Positive Mapping from Semigroup into Anti-ordered Set

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Abstract. We introduce a definition of positive mapping from semigroup with apartness into anti-ordered set and describe a connection between this notion and positive quasi-antiorder.

2010 Mathematics Subject Classification. Primary: 03F65; Secondary: 03E04, 20M10.

Key words and phrases. Constructive mathematics, semigroup with apartness, anti-order, quasi-antiorder, positive quasi-antiorder, positive mapping.

1. Introduction and preliminaries

This short investigation, in Bishop's constructive mathematics in sense of well-known book [1] and Romano's papers [5]-[8], is continuation of forthcoming papers Romano's [9], and Crvenković, Mitrović and Romano's [3]. Bishop's constructive mathematics was developed on Constructive logic ([10]) - logic without the Law of Excluded Middle \( P \lor \neg P \). Let us note that in the Constructive logic the 'Double Negation Law' \( P \iff \neg \neg P \) does not hold, but the following implication \( P \implies \neg \neg P \) holds even in the Minimal logic. Since the Constructive logic is a part of the Classical logic, results gained in the Constructive mathematics are compatible with suitable results in the Classical mathematics. Let us recall that the following deduction principle \( A \lor B, \neg B \vdash A \) acceptable in the Constructive logic.

Let \((X, =, \neq)\) be a set, where the relation "\(\neq\)" is a binary relation on \(X\), called diversity on \(X\), which satisfies the following properties:

\[
\neg(x \neq x), \quad x \neq y \implies y \neq x, \quad x \neq y \land y = z \implies x \neq z.
\]

Follows Heyting, if the following implication \(x \neq z \implies x \neq y \lor y \neq z\) holds then the diversity \(\neq\) called apartness. Let \(x\) be an element of \(X\) and \(A\) a subset of \(X\). We write \(x \bowtie A\) if and only if \((\forall a \in A)(x \neq a)\), and \(A^C = \{x \in X : x \bowtie A\}\). In \(X \times X\) the equality and diversity are defined by

\[
(x, y) = (u, v) \iff x = u \land y = v, \quad (x, y) \neq (u, v) \iff x \neq u \lor y \neq v.
\]

A relation \(q\) on \(X\) is a coequality relation ([5], [6]) on \(X\) if and only if it is consistent, symmetric and cotransitive:

\[
q \subseteq \neq, \quad q = q^{-1}, \quad q \subseteq q \ast q,
\]

where "\(\ast\)" is operation between relations \(\alpha \subseteq X \times Y\) and \(\beta \subseteq Y \times Z\) defined by

\[
(a, c) \in \beta \ast \alpha \iff (\forall b \in Y)((a, b) \in \alpha \lor (b, c) \in \beta).
\]

This operation is called filled product ([6]) of relations. For coequality \(q\) on semigroup \(S\) we say that it is an anticongruence on \(S\) if

Received January 09, 2010. Revision received February 13, 2010.

Partially supported by the Ministry of science and technology of the Republic of Srpska, Banja Luka, Bosnia and Herzegovina.
Let \( \forall a, b, x, y \in S \)(\((ax, by) \in q \implies ((a, b) \in q \lor (x, y) \in q)\)).

In articles [5] and [6] the author proved the following: If \( e \) is an equivalence on set \( X \), then there exists the maximal coequality relation \( q \) on \( X \) compatible with \( e \) in the following sense: \( e \circ q \subseteq q \) and \( q \circ e \subseteq q \). Opposite, if \( q \) is a coequality relation on set \( X \), then the relation \( q^C = \{(x, y) \in X \times X : (x, y) \vDash q\} \) is an equivalence on \( X \) compatible with \( q \) ([6], Theorem 1), and we can ([6], Theorem 2) construct the factor-set \( X/(q^C, q) = \{aq^C : a \in X\} \) with:

\[
aq^C = 1 \iff (a, b) \vDash q, aq^C \neq 1 \iff (a, b) \in q.
\]

Also, we can ([6]) construct the factor-set \( X/q = \{aq : aX\} \). If \( q \) is a coequality relation on a set \( X \), then ([6], Theorem 3) \( X/q \) is a set with:

\[
\forall a = 1, bq \iff (a, b) \vDash q, aq \neq 1 \iff (a, b) \in q.
\]

It is clear that \( X/(q^C, q) \equiv X/q \), and the mapping \( \pi : X \rightarrow X/q \), defined by \( \pi(x) = eq \), is a strongly extensional surjective mapping.

Subset \( C(x) = \{y \in S : y \neq x\} \) satisfies the following implication:

\[
y \in C(x) \land z \in X \implies y \neq z \lor z \in C(x).
\]

It is called a principal strongly extensional subset of \( X \), and besides it satisfies the following condition \( x \vDash C(x) \). If \( A \) is a subset of \( X \), we say that it is a strongly extensional subset of \( X \) if and only if the following implication

\[
x \in A \land y \in X \implies x \neq y \lor y \in A
\]

holds.

As in [7] and [8] a relation \( \alpha \) on \( X \) is an antiorder on \( X \) if and only if

\[
\forall a \in \alpha, \alpha \subseteq \alpha \ast \alpha, \neq \subseteq \alpha \cup \alpha^{-1}\quad \text{(linearity)}.
\]

Let \( \varphi \) be a strongly extensional mapping of anti-ordered sets from \((X, =, \neq, \alpha)\) into \((Y, =, \neq, \beta)\). For \( \varphi \) we say that it is reverse isotone if

\[
(\forall a, b \in X)((\varphi(a), \varphi(b)) \in \beta \implies (a, b) \in \alpha)
\]

holds. A relation \( \tau \) on \( X \) is a quasi-antiorder ([7], [8]) on \( X \) if

\[
\tau \subseteq (\alpha \subseteq \neq), \tau \subseteq \tau \ast \tau.
\]

It is clear that each coequality relation \( q \) on set \( X \) is a quasi-antiorder relation on \( X \), and the apartness is a trivial anti-order relation on \( X \). It is easy to check that if \( \tau \) is a quasi-antiorder on \( X \), then ([8]) the relation \( q = \tau \cup \tau^{-1} \) is an coequality relation on \( X \). According to [8], in that case, the relation \( \Theta \) in \( X/q \), defined by

\[
(aq, bq) \in \Theta \iff (a, b) \in \tau,
\]

is an anti-order on \( X/q \).

In the next lemma we give a description of classes of quasi-antiorder:

**Lemma 1.1.** Let \( \tau \) be a quasi-antiorder on set \( X \). Then \( x\tau \) (res. \( \tau x \)) is a strongly extensional subset of \( X \), such that \( x \vDash x\tau \) (res. \( x \vDash x\tau \)), for each \( x \in X \). Besides, the following implication \( (x, z) \in \tau \implies x\tau \cup z\tau = X \) holds for each \( x, z \) of \( X \).

**Proof.** From \( \tau \subseteq \neq \) follows \( x \vDash x\tau \). Let \( y \in x\tau \) and let \( z \) be an arbitrary element of \( X \). Thus \( (x, y) \in \tau \) and \( (x, z) \in \tau \lor (z, y) \in \tau \). So, we have \( z \in x\tau \lor y \neq z \). Therefore, \( x\tau \) is a strongly extensional subset of \( X \) such that \( x \vDash x\tau \). The proof that \( x\tau \) is a strongly extensional subset of \( X \) such that \( x \vDash x\tau \) is analogous. Besides, the following implication \( (x, z) \in \tau \implies x\tau \cup z\tau = X \) holds for each \( x, y \) of \( X \). Indeed, if \( (x, z) \in \tau \) and \( y \) is an arbitrary element of \( X \), then \( \forall y \in X \)((x, y) \in \tau \lor (y, z) \in \tau) \). Thus \( S = x\tau \cup z\tau \). □
In the next proposition we give a construction of quasi-antiorder relation.

**Proposition 1.1.** If $A$ is a strongly extensional subset of $X$, then the relation $\sigma$ on $X$, defined by $(x, y) \in \sigma \iff x \in A \land x \neq y$, is a quasi-antiorder relation on $X$.

**Proof.** It is clear that $\sigma$ is a consistent relation on $X$. Let $(x, z) \in \sigma$ and let $y$ be an arbitrary element of $X$. Then, $x \in A \land x \neq z$. Thus, $x \neq y \lor y \neq z$. If $x \neq y$ and $x \in A$, then $(x, y) \in \sigma$. If $y \neq z$ and $x \in A$, by strongly extensionality of $A$, we have $y \neq z$ and $x \in A$ and $x \neq y \lor y \in A$. In the case when $y \neq z \land x \in A \land x \neq y$ we have again $(x, y) \in \sigma$; in the case when $y \neq z$ and $x \in A$ and $y \in A$ we have $(y, z) \in \sigma$. So, the relation $\sigma$ a is cotransitive relation. Therefore, relation $\sigma$ is a quasi-antiorder relation on $X$. Further, we have:

- $x \in A \implies x \sigma = C(x)$, $(x \in A) \implies x \sigma = \emptyset$;
- $y \in A \implies y = C(y) \cap A$, $y \bowtie A \implies y \sigma = A$.

$\Box$

For undefined notions and notations we refer readers to the books [1] and [10] and to author’s papers [5]-[8]

2. Positive quasi-antiorder

According to [3], in this section we give a definition and some basic properties of positive quasi-antiorder: A quasi-antiorder $\tau$ on a semigroup $(S, =, \neq, \cdot)$ is positive if and only if

\[(\forall a, b \in S)((a, ab) \bowtie \tau \land (a, ba) \bowtie \tau).
\]

Quasi-order and positive quasi-order are important notion in the Semigroup Theory. They studied, for example, by M.S.Putcha in [4] and S.Bogdanović and M.Čirić in [2]. Quasi-antiorder is introduced and studied by Romano in [7], [8] and [9]. In the article [7], this author studied the maximal quasi-antiorder in semigroup with apartness. Positive quasi-antiorder defined and studied by Crvenković, Mitrović and Romano in their forthcoming article [3].

In the following proposition we give a construction of positive quasi-antiorder relation on semigroup with apartness using ideal of $S$:

**Proposition 2.1.** Let $J$ be a strongly extensional ideal of $S$ such that $J \subset S$. Then the relation $\sigma$ on $S$, defined by $(a, b) \in \sigma \iff a \in J \land b \bowtie J$, is a positive quasi-antiorder relation on $S$.

**Proof.** It is clearly that $\sigma \subseteq \neq$. Let $(a, c) \in \sigma$ and let $b$ be an arbitrary element of $S$. Then, $a \in J$ and $c \bowtie J$. Thus, by strongly extensionality of $J$, we have

\[a \in J \land (t \neq b \lor b \in J) \land c \bowtie J\text{ for any } t \in J.
\]

If $b \in J \land c \bowtie J$, then $(b, c) \in \sigma$. In the second case we have $(a, b) \in \sigma$. So, relation $\sigma$ is cotransitive. Let $(u, v)$ be an arbitrary element of $\sigma$ and let $a, b$ be arbitrary elements of $S$. Then, we have

\[(u, v) \in \sigma \implies (u, a) \in \sigma \lor (a, b) \in \sigma \lor (ab, v) \in \sigma
\]

\[\implies u \neq a \lor (a \in J \land ab \bowtie J) \lor ab \neq v
\]

\[\implies (a, ab) \neq (u, v) \in \sigma.
\]

The proof for $(a, ba) \bowtie \sigma$ is similar to this proof.
Immediately follows:

\[ a \in J \implies a\sigma = J^C, \quad \neg(a \in J) \implies a\sigma = \emptyset; \]
\[ b \in J \implies \sigma b = \emptyset, \quad b \nvdash J \implies \sigma b = J. \]

\[ \square \]

In the following theorem, taken from article [3], we give without proof some fundamental properties of positive quasi-antiorder in semigroup \( S \) with apartness:

**Theorem 2.1.** The following conditions for a quasi-antiorder \( \tau \) on a semigroup \( S \) are equivalent:

(1) \( \tau \) is positive;
(2) \( (\forall a, b \in S)(a\tau \cup b\tau \subseteq (ab)\tau) \);
(3) \( (\forall a, b \in S)(\tau(ab) \subseteq \tau a \cap \tau b) \);
(4) \( \tau \) is a strongly extensional consistent subset of \( S \) such that \( a \nvdash a\tau \) for each \( a \in S \); and

(5) \( \tau \) is a strongly extensional ideal of \( S \) such that \( b \nvdash \tau b \), for each \( b \in S \).

As in [6] we describe construction of the maximal positive quasi-antiorder in a semigroup \( S = S^1 \). Let \( a \) be an element of \( S \). Then ([6], Theorem 6) the set \( C(a) = \{ x \in S : x \nvdash S a S \} \) is a consistent subset of \( S \) such that \( a \nvdash C(a) \). This subset \( C(a) \) is called a principal consistent subset of \( S \) generated by \( a \). If we introduce relation \( f \), defined by \( (a, b) \in f \iff b \in C(a) \), we have ([6], Theorem 7) that the relation \( f \) is a consistent relation, and the relation \( c(f) = \bigcap_{n \in N} n^n f \) is a quasi-antiorder on \( S \). For an element \( a \) of a semigroup \( S \) and for \( n \in N \) we introduce the following notations

\[ A_n(a) = \{ x \in S : (a, x) \in n^n f \}, \quad A(a) = \{ x \in S : (a, x) \in c(f) \} \]
\[ B_n(a) = \{ y \in S : (y, a) \in n^n f \}, \quad B(a) = \{ y \in S : (y, a) \in c(f) \}. \]

In the following theorem present some characteristics of these sets.

**Theorem 2.2.** (1) The set \( A(a) = \bigcap_{n \in N} A_n(a) \) is the maximal strongly extensional consistent subset of \( S \) such that \( a \nvdash A(a) \).
(2) \( A(a) \cup A(b) \subseteq A(ab) \).
(3) The set \( B(a) = \bigcap_{n \in N} B_n(a) \) is the maximal strongly extensional ideal of \( S \) such that \( a \nvdash B(a) \).
(4) \( B(ab) \subseteq B(a) \cap B(b) \).
(5) The relation \( c(f) \) is the maximal positive quasi-antiorder relation on semigroup \( S \).
(6) A quasi-antiorder \( \tau \) on a semigroup \( S \) is positive if and only if it contained in the maximal quasi-antiorder relation \( c(f) \) on \( S \).

**Proof.** (1)-(5) Proofs for (1)-(5) immediately follows from Theorem 2, Theorem 3, Theorem 4 and Theorem 5 of [6].

(6) It is clear that if \( \tau \) is a positive quasi-antiorder relation on \( S \), then \( \tau \subseteq c(f) \), since \( c(f) \) is the maximal positive quasi-antiorder relation on \( S \). For opposite proof, let holds \( \tau \subseteq c(f) \). Then \( (x, xy) \nvdash c(f) \supseteq \tau \) and \( (x, yx) \nvdash c(f) \supseteq \tau \) for any \( x, y \) of \( S \). So, the quasi-antiorder \( \tau \) is positive.

\[ \square \]

**3. Positive mapping**

In parallel with positive quasi-antiorders there is possibility to define and investigate positive mapping from semigroup \( S \) into an anti-ordered set. For mapping \( \varphi : S \rightarrow P \), from a semigroup \( (S, =, \neq, \cdot) \) into an anti-ordered set \( (P, =, \neq, \theta) \), we say that it is positive if and only if
(φ(a), φ(ab)) ∝ θ and (φ(b), φ(ab)) ∝ θ , for all a, b ∈ S.

A connection between this mapping and positive quasi-antiorder has been given by the following theorem:

**Theorem 3.1.** If φ is a positive mapping of a semigroup S into an anti-ordered set P, then the relation τ on S, defined by (a, b) ∈ τ ⇐⇒ (φ(a), φ(b)) ∈ θ, is a positive quasi-order on S. Opposite, if τ is a positive quasi-antiorder on semigroup S, then there anti-ordered semigroup T and positive mapping φ : S → T.

**Proof.** (1) By Lemma 2 in [8], the relation τ on semigroup S is a quasi-antiorder on S. Let x, y, a and b be arbitrary elements of S such that (x, y) ∈ τ. Then:

\[(x, y) ∈ τ \Rightarrow (x, a) ∈ τ \lor (a, b) ∈ τ \lor (ab, y) ∈ τ\]

\[\Rightarrow x \neq a \lor (φ(a), φ(ab)) ∈ θ \lor ab \neq y\]

\[\Rightarrow (a, ab) \neq (x, y) ∈ τ .\]

For the fact (b, ab) ∝ τ a proof is analogous. Therefore, the relation τ on S is a positive quasi-antiorder relation on S.

(2) If τ is a positive quasi-antiorder on semigroup S, then the relation q = τ ∪ τ⁻¹ is a coequality on S. Thus, the factor-set S/q is an anti-ordered set under the antiorder τ, defined by (aq, bq) ∈ θ ⇐⇒ (a, b) ∈ τ. Expect that, let xq, yq, aq and bq be elements of S/q such that (xq, yq) ∈ θ. Thus:

\[(xq, yq) ∈ θ \Rightarrow (aq, bq) ∈ θ \lor (aq, abq) ∈ θ \lor (abq, yq) ∈ θ\]

\[\Rightarrow xq \neq aq \lor (a, abq) ∈ τ \lor yq \neq bq\]

\[\Rightarrow (aq, abq) \neq (xq, yq) ∈ θ\]

and, analogously, we have (bq, abq) ∝ θ. So, for the strongly extensional mapping π : S → S/q we have:

\[(∀a, b ∈ S)((π(a), π(ab)) ∝ θ \land (π(b), π(ab)) ∝ θ).\]

Using such connection between quasi-antiorders and mappings of a semigroup into an anti-ordered set, various notions concerning quasi-antiorders can be translated to the notions concerning the corresponding mappings.

**References**


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