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On a class of automorphic loops

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Abstract. We introduce a new class of automorphic loops $(L,*)$ which is constructed on a given group $(G,\cdot)$ by adjoining an induced binary operation $*: G\times G \longrightarrow L$ defined by the rule, $*(x,y) = x*y = x\cdot\phi(y)$ for $\phi \in Aut(G)$, for all $x,y \in G$. In this paper, we extend the study of the class of automorphic loops. We also characterize the loop $(L,*)$ by $\mathrm{Aut}(L)$.

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1. Introduction

It is well-known that a loop is a one-operational non-associative generalization of a group. The publications of Moufang [5] and Bol [1] provided a motivation to the theory of loops, which gained a ground to deviate along the research areas of algebra, geometry, topology and combinatorics. The development of loop theory remained eclipsed under the fast moving research horizon of the theory of groups. After the completion of the list of simple groups, the research environment is getting more suitability for the structures of non-associative models like those of a loop and quasigroups. In the literature of loop theory, the groups are being used to derive new families of loops. K-loops are generalizations of abelian groups [3]. In the famous paper of Moufang [5], she derived that the alternative rule in algebra implies the well-known four Moufang identities [5]. Then she considered loops satisfying these identities, now called Moufang loops. In the present research environment it is called a *Bol loop with left Bol property.* The theory of Moufang loops has been developed by Bruck [2]. The theory of loops is expanding in different fields of applied sciences.

2. Automorphic loops

In this section, we introduce a new class of loops, which is constructed, on a group $(G,\cdot)$ with identity element e, under the binary operation $*: G\times G \to G$, defined by the rule $x*y = x\cdot\psi(y)$ for a non identity ψ in $Aut(G)$. It shall be denoted by $(L,*) = (G,\cdot,*,\psi)$ in short. Thus we define it as follow:

Definition 2.1. *Let $(G,\cdot)$ be a finite group of order $|G|>2$ with identity element e and a non identity $\psi \in Aut(G)$. Then $(L,*) = (G,\cdot,*)$ forms an automorphic loop such that $x*y = x\cdot\psi(y)$, for $x,y \in (L,*)$*

Remark 2.1. 1. *The group operation '$\cdot$' is denoted by juxtaposition for the sake of convenience and $(G,\cdot)$ is denoted by G and $x\cdot y = xy, for x,y \in G$.*

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2. *In this paper the automorphic loop $(L,*)$ shall be understood as a loop unless stated otherwise.*
3. *The automorphic loop is proper if ψ is not identity of $Aut(G)$.*
4. *The automorphic loop is not commutative, in general.*

Example 2.1. *If $G = V_4 = \{<x,y>: x^2 = e = y^2 = z^2; xy = z\}$ and $\psi = (x\ y) \in Aut(V_4)$ then the loop $(L,*)$ on V_4 by ψ in $Aut(V_4)$ is represented by the following Cayley's table:*

$*$	e	x	y	z
e	e	y	x	z
x	x	z	e	y
y	y	e	z	x
z	z	x	y	e

The loop $(L,*)$ is proper since, $(x*z)*y = z \neq e = x*(z*y)$ and is clearly non-commutative.

We notice that the loop $(L,*)$ on a group G by an identity automorphism $I \in Aut(G)$ forms a group. Thus we characterize that:

Theorem 2.1. *A loop $(L,*)$ on G by ψ in Aut(G) is a group if and only if ψ is the identity of Aut(G).*

Proposition 2.1. *A loop $(L,*)$ by ψ in Aut(G) is an abelian group G if and only, if $\psi(x)*y = \psi(y)*x$, for all x,y.*

Proof. If $x,y \in G$ and G is an abelian group, then

$$\begin{aligned} xy = yx \quad &\Rightarrow \quad \psi(x)\psi(y) = \psi(y)\psi(x) \\ &\Rightarrow \quad \psi(x)*\psi^{-1}(\psi(y)) = \psi(y)*\psi^{-1}(\psi(x)) \\ &\Rightarrow \quad \psi(x)*y = \psi(y)*x \end{aligned}$$

for all $x,y \in L$. The converse is easy to verify. □

Definition 2.2. *Let $(L,*)$ be an automorphic loop by ψ in $Aut(G)$. Then $(L,*)$ is said to be an automorphic left Bol loop, if it fulfils the left Bol property(l.b.p) i.e.,*

$$(x*(y*x))*z = x*(y*(x*z)) \qquad \cdots (l.b.p).$$

Similarly $(L,*)$ is said to be an automorphic right Bol loop, if it fulfils the right Bol property(r.b.p) i.e.,

$$((z*x)*y)*x = z*((x*y)*x) \qquad \cdots (l.b.p).$$

Definition 2.3. *Let $(L,*)$ be an automorphic loop by ψ in $Aut(G)$. Then $(L,*)$ is said to be an automorphic Moufang loop if it is both automorphic left as well as right Bol loop.*

We now relate a class of automorphic loops to well-known classes of Bol or Moufang loops:

Lemma 2.1. *Let $(L,*)$ be an automorphic loop on a group G by non-identity ψ in Aut(G). Then $(L,*)$ is an automorphic left Bol loop if and only, if $\psi^2 = I$.*

Proof. If $(L, *)$ is an automorphic left Bol loop then, for every $x, y, z \in L$,

$$\begin{aligned} (x*(y*x))*z &= x*(y*(x*z)) \\ \Rightarrow (x\psi(y\psi(x)))\psi(z) &= x\psi(y\psi(x\psi(z))) \ \ [definition] \\ \Rightarrow x\psi(y)\psi^2(x)\psi(z) &= x\psi(y)\psi^2(x)\psi^3(z) \\ \Rightarrow \psi(z) &= \psi^3(z),\ for\ all\ z \in G \\ \Rightarrow \psi^2 &= I. \end{aligned}$$

The converse follows easily. Thus the lemma is proved. □

Remark 2.2. *It is easy to verify that example (2.2) is an automorphic left Bol loop, while example 3.6 is not an automorphic Bol loop.*

Lemma 2.2. *Let $(L, *)$ be an automorphic loop on a group G by any non-identity ψ in $Aut(G)$. If $(L, *)$ is an automorphic right Bol loop then $(L, *)$ is a group.*

Thus we conclude that:

Theorem 2.2. *Any automorphic loop $(L, *)$ on a group G by ψ in Aut(G) is an automorphic Moufang loop if and only if $(L, *)$ is a group.*

3. Structure of autmorphisms of loop $(L, *)$

In this section we determine the structure $\mathrm{Aut}(L)$ of all automorphisms of $(L, *)$ in relation to that of $\mathrm{Aut}(G)$.

Lemma 3.1. [6] *If $(L, *)$ is a loop on G by ψ in Aut(G) then $\psi \in Aut(L)$.*

Corollary 3.1. *If $(L, *)$ is a loop on G by ψ in Aut(G) then the cyclic group $< \psi >$ is a subgroup of the Aut(L), i.e., $< \psi > \subseteq Aut(L)$.*

Corollary 3.2. [6] *Let $(L, *)$ be a loop on a group G by ψ in Aut(G). If α in Aut(L) is contained in Aut(G) then $\alpha \circ \psi = \psi \circ \alpha$.*

Proof. Let $\alpha(\neq \psi) \in Aut(L)$. Then for $x, y \in G$

$$\begin{aligned} \alpha(xy) &= \alpha(x*\psi^{-1}(y))\ (definition) \\ &= \alpha(x)*\alpha(\psi^{-1}(y))\ (supposition) \\ &= \alpha(x)\psi\alpha\psi^{-1}(y)\ (definition) \end{aligned}$$

Hence

$$\alpha(x)\alpha(y) = \alpha(x)\psi\alpha\psi^{-1}(y) \Rightarrow \psi \circ \alpha = \alpha \circ \psi.$$

□

Thus α is contained in the centralizer of ψ in $\mathrm{Aut}(G)$ and hence

$$Aut(L) \subseteq Aut(G) \qquad \cdots (a).$$

We generalize the following:

Theorem 3.1. *If $(L, *)$ is a loop on G by $\psi \in Aut(G)$ then Aut(L)=Aut(G) if and only if $\beta \circ \psi = \psi \circ \beta$, for all β in Aut(G).*

Proof. Suppose that $\beta \in Aut(G)$ and $\beta \circ \psi = \psi \circ \beta$. Then,

$$\begin{aligned} \beta(x * y) &= \beta(x\psi(y)) = \beta(x)\beta(\psi(y)) \ (supposition) \\ &= \beta(x) * \psi^{-1}\beta\psi(y) \ (definition) \\ &= \beta(x) * \beta(y) \ (supposition) \end{aligned}$$

for all x, y, $z \in L$, which proves that $\beta \in Aut(L)$. Hence

$$Aut(G) \subseteq Aut(L) \qquad \cdots (b)$$

The converse of (b) is in Corollary 2 of Lemma 3.1. Thus Aut(G)= Aut(L) if and only if $\beta \circ \psi = \psi \circ \beta$, for all β in Aut(G). □

Since inversion $i : G \to G$ on G defined by $i(g) = g^{-1}$ commutes with every automorphism of G, therefore

Theorem 3.2. [6] *Let G be a group with inversion $i : G \to G$ as an automorphism of G such that $i(g) = g^{-1}$ for all $g \in G$. Then $Aut(L)= Aut(G)$.*

Example 3.1. *In example (2.2) if $\psi = (x\ y\ z)$ then the loop $(L, *)$ is represented by the following Cayley's table:*

$*$	e	x	y	z
e	e	y	z	x
x	x	z	y	e
y	y	e	x	z
z	z	x	e	y

is a proper loop since, $(x * z) * y = z \neq x = x * (z * y)$.
It is easy to verify that $\alpha = (x\ y)$, $\beta = (x\ z)$ and $\gamma = (y\ z)$ in Aut(V_4) are not in Aut(L) as none of α, β and γ lies in the centralizer of ψ in V_4.

Thus we characterize the class of loops $(L, *)$ on an abelian group G in the following;

Theorem 3.3. *Let G be a finite abelian group and not an elementary abelain p-group, for a prime number p. Then Aut(G)=Aut(L) if and only if $(L, *)$ is a loop on G by any ψ in Aut(G).*

Now we consider, if $(L, *)$ is a loop on G by ψ in Aut(G) and α in Aut(G) which is not in Aut(L), i.e. $\alpha \notin <\psi>$, then we see that $\alpha \in$ Aut($\alpha(L)$), where, $\alpha(L) = \{\alpha(x) : x \in L\}$ and $(\alpha(L), *)$ is a loop on G by $\alpha^{-1}\psi\alpha$ in Aut(G). We proceed to prove it in the following theorem:

Theorem 3.4. *Let $(L, *)$ be a loop on a group G by ψ in Aut(G). If α in Aut(G) is not in Aut(L) then $\alpha \in Aut(\alpha(L))$, where $(\alpha(L), *)$ is a loop on G by $\alpha\psi\alpha^{-1}$ in Aut(G).*

Proof. Let $\alpha(L) = \{\alpha(x) : x \in L\}$. Then we prove that $(\alpha(L), *)$ is a loop on G. Since $\alpha(x), \alpha(y) \in G$ and $\alpha(x) * \alpha(y) = \alpha(x)\psi(\alpha(y)) \in \alpha(L)$.Then, $\alpha(x) * \alpha(y) = \alpha(x)\alpha(\alpha^{-1}\psi\alpha)(y) = \alpha(x\alpha^{-1}\psi\alpha(y)) = \alpha(x * y)$, if $(x * y) = x\alpha^{-1}\psi\alpha(y)$. i.e. $\alpha \in Aut(L)$, if $(L, *)$ is a loop on G by $\alpha^{-1}\psi\alpha$ in Aut(G). □

Corollary 3.3. *The structures of loops $(L, *)$ and $(\alpha(L), *)$ on G by ψ and $\alpha^{-1}\psi\alpha$ in Aut(G) respectively commute the following diagram:*

$$\begin{tikzcd} G \arrow[r, "\alpha^{-1}\psi\alpha"] \arrow[d, "\psi"'] & (\alpha(L), *) \\ (L, *) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Definition 3.1. *If $(L, *)$ is a loop on G by ψ in Aut(G) and $\alpha \in <\psi>$ in Aut(G). Then $(\alpha(L), *)$ is conjugate to $(L, *)$ by α as $(\alpha(L), *)$ is a loop on G by $\alpha^{-1}\psi\alpha$. Thus we conclude the following.*

Theorem 3.5. *Let G be a finite group with the group Aut(G) of all automorphisms of G. If $(L, *)$ is a loop on G by ψ in Aut(G) then there is one-to-one correspondence between the conjugate classes of Aut(G) and the conjugate classes of the loops on G.*

4. The duality $(\hat{L}, \circ)$ of $(L, *)$

Let $M_\ell(G) = \{< L_x >: x \in G\}$ be the left multiplicative group. If ψ is any automorphism of $M_\ell(G)$, then the automorphic loop $(\hat{L}, \circ)$ on $M_\ell(G)$ by ψ is called dual of $(L, *)$. $(\hat{L}, \circ)$ is given as follows:
$(\hat{L}, \circ) = \{< \hat{L}_x >: x \in M_\ell(G)\}$

Remark 4.1. 1. $Aut(M_\ell(G)) \cong Aut(G)$
2. $(\hat{L}, \circ) \cong (L, *)$

Theorem 4.1. *If $(L, *)$ is a loop on a group G by ψ in Aut(G) then the set $\hat{L} = \{< \hat{L}_x >: x \in L\}$ consisting of all mappings of $M_\ell(L)$ forms a loop on $M_\ell(G)$ under the binary operation of composition defined by $\hat{L}_x \circ \hat{L}_y = \hat{L}_{x*y}$.*

Corollary 4.1. *If $(L, *)$ is a loop on a group G by $\psi \in Aut(G)$ then $(\hat{L}, \circ)(\subseteq (M_\ell(L), \circ))$ is a dual of $(L, *)$.*

Now we demonstrate the above by the following example.

Lemma 4.1. *If $(L, *)$ is a loop on G by ψ in Aut(G) then, $\alpha \in Aut(G)$ if and only if, $\alpha \hat{L}_x \alpha^{-1} = \hat{L}_{\alpha(x)}$, for all $\hat{L}_x \in M_\ell(L)$.*

Proposition 4.1. *In a loop $(L, *)$ on a group G by ψ in Aut(G), the following are equivalent:*
(i) *$(L, *)$ is a loop on a group G by $\psi \in Aut(G)$.*
(ii) $\psi_{(x,y)} = \hat{L}^{-1}_{x*y} \circ \hat{L}_x \circ \hat{L}_y$.
(iii) $\hat{L}_x \circ \hat{L}_y = \hat{L}_{x*y} \circ \psi_{(x,y)}$.
(iv) $\hat{L}^2_x = \hat{L}_{x^2} \circ \psi$.
(v) $\alpha \circ \psi_{(x,y)} \circ \alpha^{-1} = \hat{L}_{\alpha(x)} \circ \hat{L}_{\alpha(y)}$.

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