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On the Hermite interpolation polynomial

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Abstract. The Newton form for the Hermite interpolation polynomial using the divided differences with multiple knots is proved. Using this representation, sufficient conditions for the convergence of the sequence of Hermite interpolation polynomials are established. One extends this way a result obtained by M. Ivan, regarding to sufficient conditions for the uniform convergence of the sequence of Lagrange polynomials.

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1. Introduction

G. Faber [2] proved that there exists a function $f \in C[0,1]$ for which the sequence of Lagrange polynomials $(L_m f)_{m\geq 0}$ doesn't converges uniformly to f on $[0,1]$.
I. Muntean [7] generalized the result of G. Faber.

M. Ivan [5] established sufficient conditions for the uniform convergence of the sequence of the Lagrange polynomials $(L_m f)_{m\geq 0}$ associated the function $f \in C[a,b]$.

The focus of the present paper is to establish sufficient conditions for the uniform convergence of the sequence of Hermite polynomials.

First, let us to introduce the following notations: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}_0$, $r_0, r_1, \dots, r_m \in \mathbb{N}$, $r_0 + r_1 + \cdots + r_m = n+1$, $\alpha = \max\{r_0 - 1, r_1 - 1, \dots, r_m - 1\}$. Next, if $I \subseteq \mathbb{R}$ is an interval then $D^\alpha(I)$ denotes the set of real valued functions α-times differentiable on I. If $\alpha = 0$, then $D^0(I) = \mathcal{F}(I) = \{f : I \to \mathbb{R}\}$.

Let $x_0, x_1, \dots, x_m \in I$ be distinct knots. If $f \in D^\alpha(I)$, the divided difference with multiple knots

$$\big[\underbrace{x_0, x_0, \dots, x_0}_{r_0 \text{ times}}, \underbrace{x_1, x_1, \dots, x_1}_{r_1 \text{ times}}, \dots, \underbrace{x_m, x_m, \dots, x_m}_{r_m \text{ times}}; f\big]$$

will be denoted by $[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f]$.

It is known (see [4], [5], [7], [9]) that the divided difference with multiple knots can be represented under the form

$$\big[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f\big] = \frac{(Wf)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)}{V\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)}, \tag{1}$$

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where

$$(Wf)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big) \tag{2}$$

$$= \begin{vmatrix} 1 & x_0 & x_0^2 \ldots & x_0^{n-1} & f(x_0) \\ 0 & 1 & 2x_0 \ldots & (n-1)x_0^{n-2} & f'(x_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & (n-1)(n-2)\cdot\ldots\cdot(n-r_0+1)x_0^{n-r_0} & f_{(x_0)}^{(r_0-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_m & x_m^2 \ldots & x_m^{n-1} & f(x_m) \\ 0 & 1 & 2x_m \ldots & (n-1)x_m^{n-2} & f'(x_m) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & (n-1)(n-2)\cdot\ldots\cdot(n-r_m+1)x_m^{n-r_m} & f_{(x_m)}^{(r_m-1)} \end{vmatrix}$$

and

$$V\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big) \tag{3}$$

$$= \begin{vmatrix} 1 & x_0 & x_0^2 \ldots & x_0^{n} \\ 0 & 1 & 2x_0 \ldots & nx_0^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & n(n-1)\cdot\ldots\cdot(n-r_0+2)x_0^{n-r_0+1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_m & x_m^2 \ldots & x_m^{n} \\ 0 & 1 & 2x_m \ldots & nx_m^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & n(n-1)\cdot\ldots\cdot(n-r_m+2)x_m^{n-r_m+1} \end{vmatrix}.$$

In [4], the following mean-value theorem for divided differences with multiple knots was proved:

Theorem 1.1. *Let $a, b \in \mathbb{R}$ be given such that $a < b$. If $f \in C^{n-1}[a, b]$ and exists $f^{(n)}$ on (a, b), then there exists $\xi \in (a, b)$ such that the following*

$$\Big[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\Big] = \frac{1}{n!}\, f^{(n)}(\xi) \tag{4}$$

holds.

2. The Hermite interpolation polynomial

In 1878 Charles Hermite proved that for $f \in D^{(\alpha)}(I)$ there exists a unique polynomial of degree at most n, denoted $(H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)(x)$ and called Hermite interpolation polynomial, such that the following conditions

$$\begin{cases} a_0 + a_1x_k + a_2x_k^2 + \ldots + a_{r_k-1}x_k^{r_k-1} + \ldots + a_nx_k^n = f(x_k) \\ a_1 + 2a_2x_k + \ldots + (r_k-1)a_{r_k-1}x^{r_k-2} + \ldots + na_nx_k^{n-1} = f'(x_k) \\ \ldots \\ (r_k-1)!a_{r_k-1} + \ldots + n(n-1)\ldots(n-r_k+2)a_nx_k^{n-r_k+1} = f_{(x_k)}^{(r_k-1)} \\ k \in \{0, 1, \ldots, m\} \\ a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = (H_nf)(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)})(x) \end{cases} \tag{5}$$

are verified.

Note that (5) is a linear system in the unknowns $a_0, a_1, \ldots, a_n$ and it has solutions if and only if the following

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n & f(x_0) \\ 0 & 1 & 2x_0 & \ldots & nx_0^{n-1} & f'(x_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n(n-1)\cdot\ldots\cdot(n-r_m+2)x_m^{n-r_m+1} & f_{(x_m)}^{(r_m-1)} \\ 1 & x & x^2 & \ldots & x^n & (H_nf)(x_0^{(r_0)},\ldots,x_m^{(r_m)})(x) \end{vmatrix} = 0 \tag{6}$$

holds.

But (6) can be expressed in the equivalent form

$$\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n & f(x_0) \\ 0 & 1 & 2x_0 & \ldots & nx_0^{n-1} & f'(x_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n(n-1)\cdot\ldots\cdot(n-r_m+2)x_m^{n-r_m+1} & f_{(x_m)}^{(r_m-1)} \\ 1 & x & x^2 & \ldots & x^n & 0 \end{vmatrix}$$

$$+\begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n & 0 \\ 0 & 1 & 2x_0 & \ldots & nx_0^{n-1} & 0 \\ \ldots & \ldots & \ldots & \ldots & & \\ 0 & 0 & 0 & \ldots & n(n-1)\cdot\ldots\cdot(n-r_m+2)x_m^{n-r_m+1} & 0 \\ 1 & x & x^2 & \ldots & x^n & (H_nf)(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)})(x) \end{vmatrix} = 0.$$

From the above identity, it follows

$$(H_nf)\big(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)}\big)(x) = -\frac{(Uf)\big(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)}\big)(x)}{V\big(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)}\big)} \tag{7}$$

where

$$(Uf)\big(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)}\big)(x) \tag{8}$$

$$= \begin{vmatrix} 1 & x_0 & x_0^2 & \ldots & x_0^n & f(x_0) \\ 0 & 1 & 2x_0 & \ldots & nx_0^{n-1} & f'(x_0) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & n(n-1)\cdot\ldots\cdot(n-r_m+2)x^{n-r_m+1} & f_{(x_m)}^{(r_m-1} \\ 1 & x & x^2 & \ldots & x^n & 0 \end{vmatrix}.$$

Taking (7) and (8) into account, yields that the coefficient of x^n in the Hermite polynomial is

$$\frac{(Wf)\big(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)}\big)}{V\big(x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)}\big)},$$

i.e. it is $\big[x_0^{(r_0)},x_1^{(r_1)},\ldots,x_m^{(r_m)};f\big]$.

3. The Hermite interpolatory divided difference formula

In the following we use the ideas of M. Ivan [5], for the construction of Newton's interpolatory divided difference formula.

Let $Q(x)$ the polynomial defined by

$$Q(x) = (H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)(x)$$
$$-(x-x_0)^{r_0}(x-x_1)^{r_1}\cdot\ldots\cdot(x-x_{m-1})^{r_{m-1}}(x-x_m)^{r_m-1}\big[x_0^{(r_0)}, x_1^{(r_1)}, ..., x_m^{(r_m)}; f\big].$$

The polynomial $Q(x)$ has the degree at most $(n-1)$ and it satisfies the relations

$$Q^{(i_k)}(x_k) = 0,\ \ i_k \in \{0, 1, \dots, r_k - 1\}, k \in \{0, 1, \dots, m-1\};$$
$$Q^{(i)}(x_m) = 0,\ \ i \in \{0, 1, \dots, r_m - 2\}.$$

Taking the uniqueness of the Hermite interpolation polynomial into account, from the above identities it follows

$$Q(x) = (H_{n-1} f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m-1)}\big)(x).$$

This way is obtained the following recurrence formula

$$(H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)(x) \tag{9}$$
$$= (H_{n-1} f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m-1)}\big)(x)$$
$$+(x-x_0)^{r_0}(x-x_1)^{r_1}\cdot\ldots\cdot(x-x_{m-1})^{r_{m-1}}(x-x_m)^{r_m-1}\big[x_0^{(r_0)}, x_1^{(r_1)}, ..., x_m^{(r_m)}; f\big].$$

From (9), yields

$$(H_2 f)(x_0^{(2)})(x) = (H_1 f)(x_0)(x) + (x - x_0)\big[x_0^{(2)}; f\big],$$
$$(H_3 f)(x_0^{(3)})(x) = (H_2 f)(x_0^{(2)})(x) + (x - x_0)^2\big[x_0^{(3)}; f\big],$$
$$\dots$$
$$(H_{n-1} f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_{m-1}^{(r_{m-1})}, x_m^{(r_m-1)}\big)(x)$$
$$= (H_{n-2} f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_{m-1}^{(r_{m-1})}, x_m^{(r_m-2)}\big)(x)$$
$$+ (x-x_0)^{r_0}(x-x_1)^{r_1}\cdot\ldots\cdot(x-x_{m-1})^{r_{m-1}}(x-x_m)^{r_m-2}$$
$$\cdot\big[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_{m-1}^{(r_{m-1})}, x_m^{(r_m-1)}; f\big],$$
$$(H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)(x)$$
$$= (H_{n-1} f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_{m-1}^{(r_{m-1})}, x_m^{(r_m-1)}\big)(x)$$
$$+ (x - x_0)^{r_0}(x - x_1)^{r_1}\cdot\ldots\cdot(x - x_{m-1})^{r_{m-1}}(x - x_m)^{r_m-1}$$
$$\cdot\big[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f\big].$$

Adding the above identities, it follows:

Theorem 3.1. *The Hermite interpolation polynomial can be represented under the form*

$$(H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)(x) = \sum_{i=0}^{r_0-1}(x - x_0)^i f^{(i)}(x_0) \tag{10}$$
$$+\sum_{k=1}^{m}(x-x_0)^{r_0}(x-x_1)^{r_1}...(x-x_{k-1})^{r_{k-1}}\sum_{i=1}^{r_k}(x-x_k)^{i-1}\big[x_0^{(r_0)}, x_1^{(r_1)}, ..., x_k^{(i)}; f\big].$$

Remark 3.1. For $r_0 = r_1 = \cdots = r_m = 1$, from (10) follows the Newton interpolation divided difference formula, so (10) can be called the Newton form of the Hermite interpolation polynomial.

4. Uniform approximation via Hermite interpolation

In [5], M. Ivan establish sufficient conditions for the uniform convergence of the sequence of Lagrange interpolation polynomials. Using his ideas, we shall gave sufficient conditions for the uniform convergence of the Hermite interpolation polynomials.

First, let us to recall the following

Theorem 4.1. [5] *Suppose that $f \in C^n[a,b]$ and $f^{(n+1)}$ exists on (a,b). Then, there exists $\xi \in (a,b)$ such that*

$$\begin{aligned} & f(x) - (H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)(x) \\ & = (x-x_0)^{r_0}(x-x_1)^{r_1}\ldots(x-x_m)^{r_m}\frac{f^{(n+1)}(\xi)}{(n+1)!}. \end{aligned} \tag{11}$$

Next, using the ideas from [5], suppose that $f \in C^\infty[a,b]$ possesses uniform bounded derivatives such that there exists $M > 0$ so that

$$|f^{(k)}(x)| \le M, \tag{12}$$

for any $x \in [a,b]$ and any $k \in \mathbb{N}_0$.

Remark 4.1. Because $r_0, r_1, \ldots, r_m \in \mathbb{N}$ and $r_0 + r_1 + \cdots + r_m = n+1$, it follows $n \ge m$, so if m tends to infinity then n also tends to infinity.

Now, we are ready to prove the main result of this section which is the following

Theorem 4.2. *Suppose that $(x_m)_{m\ge 0}$ is a sequence of distinct knots, $x_m \in [a,b]$ for any $m \in \mathbb{N}_0$. Then the inequality*

$$\left|(H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)(x) - f(x)\right| \le \frac{(b-a)^{n+1}}{(n+1)!}M \tag{13}$$

holds for any $x \in [a,b]$, any $m \in \mathbb{N}_0$ and the sequence

$$\Big((H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)(x)\Big)_{n\ge 0}$$

converges to f, uniformly on $[a,b]$.

Proof. Taking (11) and (12) into account, yields

$$\begin{aligned} & \left|(H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)(x) - f(x)\right| \\ & = \frac{1}{(n+1)!}\left|(x-x_0)^{r_0}(x-x_1)^{r_1}\cdot\ldots\cdot(x-x_m)^{r_m}\right|\left|f^{(n+1)}_{(\xi)}\right| \le \frac{(b-a)^{n+1}}{(n+1)!}M, \end{aligned}$$

i.e. (13) holds. Next, because $\lim_{n\to\infty}\frac{(b-a)^{n+1}}{(n+1)!} = 0$, from (13) it follows that the sequence $\Big((H_n f)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)(x)\Big)_{n\ge 0}$ converges to f, uniformly on $[a,b]$. $\square$

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