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Positive Mapping from Semigroup into Anti-ordered Set

Daniel A. Romano

Abstract. We introduce a definition of positive mapping from semigroup with apartness into anti-ordered set and describe a connection between this notion and positive quasi-antiorder.

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1. Introduction and preliminaries

This short investigation, in Bishop's constructive mathematics in sense of well-known book [1] and Romano's papers [5]-[8], is continuation of forthcoming papers Romano's [9], and Crvenković, Mitrović and Romano's [3]. Bishop's constructive mathematics was developed on Constructive logic ([10]) - logic without the Law of Excluded Middle $P \vee \neg P$. Let us note that in the Constructive logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in the Minimal logic. Since the Constructive logic is a part of the Classical logic, results gained in the Constructive mathematics are compatible with suitable results in the Classical mathematics. Let us recall that the following deduction principle $A \vee B, \neg B \vdash A$ acceptable in the Constructive logic.

Let $(X, =, \neq)$ be a set, where the relation "$\neq$" is a binary relation on X, called *diversity* on X, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z.$$

Follows Heyting, if the following implication $x \neq z \Longrightarrow x \neq y \vee y \neq z$ holds then the diversity $\neq$ called *apartness*. Let x be an element of X and A a subset of X. We write $x \bowtie A$ if and only if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$. In $X \times X$ the equality and diversity are defined by

$$(x, y) = (u, v) \Longleftrightarrow x = u \wedge y = v,\ (x, y) \neq (u, v) \Longleftrightarrow x \neq u \vee y \neq v.$$

A relation q on X is a coequality relation ([5], [6]) on X if and only if it is consistent, symmetric and cotransitive:

$$q \subseteq \neq,\ q = q^{-1},\ q \subseteq q * q,$$

where "$*$ " is operation between relations $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ defined by

$$(a, c) \in \beta * \alpha \Longleftrightarrow (\forall b \in Y)((a, b) \in \alpha \vee (b, c) \in \beta).$$

This operation is called *filled product* ([6]) of relations. For coequality q on semigroup S we say that it is an *anticongruence* on S if

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$$(\forall a, b, x, y \in S)((ax, by) \in q \Longrightarrow ((a, b) \in q \vee (x, y) \in q)).$$

In articles [5] and [6] the author proved the following: If e is an equivalence on set X, then there exists the maximal coequality relation q on X compatible with e in the following sense: $e \circ q \subseteq q$ and $q \circ e \subseteq q$. Opposite, if q is a coequality relation on set X, then the relation $q^C = \{(x, y) \in X \times X : (x, y) \bowtie q\}$ is an equivalence on X compatible with q ([6], Theorem 1), and we can ([6], Theorem 2) construct the factor-set $X/(q^C, q) = \{aq^C : a \in X\}$ with:

$$aq^C =_1 bq^C \Longleftrightarrow (a, b) \bowtie q, \; aq^C \neq_1 bq^C \Longleftrightarrow (a, b) \in q.$$

Also, we can ([6]) construct the factor-set $X/q = \{aq : aX\}$. If q is a coequality relation on a set X, then ([6], Theorem 3) X/q is a set with:

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q, \; aq \neq_1 bq \Longleftrightarrow (a, b) \in q.$$

It is clear that $X/(q^C, q) \cong X/q$, and the mapping $\pi : X \longrightarrow X/q$, defined by $\pi(x) = xq$, a strongly extensional surjective mapping.

Subset $C(x) = \{y \in S : y \neq x\}$ satisfies the following implication:

$$y \in C(x) \wedge z \in X \Longrightarrow y \neq z \vee z \in C(x).$$

It is called a *principal strongly extensional subset* of X, and besides it satisfies the following condition $x \bowtie C(x)$. If A is a subset of X, we say that it is a *strongly extensional subset* of X if and only if the following implication

$$x \in A \wedge y \in X \Longrightarrow x \neq y \vee y \in A$$

holds.

As in [7] and [8] a relation α on X is an *antiorder* on X if and only if

$$\alpha \subseteq \neq, \; \alpha \subseteq \alpha * \alpha, \; \neq \subseteq \alpha \cup \alpha^{-1} \text{ (linearity).}$$

Let φ be a strongly extensional mapping of anti-ordered sets from $(X, =, \neq, \alpha)$ into $(Y, =, \neq, \beta)$. For φ we say that it is *reverse isotone* if

$$(\forall a, b \in X)((\varphi(a), \varphi(b)) \in \beta \Longrightarrow (a, b) \in \alpha)$$

holds. A relation τ on X is a *quasi-antiorder* ([7], [8]) on X if

$$\tau \subseteq (\alpha \subseteq) \neq, \; \tau \subseteq \tau * \tau.$$

It is clear that each coequality relation q on set X is a quasi-antiorder relation on X, and the apartness is a trivial anti-order relation on X. It is easy to check that if τ is a quasi-antiorder on X, then ([8]) the relation $q = \tau \cup \tau^{-1}$ is an coequality relation on X. According to [8], in that case, the relation Θ in X/q, defined by

$$(aq, bq) \in \Theta \Longleftrightarrow (a, b) \in \tau \;,$$

is an anti-order on X/q.

In the next lemma we give a description of classes of quasi-antiorder:

Lemma 1.1. *Let τ be a quasi-antiorder on set X. Then $x\tau$ (res. τx) is a strongly extensional subset of X, such that $x \bowtie x\tau$ (res. $x \bowtie \tau x$), for each $x \in X$. Besides, the following implication $(x, z) \in \tau \Longrightarrow x\tau \cup \tau z = X$ holds for each x, z of X.*

Proof. From $\tau \subseteq \neq$ follows $x \bowtie x\tau$. Let $y \in x\tau$ and let z be an arbitrary element of X. Thus $(x, y) \in \tau$ and $(x, z) \in \tau \vee (z, y) \in \tau$. So, we have $z \in x\tau \vee y \neq z$. Therefore, $x\tau$ is a strongly extensional subset of X such that $x \bowtie x\tau$. The proof that τx is a strongly extensional subset of X such that $x \bowtie \tau x$ is analogous. Besides, the following implication $(x, z) \in \tau \Longrightarrow x\tau \cup z\tau = X$ holds for each x, y of X. Indeed, if $(x, z) \in \tau$ and y is an arbitrary element of X, then $(\forall y \in X)((x, y) \in \tau \vee (y, z) \in \tau)$. Thus $S = x\tau \cup \tau z$. □

In the next proposition we give a construction of quasi-antiorder relation.

Proposition 1.1. *If A is a strongly extensional subset of X, then the relation σ on X, defined by $(x,y) \in \sigma \Longleftrightarrow x \in A \wedge x \neq y$, is a quasi-antiorder relation on X.*

Proof. It is clear that σ is a consistent relation on X. Let $(x,z) \in \sigma$ and let y be an arbitrary element of X. Then, $x \in A \wedge x \neq z$. Thus, $x \neq y \vee y \neq z$. If $x \neq y$ and $x \in A$, then $(x,y) \in \sigma$. If $y \neq z$ and $x \in A$, by strongly extensionality of A, we have $y \neq z$ and $x \in A$ and $x \neq y \vee y \in A$. In the case when $y \neq z \wedge x \in A \wedge x \neq y$ we have again $(x,y) \in \sigma$; in the case when $y \neq z$ and $x \in A$ and $y \in A$ we have $(y,z) \in \sigma$. So, the relation σ a is cotransitive relation. Therefore, relation σ is a quasi-antiorder relation on X. Further, we have:

$$x \in A \Longrightarrow x\sigma = C(x), \;\; \neg(x \in A) \Longrightarrow x\sigma = \emptyset;$$
$$y \in A \Longrightarrow y = C(y) \cap A, \;\; y \bowtie A \Longrightarrow \sigma y = A.$$

□

For undefined notions and notations we refer readers to the books [1] and [10] and to author's papers [5]-[8]

2. Positive quasi-antiorder

According to [3], in this section we give a definition and some basic properties of positive quasi-antiorder: A quasi-antiorder τ on a semigroup $(S,=,\neq,\cdot)$ is positive if and only if

$$(\forall a,b \in S)((a,ab) \bowtie \tau \wedge (a,ba) \bowtie \tau).$$

Quasi-order and positive quasi-order are important notion in the Semigroup Theory. They studied, for example, by M.S.Putcha in [4] and S.Bogdanović and M.Ćirić in [2]. Quasi-antiorder is introduced and studied by Romano in [7], [8] and [9]. In the article [7], this author studied the maximal quasi-antiorder in semigroup with apartness. Positive quasi-antiorder defined and studied by Crvenković, Mitrović and Romano in their forthcoming article [3].

In the following proposition we give a construction of positive quasi-antiorder relation on semigroup with apartness using ideal of S:

Proposition 2.1. *Let J be a strongly extensional ideal of S such that $J \subset S$. Then the relation σ on S, defined by $(a,b) \in \sigma \Longleftrightarrow a \in J \wedge b \bowtie J$, is a positive quasi-antiorder relation on S.*

Proof. It is clearly that $\sigma \subseteq \neq$. Let $(a,c) \in \sigma$ and let b be an arbitrary element of S. Then, $a \in J$ and $c \bowtie J$. Thus, by strongly extensionality of J, we have

$$a \in J \wedge (t \neq b \vee b \in J) \wedge c \bowtie J \text{ for any } t \in J.$$

If $b \in J \wedge c \bowtie J$, then $(b,c) \in \sigma$. In the second case we have $(a,b) \in \sigma$. So, relation σ is cotransitive. Let (u,v) be an arbitrary element of σ and let a,b be arbitrary elements of S. Then, we have

$$\begin{aligned}(u,v) \in \sigma &\Longrightarrow (u,a) \in \sigma \vee (a,ab) \in \sigma \vee (ab,v) \in \sigma \\ &\Longrightarrow u \neq a \vee (a \in J \wedge ab \bowtie J) \vee ab \neq v \\ &\Longrightarrow (a,ab) \neq (u,v) \in \sigma \;.\end{aligned}$$

The proof for $(a,ba) \bowtie \sigma$ is similar to this proof.

Immediately follows:

$$a \in J \Longrightarrow a\sigma = J^C, \ \neg(a \in J) \Longrightarrow a\sigma = \emptyset;$$
$$b \in J \Longrightarrow \sigma b = \emptyset \ , \ b \bowtie J \Longrightarrow \sigma b = J.$$

□

In the following theorem, taken from article [3], we give without proof some fundamental properties of positive quasi-antiorder in semigroup S with apartness:

Theorem 2.1. *The following conditions for a quasi-antiorder τ on a semigroup S are equivalent:*
(1) τ is positive;
(2) $(\forall a, b \in S)(a\tau \cup b\tau \subseteq (ab)\tau)$;
(3) $(\forall a, b \in S)(\tau(ab) \subseteq \tau a \cap \tau b)$;
(4) $a\tau$ is a strongly extensional consistent subset of S such that $a \bowtie a\tau$ for each $a \in S$; and
(5) τb is a strongly extensional ideal of S such that $b \bowtie \tau b$, for each $b \in S$.

As in [6] we describe construction of the maximal positive quasi-antiorder in a semigroup $S = S^1$. Let a be an element of S. Then ([6], Theorem 6) the set $C_{(a)} = \{x \in S : x \bowtie SaS\}$ is a consistent subset of S such that $a \bowtie C_{(a)}$. This subset $C_{(a)}$ is called a *principal* consistent subset of S generated by a. If we introduce relation f, defined by $(a, b) \in f \Longleftrightarrow b \in C_{(a)}$, we have ([6], Theorem 7) that the relation f is a consistent relation, and the relation $c(f) = \bigcap_{n \in N} {}^n f$ is a quasi-antiorder on S. For an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A_n(a) = \{x \in S : (a, x) \in {}^n f\}, \ A(a) = \{x \in S : (a, x) \in c(f)\}$$
$$B_n(a) = \{y \in S : (y, a) \in {}^n f\}, \ B(a) = y \in S : (y, a) \in c(f).$$

In the following theorem present some characteristics of these sets.

Theorem 2.2. *(1) The set $A(a) = \bigcap_{n \in N} A_n(a)$ is the maximal strongly extensional consistent subset of S such that $a \bowtie A(a)$.*
(2) $A(a) \cup A(b) \subseteq A(ab)$.
(3) The set $B(a) = \bigcap_{n \in N} B_n(a)$ is the maximal strongly extensional ideal of S such that $a \bowtie B(a)$.
(4) $B(ab) \subseteq B(a) \cap B(b)$.
(5)The relation $c(f)$ is the maximal positive quasi-antiorder relation on semigroup S.
(6) A quasi-antiorder τ on a semigroup S is positive if and only if it contained in the maximal quasi-antiorder relation $c(f)$ on S.

Proof. (1)-(5) Proofs for (1)-(5) immediately follows from Theorem 2, Theorem 3, Theorem 4 and Theorem 5 of [6].
(6) It is clear that if τ is a positive quasi-antiorder relation on S, then $\tau \subseteq c(f)$, since $c(f)$ is the maximal positive quasi-antiorder relation on S. For opposite proof, let holds $\tau \subseteq c(f)$. Then $(x, xy) \bowtie c(f) \supseteq \tau$ and $(x, yx) \bowtie c(f) \supseteq \tau$ for any x, y of S. So, the quasi-antiorder τ is positive. □

3. Positive mapping

In parallel with positive quasi-antiorders there is possibility to define and investigate positive mapping from semigroup S into an anti-ordered set. For mapping $\varphi : S \longrightarrow P$, from a semigroup $(S, =, \neq, \cdot)$ into an anti-ordered set $(P, =, \neq, \theta)$, we say that it is *positive* if and only if

$$(\varphi(a), \varphi(ab)) \bowtie \theta \text{ and } (\varphi(b), \varphi(ab)) \bowtie \theta \text{ , for all } a, b \in S.$$

A connection between this mapping and positive quasi-antiorder has been given by the following theorem:

Theorem 3.1. *If φ is a positive mapping of a semigroup S into an anti-ordered set P, then the relation τ on S, defined by $(a, b) \in \tau \Longleftrightarrow (\varphi(a), \varphi(b)) \in \theta$, is a positive quasi-order on S. Opposite, if τ is a positive quasi-antiorder on semigroup S, then there anti-ordered semigroup T and positive mapping $\varphi : S \longrightarrow T$.*

Proof. (1) By Lemma 2 in [8], the relation τ on semigroup S is a quasi-antiorder on S. Let x, y, a and b be arbitrary elements of S such that $(x, y) \in \tau$. Then:

$$\begin{aligned}(x, y) \in \tau &\Longrightarrow (x, a) \in \tau \vee (a, ab) \in \tau \vee (ab, y) \in \tau \\ &\Longrightarrow x \neq a \vee (\varphi(a), \varphi(ab)) \in \theta \vee ab \neq y \\ &\Longrightarrow (a, ab) \neq (x, y) \in \tau \ .\end{aligned}$$

For the fact $(b, ab) \bowtie \tau$ a proof is analogous. Therefore, the relation τ on S is a positive quasi-antiorder relation on S.

(2) If τ is a positive quasi-antiorder on semigroup S, then the relation $q = \tau \cup \tau^{-1}$ is a coequality on S. Thus, the factor-set S/q is an anti-ordered set under the antiorder θ, defined by $(aq, bq) \in \theta \Longleftrightarrow (a, b) \in \tau$. Expect that, let xq, yq, aq and bq be elements of S/q such that $(xq, yq) \in \theta$. Thus:

$$\begin{aligned}(xq, yq) \in \theta &\Longrightarrow (xq, aq) \in \theta \vee (aq, abq) \in \theta \vee (abq, yq) \in \theta \\ &\Longrightarrow xq \neq aq \vee (a, ab) \in \tau \vee abq \neq yq \\ &\Longrightarrow (aq, abq) \neq (xq, yq) \in \theta\end{aligned}$$

and, analogously, we have $(bq, abq) \bowtie \theta$. So, for the strongly extensional mapping $\pi : S \longrightarrow S/q$ we have:

$$(\forall a, b \in S)((\pi(a), \pi(ab)) \bowtie \theta \wedge (\pi(b), \pi(ab)) \bowtie \theta).$$

□

Using such connection between quasi-antiorders and mappings of a semigroup into a anti-ordered set, various notions concerning quasi-antiorders can be translated to the notions concerning the corresponding mappings.

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(Daniel A. Romano) Faculty of Education Bijeljina, Department of Mathematics, East Sarajevo University 76300 Bijeljina, Semberskih Ratara Street, Bosnia and Herzegovina
E-mail address: bato49@hotmail.com