-} = y^{\sim} \rightarrow x^{\sim} = x^{\sim-} \rightsquigarrow y^{\sim-}$;
(c_{18}) $x \rightarrow y^{\sim} = y^{\sim-} \rightsquigarrow x^{-} = x^{-\sim} \rightarrow y^{\sim}$ *and* $x \rightsquigarrow y^{-} = y^{-\sim} \rightarrow x^{\sim} = x^{\sim-} \rightsquigarrow y^{-}$;
(c_{19}) $(x \rightarrow y^{\sim-})^{\sim-} = x \rightarrow y^{\sim-}$ *and* $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$.

Proof. (c_{17}): By (c_{15}) we have : $y \rightsquigarrow x^{-} = x \rightarrow y^{\sim}$. Replacing y with y^{-} we get : $y^{-} \rightsquigarrow x^{-} = x \rightarrow y^{-\sim}$. Replacing x with $x^{-\sim}$ in the last equality we get: $y^{-} \rightsquigarrow x^{-\sim-} = x^{-\sim} \rightarrow y^{-\sim}$. Hence, applying (c_{16}) it follows that: $y^{-} \rightsquigarrow x^{-} = x^{-\sim} \rightarrow y^{-\sim}$.
Thus, $x \rightarrow y^{-\sim} = y^{-} \rightsquigarrow x^{-} = x^{-\sim} \rightarrow y^{-\sim}$.
Similarly, $x \rightsquigarrow y^{\sim-} = y^{\sim} \rightarrow x^{\sim} = x^{\sim-} \rightsquigarrow y^{\sim-}$.
(c_{18}): The assertions follow replacing in (c_{17}) y with $y^{\sim}$ and respectively y with y^{-} and applying (c_{16}).
(c_{19}): Applying (c_3) and (c_{18}) we have:
$$1 = (x \rightarrow y^{\sim-}) \rightsquigarrow (x \rightarrow y^{\sim-}) = x \rightarrow ((x \rightarrow y^{\sim-}) \rightsquigarrow y^{\sim-}) =$$
$$x \rightarrow ((x \rightarrow y^{\sim-})^{\sim-} \rightsquigarrow y^{\sim-}) = (x \rightarrow y^{\sim-})^{\sim-} \rightsquigarrow (x \rightarrow y^{\sim-}).$$

Hence, $(x \to y^{\sim -})^{\sim -} \leq x \to y^{\sim -}$. On the other hand, by (c_{12}) we have $x \to y^{\sim -} \leq (x \to y^{\sim -})^{\sim -}$, so $(x \to y^{\sim -})^{\sim -} = x \to y^{\sim -}$.
Similarly, $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$. □

Proposition 2.7. *In every bounded pseudo-BCK lattice A we have:*
(c_{20}) $(x \vee y)^{-} = x^{-} \wedge y^{-}, \quad (x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}$.

Proof. According to (c_{10}), for all $x, y, z \in A$ we have:
$(x \vee y) \to z = (x \to z) \wedge (y \to z)$ and $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$.
Taking $z = 0$ we get $(x \vee y)^{-} = x^{-} \wedge y^{-}$ and $(x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}$. □

3. On pseudo-BCK algebras with pseudo-double negation

In this section we prove equivalent definitions for pseudo-BCK algebras with pseudo-double negation and we present some conditions for a pseudo-BCK lattice with pseudo-double negation to be distributive. For the case of a BCK algebra, some of these results were established in [6]. We also prove that every bounded locally finite pseudo-hoop satisfies the pseudo-double negation condition.

Definition 3.1. *([7]) A pseudo-BCK algebra with (pDN) condition (i.e. with* pseudo-double negation *condition) or a pseudo-BCK(pDN) algebra for short is a bounded pseudo-BCK algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ satisfying the condition:*
(pDN) $(x^{-})^{\sim} = (x^{\sim})^{-} = x$ *for all* $x \in A$.

Example 3.1. *([8]) Let $(G, \vee, \wedge, +, -, 0)$ be a linearly ordered ℓ-group and let $u \in G$, $u < 0$. Define*

$$x \to y = \begin{cases} 0, \text{ if } x \leq y \\ (u - x) \vee y, \text{ if } x > y \end{cases}$$

$$x \rightsquigarrow y = \begin{cases} 0, \text{ if } x \leq y \\ (-x + u) \vee y, \text{ if } x > y. \end{cases}$$

Then, $\mathcal{A} = ([u, 0], \to, \rightsquigarrow, 0 = u, 1 = 0)$ is a pseudo-BCK(pDN) algebra.

Proposition 3.1. *([7]) Let $\mathcal{A}$ be a pseudo-BCK algebra with (pDN) condition. Then, for all $x, y \in A$ the following hold:*
(c_{21}) $x \leq y$ *iff* $y^{-} \leq x^{-}$ *iff* $y^{\sim} \leq x^{\sim}$;
(c_{22}) $x \to y = y^{-} \rightsquigarrow x^{-}, \quad x \rightsquigarrow y = y^{\sim} \to x^{\sim}$;
(c_{23}) $x^{\sim} \to y = y^{-} \rightsquigarrow x$;
(c_{24}) $(x \to y^{-})^{\sim} = (y \rightsquigarrow x^{\sim})^{-}$.

Proposition 3.2. *In every bounded pseudo-BCK(pDN) lattice A we have:*
(c_{25}) $(x^{-} \vee y^{-})^{\sim} = (x^{\sim} \vee y^{\sim})^{-} = x \wedge y$.

Proof. By (c_{20}) we have $(x^{-} \vee y^{-})^{\sim} = x^{-\sim} \wedge y^{-\sim} = x \wedge y$.
Similarly, $(x^{\sim} \vee y^{\sim})^{-} = x \wedge y$. □

Let $\mathcal{A}$ be a pseudo-BCK algebra. For all $x, y \in A$, define (see [4], [10]):

$$x \vee y = (x \to y) \rightsquigarrow y, \quad x \cup y = (x \rightsquigarrow y) \to y.$$

As a consequence of the property (c_9), we can see that in every pseudo-BCK algebra the following hold:

$$x \vee y \to y = x \to y \text{ and } \mathrm{x} \cup \mathrm{y} \rightsquigarrow \mathrm{y} = \mathrm{x} \rightsquigarrow \mathrm{y}$$

for all $x, y \in A$.
According to [4], a pseudo-BCK algebra A is said to be *sup-commutative* if:

$$x \vee y = y \vee x \text{ and } x \cup y = y \cup x \text{ for all } x, y \in A.$$

It is easy to check that a sup-commutative pseudo-BCK algebra is a pseudo-BCK(pDN) algebra. It was proved in [7] that the bounded sup-commutative pseudo-BCK algebras are categorically isomorphic with pseudo-MV algebras. It also was proved in [7] that a bounded sup-commutative pseudo-BCK algebra is an equivalent definition of a pseudo-Wajsberg algebra. We also mention that the sup-commutative pseudo-BCK algebras are called in [11]) commutative pseudo-BCK algebras.

Proposition 3.3. *Let A be a bounded pseudo-BCK(pDN) algebra and $x, y \in A$. If $x \wedge y$ exists, then $x^- \vee y^-$, $x^\sim \vee y^\sim$ exist and:*
(c_{26}) $(x \wedge y)^- = x^- \vee y^-$, $\quad (x \wedge y)^\sim = x^\sim \vee y^\sim$.

Proof. Since $x \wedge y \leq x, y$, we get $x^-, y^- \leq (x \wedge y)^-$. It follows that $(x \wedge y)^-$ is an upper bound of x^- and y^-. Let u be an arbitrary upper bound of x^- and y^-, that is $x^-, y^- \leq u$. Since A is with (pDN), we get $u^\sim \leq x, y$, so $u^\sim \leq x \wedge y$. Finally we get $(x \wedge y)^- \leq u$, so $(x \wedge y)^-$ is the least upper bound of x^- and y^-. Thus, $x^- \vee y^-$ exists and $(x \wedge y)^- = x^- \vee y^-$.
Similarly, $x^\sim \vee y^\sim$ exists and $(x \wedge y)^\sim = x^\sim \vee y^\sim$. □

Corollary 3.1. *In every bounded pseudo-BCK(pDN) lattice A we have:*
(c_{27}) $(x^- \wedge y^-)^\sim = (x^\sim \wedge y^\sim)^- = x \vee y$.

Theorem 3.1. *Every bounded locally finite pseudo-hoop is with (pDN).*

Proof. Let A be a bounded locally finite pseudo-hoop and $x \in A$. If $x = 0$, then $0^{-\sim} = 0^{\sim -} = 0$. Suppose $x \neq 0$ and we prove that $x^{-\sim} = x$. By (c_{21}) we have $x \leq x^{-\sim}$. Suppose that $x^{-\sim} \not\leq x$, hence $x^{-\sim} \to x \neq 1$. Since A is locally finite, there is $n \in \mathbb{N}$, $n \geq 1$ such that $(x^{-\sim} \to x)^n = 0$. We have:
$(x^{-\sim} \to x) \to x^- = (x^{-\sim} \to x) \to x^{-\sim -} = (x^{-\sim} \to x) \to (x^{-\sim} \to 0) =$
$(x^{-\sim} \to x) \odot x^{-\sim} \to 0 = (x \wedge x^{-\sim}) \to 0 = x \to 0 = x^-$.
$(x^{-\sim} \to x)^2 \to x^- = (x^{-\sim} \to x) \to ((x^{-\sim} \to x) \to x^-) = (x^{-\sim} \to x) \to x^- = x^-$.
By induction we get $(x^{-\sim} \to x)^n \to x^- = x^-$. Thus, $0 \to x^- = x^-$, so $x^- = 1$. Hence, $x = 0$, a contradiction. Therefore, $x^{-\sim} \leq x$, so $x^{-\sim} = x$.
Similarly $x^{\sim -} = x$. □

Theorem 3.2. *Let $(A, \to, \rightsquigarrow, 0, 1)$ a bounded pseudo-BCK algebra. The following are equivalent:*
(a) A is with (pDN) condition;
(b) $x \to y = y^- \rightsquigarrow x^-$ and $x \rightsquigarrow y = y^\sim \to x^\sim$;
(c) $x^\sim \to y = y^- \rightsquigarrow x$ and $x^- \rightsquigarrow y = y^\sim \to x$;
(d) $x^- \leq y$ implies $y^\sim \leq x$ and $x^\sim \leq y$ implies $y^- \leq x$.

Proof. $(a) \Rightarrow (b)$: By (c_{15}) we have:
$$x \to y = x \to y^{-\sim} = y^- \rightsquigarrow x^- \text{ and } x \rightsquigarrow y = x \rightsquigarrow y^{\sim -} = y^\sim \to x^\sim.$$
$(b) \Rightarrow (c)$: By (c_{15}) we have: $x^\sim \to y^{-\sim} = y^- \rightsquigarrow x^{\sim -}$.
Applying (b) we get: $x^\sim \to y = y^- \rightsquigarrow x^{\sim -}$ and $y^- \rightsquigarrow x = x^\sim \to y^{-\sim}$.
Thus, $x^\sim \to y = y^- \rightsquigarrow x$. Similarly, $x^- \rightsquigarrow y = y^\sim \to x$.
$(c) \Rightarrow (d)$: If $x^- \leq y$, then $x^- \rightsquigarrow y = 1$. Applying (c) we get $y^\sim \to x = 1$, that is $y^\sim \leq x$.
Similarly, $x^\sim \leq y$ implies $y^- \leq x$.
$(d) \Rightarrow (a)$: From $x^- \leq x^-$ and (d) we have $x^{-\sim} \leq x$. Taking into consideration (c_{12}) we get $x^{-\sim} = x$.
Similarly, $x^{\sim -} = x$. Thus, A is with (pDN) condition. □

Theorem 3.3. *If* $(A, \to, \rightsquigarrow, 0, 1)$ *a bounded pseudo-BCK(pDN) algebra, then the following are equivalent:*
(a) $(A, \leq)$ *is a meet-semilattice;*
(b) $(A, \leq)$ *is a join-semilattice;*
(c) $(A, \leq)$ *is a lattice.*

Proof. $(a) \Rightarrow (b)$: Consider $x, y \in A$. Since A is a meet-semilattice, then $x^- \wedge y^-$ exists. Applying (c_{26}), it follows that $x^{-\sim} \vee y^{-\sim}$ exists, that is $x \vee y$ exists.
Thus, A is a join-semilattice.
$(b) \Rightarrow (c)$: Because A is a join-semilattice it follows that $x^- \vee y^-$ exists for all $x, y \in A$. Hence, by (c_{25}), $x \wedge y = (x^- \vee y^-)^\sim$. Thus, $x \wedge y$ exists, so A is a lattice.
$(c) \Rightarrow (a)$: It is obvious, since A is a lattice. □

Proposition 3.4. *In every bounded pseudo-BCK(pDN) lattice the following hold:*
(1) $y \to (\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I}(y \to x_i)$;
(2) $y \rightsquigarrow (\bigwedge_{i\in I} x_i) = \bigwedge_{i\in I}(y \rightsquigarrow x_i)$

Proof. By Proposition 2.2 we have: $(x^- \vee y^-) \rightsquigarrow z^- = (x^- \rightsquigarrow z^-) \wedge (y^- \rightsquigarrow z^-)$.
Applying (c_{15}) we get: $z \to (x^- \vee y^-)^\sim = (z \to x^{-\sim}) \wedge (z \to y^{-\sim})$.
By (c_{25}) we have: $(x^- \vee y^-)^\sim = x \wedge y$. Hence, $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$.
By induction we get assertion (1).
(2) Similarly as (1). □

Remark 3.1. *If the pseudo-BCK lattice A is without (pDN), then the results of Proposition 3.4 do not hold. Indeed, in the pseudo-BCK lattice A from Example 2.2 we have* $a \to (a \wedge b) = a \to 0 = 0$, *while* $(a \to a) \wedge (a \to b) = 1 \wedge b = b$.
Thus, $a \to (a \wedge b) \neq (a \to a) \wedge (a \to b)$.

Proposition 3.5. *In every pseudo-BCK(pDN) lattice the following conditions are equivalent:*
(C_1) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$ *and* $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$;
(C_2) $z \to (x \vee y) = (z \to x) \vee (z \to y)$ *and* $z \rightsquigarrow (x \vee y) = (z \rightsquigarrow x) \vee (z \rightsquigarrow y)$.

Proof. $(C_1) \Rightarrow (C_2)$: By the second identity from (C_1) we have:
$$(x^- \wedge y^-) \rightsquigarrow z^- = (x^- \rightsquigarrow z^-) \vee (y^- \rightsquigarrow z^-).$$
Applying (c_{15}) we get: $(x^- \wedge y^-) \rightsquigarrow z^- = z \to (x^- \wedge y^-)^\sim = z \to (x \vee y)$.
By (c_{22}) we have: $(x^- \rightsquigarrow z^-) \vee (y^- \rightsquigarrow z^-) = (z \to x) \vee (z \to y)$.
Thus, $z \to (x \vee y) = (z \to x) \vee (z \to y)$.
Similarly, from the first identity of (C_1) we get the second identity from (C_2).
$(C_2) \Rightarrow (C_1)$: By the second identity from (C_2) we get:
$$z^- \rightsquigarrow (x^- \vee y^-) = (z^- \rightsquigarrow x^-) \vee (z^- \rightsquigarrow y^-).$$
Applying (c_{23}) we have:
$$(x^- \vee y^-)^\sim \to z = (x \to z) \vee (y \to z).$$
Thus, $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.
Similarly, from the first identity of (C_2) we get the second identity from (C_1). □

Remark 3.2. *The class of pseudo-BCK(pDN) lattices satisfying the conditions* (C_1) *and* (C_2) *is not empty. Indeed, one can see that every pseudo-MV algebra satisfies these conditions.*

Theorem 3.4. *Let A be a pseudo-BCK lattice such that at least one of the following identities holds:*
(C_1^1) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

(C_1^2) $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$.
Then $(A, \leq)$ *is distributive.*

Proof. Let's denote $u = (x \vee y) \wedge (x \vee z)$. Obviously, $x \leq u$ and $y \wedge z \leq u$.
It follows that u is an upper bound of x and $y \wedge z$.
Let's consider v an arbitrary upper bound of x and $y \wedge z$, that is $x \leq v$ and $y \wedge z \leq v$.
By Proposition 2.2 we get:

$$(x \vee y) \to v = (x \to v) \wedge (y \to v) = y \to v \text{ and}$$
$$(x \vee z) \to v = (x \to v) \wedge (z \to v) = z \to v.$$

If the identity (C_1^1) is satisfied, then we have:

$$[(x \vee y) \to v] \vee [(x \vee z) \to v] = (y \to v) \vee (z \to v) = (y \wedge z) \to v = 1 \text{ and}$$
$$[(x \vee y) \wedge (x \vee z)] \to v = [(x \vee y) \to v] \vee [(x \vee z) \to v] = 1,$$

that is $(x \vee y) \wedge (x \vee z) \leq v$, so $u \leq v$.
Thus, u is the least upper bound of x and $y \wedge z$.
We conclude that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, that is $(A, \leq)$ is distributive.
Similary, if (C_1^2) is satisfied, we get the same conclusion. □

Corollary 3.2. *If A is a pseudo-BCK(pDN) lattice satisfying* (C_1) *or* (C_2)*, then* $(A, \leq)$ *is distributive.*

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