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Degree preservation for the p-Laplace operator and applications

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Abstract. We try to investigate in this paper the behaviour of a non-linear perturbation of the p-Laplace operator, under a variation of p. Where we can show conservation of the degree under suitable assumption on the non-linearity.

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1. Introduction

In this paper we try to prove some existence results and the behaviour of a type of equations involving the $p-$laplacian when we try to vary p. In fact all the proofs are based on a homotopy invariance argument of the Leray-Schauder degree. In the first part we will try to generalize the result of of Del Pino in [5] for higher dimensions, also among the proof we will see the behaviour of the $p-$laplacian when we perturb p, so the proof consists of moving the p to 2 which is the linear case, and by a topological degree argument, we show the existence of solutions. In the second part we try to see the behaviour of a curve of solutions of a Dirichlet problem involving an exponential non-linearity. In fact the problem was treated in [11] where we show the existence of solution under some assumption on the weight of the non-linearity, here we prove existence and stability by moving p to 2 and the case $p = 2$ was deeply studied by Lassoued in [10]. The last part consist of a proof of a conservation of the degree when dealing with a Dirichlet problem without assumptions on the growth, actually in almost all the paper we do not assume polynomial growth in the non-linearity but some splitting property in the blow-up case, also we prove an existence result for those kind of problem if we add in this case a growth condition on the non-linearity.

2. Main results

In all this paper Ω is a bounded smooth domain of $\mathbb{R}^n$ where $n \geq 3$ and $p \geq 1$. Also $C^0(\Omega)$ denote the space of continuous functions on $\overline{\Omega}$ that vanish on the boundary $\partial\Omega$.
The first result deals with Dirichlet problems having the following form :

$$\begin{cases} -\Delta_p u &= f(x,u) \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega \end{cases}, \tag{1}$$

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Theorem 2.1. *Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Caratheodory function such that*

$$\lambda_{1,p} < \liminf_{s \longrightarrow \infty} \frac{f(x,s)}{s\,|s|^{p-2}} \leq \limsup_{s \longrightarrow \infty} \frac{f(x,s)}{s\,|s|^{p-2}} < \lambda_{2,p} \tag{2}$$

Then Problem (1) has at least one solution.

The second result deals with Dirichlet problems with a strong non-linearity in the right-hand side, mainly an exponential non-linearity. So if we consider the following problem :

$$\begin{cases} -\Delta_p u &=& V(x)f(u) \text{ in } \Omega \\ u &=& 0 \text{ in } \partial\Omega \end{cases}, \tag{3}$$

Finding a solution to this problem is equivalent to find a fixed point to the operator T_p defined by, $T_p(v) = u$ if and only if

$$\begin{cases} -\Delta_p u &=& V(x)f(v) \text{ in } \Omega \\ u &=& 0 \text{ in } \partial\Omega \end{cases}, \tag{4}$$

According to this definition we have the following Theorem :

Theorem 2.2. *Consider $V \in L^q(\Omega)$ where $q > fracp_0n$, then there exists $c = c(n, p_0, p_1, \Omega) > 0$ so that if*

$$\|V\|_{L^q} \leq \min_{p \in [p_0,p_1]} \left\{ \frac{1}{c^{p-1}} \frac{(p_1-1)^{p-1}}{e^{p_1-1}} \right\} \tag{5}$$

we have

$$1 = \deg(id - T_p, 0, B(0, p_1-1)) = \deg(id - T_{p_1}, 0, B(0, p_1-1)), \forall p \in [p_0, p_1] \tag{6}$$

Corollary 2.1. *If $\underline{u}(p)$ denote the minimal solution of (3) in $B(0, p_1 - 1)$ then $p \longrightarrow \underline{u}(p)$ is upper semi-continuous.*

Theorem 2.3. *Let f be a continuous Lipschitz function such that,*
i) $\lim_{\substack{p \longrightarrow p_0 \\ s \longrightarrow \infty}} \frac{f(x,st)}{|s|^{p-1}} = V_{p_0}(x)g(t)$ *uniformly in x. Where $V_{p_0} \in L^q(\Omega)$ and g monotone non-decreasing.*
ii) There exist $\psi \in L^q(\Omega)$ and $\Omega' \subset \Omega$ such that $\frac{V_{p_0}}{\|V_{p_0}\|_{L^q}} \in \mathcal{H}_\psi(\Omega')$.

Then $\deg(id - T_p, 0, B(0, R)) = \deg(id - T_2, 0, B(0, R))$ *for $R > 0$ large enough.*

Corollary 2.2. *If in addition we have*

$$\sup(\liminf_{s \longrightarrow -\infty} \frac{f(x,s)}{s}, \limsup_{s \longrightarrow +\infty} \frac{f(x,s)}{s}) < \lambda_{1,2} \tag{7}$$

then $\deg(id - T_p, 0, B(0, R)) = 1$.

3. Preliminary Results

First let us recall some regularity theorem about the $p - Laplace$ operator :

Proposition 3.1. *Let u be a solution of*

$$\begin{cases} -\Delta_p u &=& f \text{ in } \Omega \\ u &=& 0 \text{ in } \partial\Omega \end{cases} \tag{8}$$

Where $f \in L^q(\Omega)$. Then If $q > \frac{p}{n}$ Then there exist $C = C(n, p, \Omega) > 0$ such that $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty} \leq C \|f\|_{L^q}. \tag{9}$$

This result is proved using the classical Stampachia technique (see [12],[2],[11])

Remark 3.1. *If we take a closer look to the proof, one can see that it is possible to take C independent of $p \in [p_0, p_1] \subset]1, +\infty[$.*

Now we know from the result of Di Benedetto [7] that in fact that the last inequality holds for the $C^{1,\alpha}(\Omega)$ norm in stead of the L^∞ norm, and same for the constant C, α could be chosen independently of p in the same range.
Also it is well known that the $p-$Laplacian has a sequence of eigenvalues $(\lambda_{k,p})_{k\geq 1}$ such that $\lambda_{k,p} \longrightarrow \infty$ when $k \longrightarrow \infty$. In fact, If we take

$$\Gamma_k(p) = \{A \subset S_{L^p}(0,1)\,;\ A \text{ is symmetric and } \gamma(A) = k\} \tag{10}$$

where γ and $S_{L^p}(0,1)$ denote respectively the genus and the unit sphere of $L^p(\Omega)$, then

Proposition 3.2. *There exists $u(k,p) \in W_0^{1,p}(\Omega)$ and $\lambda_{k,p}$ such that*

i)

$$\lambda_{k,p} = \int_\Omega |\nabla u(k,p)|^p = \sup_{A\in\Gamma_k(p)} \min_{u\in A} \int_\Omega |\nabla u|^p \tag{11}$$

ii)

$$\begin{cases} -\Delta_p u(k,p) &= \lambda_{k,p} |u(k,p)|^{p-2} u(k,p) \ \textit{in } \Omega \\ u &= 0 \ \textit{in } \partial\Omega \end{cases}. \tag{12}$$

In particular the first eigenvalue is obtained by minimization i.e.

$$\lambda_{1,p} = \min_{\substack{\int_\Omega |u|^p = 1 \\ u \in W_0^{1,p}}} \int_\Omega |\nabla u|^p. \tag{13}$$

The first eigenvalue of the $p-$Laplacian is very special since that the non-linear operator $-\Delta_p$ behaves like a linear one near that eigenvalue.

Proposition 3.3. *The first eigenvalue of the $p-$Laplacian is the only one with positive eigenfunction, furthermore it is simple and isolated.*

Those results cannot be extended to the other eigenvalues and we do not know if they are the only eigenvalues, although, we know due to Huang [9] that $\lambda_{2,p}$ is the second eigenvalue, that is, there is no eigenvalues in $]\lambda_{1,p}, \lambda_{2,p}[$.
Also we will use the following non existence result of [11], about problems of this type :

$$\begin{cases} -\Delta_p u &= V(x) f(u) \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega \end{cases} \tag{14}$$

Proposition 3.4. *Assume that $\widetilde{V} = \frac{V}{\|V\|_{L^q}} \in \mathcal{H}_\psi(\Omega') = \{v \in E;\ v > \psi \text{ on } \Omega'\}$, where $\psi \in L^q(\Omega)$ is a positive function, then there exist $\lambda^* \in [0, +\infty]$ such that if $\|V\|_E > \lambda^*$, problem have no positive solution. Moreover, we have the following estimation :*

$$c(p,d,\Omega) \sup_{\alpha>0} \frac{\alpha^{p-1}}{f(\alpha)} \leq \lambda^* \leq \lambda_{1,p}(\Omega', \psi) \sup_{\alpha>0} \frac{\alpha^{p-1}}{f(\alpha)}. \tag{15}$$

Where f in here is monotone non-decreasing and $q > \frac{n}{p}$ and

$$\lambda_{1,p}(\Omega', \psi) = \inf\left\{ \int_{\Omega'} |\nabla u|^p \ ;\int_{\Omega'} |u|^p \psi = 1,\ u \in W_0^{1,p}(\Omega') \right\}. \tag{16}$$

4. Proof of Theorem 2.1

Define the function $\Phi : [1, +\infty[\times C^0(\Omega) \longrightarrow C^0(\Omega)$ by $\Phi(p, u) = |u|^{p-2} u$ and the operator $T :]1, +\infty[\times C^0(\Omega) \longrightarrow C^0(\Omega)$ by $T(p, v) = u$ if and only if

$$\begin{cases} -\Delta_p u &= v \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega \end{cases} \tag{17}$$

Lemma 4.1. *T is a completely continuous operator.*

Proof. Let $q > \frac{n}{p_1}$, then $T : [p_0, p_1] \times L^q(\Omega) \longrightarrow C^0(\Omega)$ is completely continuous. Take $h_k \rightharpoonup h$ in $L^q(\Omega)$ and $p_k \longrightarrow p$ as $k \longrightarrow \infty$, then we have

$$\begin{cases} -\Delta_{p_k} u_k &= h_k \text{ in } \Omega \\ u_k &= 0 \text{ in } \partial\Omega \end{cases}, \tag{18}$$

where $u_k = T(p_k, h_k)$. Therefore, using Proposition (3.1) and the remark that follows, there exist C and α independent of k so that

$$\|u_k\|_{C^{1,\alpha}} \leq C \|h_k\|_{L^q} \tag{19}$$

Thus (u_k) is uniformly bounded in $C^{1,\alpha}(\Omega)$ so there exist a subsequence of (u_k) which we will denote it also (u_k) that converges to $u \in C^{1,\frac{\alpha}{2}}(\Omega)$ in the $C^{1,\frac{\alpha}{2}}(\Omega)$ norm. So we have

$$\int_\Omega |\nabla u_k|^{p_k-2} \nabla u_k \nabla\varphi = \int_\Omega h_k\varphi, \ \forall\varphi \in C_0^\infty(\Omega), \tag{20}$$

after passing to the limit we get

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla\varphi = \int_\Omega h\varphi, \forall\varphi \in C_0^\infty(\Omega), \tag{21}$$

That yields to $u = T(p, h)$ using the previous regularity theorems. □

Lemma 4.2. *The function $p \longmapsto \lambda_1(p)$ is continuous in $[p_0, p_1]$.*

This was already been proved in [6] and [9] in the general case but here we are giving a simpler proof.

Proof of Lemma. Let $p_k \longrightarrow p$, then we have

$$T(p_k, \Phi(p_k, \varphi_{1,p_k})) = \left(\frac{1}{\lambda_1(p_k)}\right)^{\frac{1}{p_k-1}} \varphi_{1,p_k} \tag{22}$$

Since (φ_{1,p_k}) is bounded in $C^{1,\alpha}(\Omega)$ we have $T(p_k, \Phi(p_k, \varphi_{1,p_k}))$ converges to $T(p, \Phi(p, u))$, which is equal to $\left(\frac{1}{\widetilde{\lambda}}\right)^{\frac{1}{p-1}} u$. Therefore we have

$$\begin{cases} -\Delta_p u &= \widetilde{\lambda} |u|^{p-2} u \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega \end{cases} \tag{23}$$

And since $u \geq 0$ by the characterization of the first eigenvalue of the $p-$Laplacian, we get $u = \varphi_{1,p}$ and $\widetilde{\lambda} = \lambda_1(p)$. □

Lemma 4.3. *The function $p \longmapsto \lambda_2(p)$ is continuous in $[p_0, p_1]$.*

The proof here is not as simple as the previous one, since we do not have such characterization of the second eigenvalue though it can be found in [9].

Now if we consider the operator $H(p,u) = T(p, (\lambda_1(p) + t(p))\,\Phi(p,u))$ then it is a completely continuous operator. Where $t(p) = \frac{1}{2}(\lambda_2(p) - \lambda_1(p))$

Proposition 4.1. *For every $p > 1$,*

$$deg(Id - H, 0, B(0,r)) = -1, \tag{24}$$

and

$$d(Id - T(p, (\lambda_1(p) - t)\,\Phi(p,u)), 0, B(0,r)) = 1, \tag{25}$$

for every $t > 0$ and $r > 0$.

Proof of Proposition. The proof follows from the invariance under homotopy of the degree by taking p to 2.

Now consider the following homotopy

$$K(s,u) = (-\Delta_p)^{-1}\left[s(\lambda_1(p) + t(p))\,|u|^{p-2}\,u + (1-s)f(x,u)\right], \tag{26}$$

Then we have $\deg(u - K(s,u), 0, B(0,R)) = -1$ for R large enough. In fact K is admissible because if we have a sequence $(u_k)_k$ in C^0 such that $\|u_k\|_{L^\infty} \longrightarrow \infty$ and $K(s_k, u_k) = u_k$, then after rescaling by $\|u_k\|_{L^\infty}^{1-p}$ we get

$$\left\{\begin{array}{rcl} -\Delta_p v_k &=& s_k(\lambda_1(p) + t(p))\,|v_k|^{p-2}\,v_k + (1-s_k)\frac{f(x,u_k)}{\|u_k\|_{L^\infty}^{p-1}} \text{ in } \Omega \\ v_k &=& 0 \text{ in } \partial\Omega \end{array}\right. , \tag{27}$$

where $v_k = \frac{u_k}{\|u_k\|_{L^\infty}}$, using Lemma(4.1) we can extract a convergent subsequence of (v_k) that converges to v in $C^1(\Omega)$, therefore, passing to the limit in (27) we find that v satisfies the following equation

$$\left\{\begin{array}{rcl} -\Delta_p v &=& s(\lambda_1(p) + t(p))\,|v|^{p-2}\,v + (1-s)\lambda\,|v|^{p-2}\,v \text{ in } \Omega \\ v &=& 0 \text{ in } \partial\Omega \end{array}\right. \tag{28}$$

where $\lambda \in]\lambda_1(p), \lambda_2(p)[$ therefore $v = 0$ which is impossible since $\|v\|_{L^\infty} = 1$. □

If $f(x,0) = 0$ then we can add the following assumption on f so we can get a non trivial solution $\lim\limits_{s \longrightarrow 0} \frac{f(x,s)}{s^{p-1}} < \lambda_1(p)$. In fact consider $\varepsilon > 0$ small enough so that

$$f(x,s) \leq (\lambda_1(p) - \delta)\,|s|^{p-2}\,s, \text{ for } 0 < s < \varepsilon \tag{29}$$

and take

$$\widetilde{f}(x,s) = \min(f(x,s), (\lambda_1(p) - \delta)\,|s|^{p-2}\,s). \tag{30}$$

Now if we define $\widetilde{T}$ by $\widetilde{T}v = u$ iff

$$\left\{\begin{array}{rcl} -\Delta_p u &=& \widetilde{f}(x,v) \text{ in } \Omega \\ u &=& 0 \text{ in } \partial\Omega \end{array}\right. \tag{31}$$

then one can notice that $\widetilde{T}$ stabilizes $B(0,\varepsilon)$ therefore T is just the operator defined by $Tv = u$ iff

$$\left\{\begin{array}{rcl} -\Delta_p u &=& f(x,v) \text{ in } \Omega \\ u &=& 0 \text{ in } \partial\Omega \end{array}\right. \tag{32}$$

then

$$\deg(id - T, 0, B(0,\varepsilon)) = \deg\left(id - \widetilde{T}, 0, B(0,\varepsilon)\right) \tag{33}$$

and using Proposition (4.4) and the same procedure of the previous proof, we get

$$\deg\left(id - T, 0, B(0,\varepsilon)\right) = \deg\left(id - \widetilde{T}, 0, B(0,\varepsilon)\right) = 1. \tag{34}$$

Using the excision property of the degree we get the existence of a non-trivial solution.

5. Proof of Theorem 2.2

Here we fix $p \in [p_0, p_1] \subset]1, +\infty[$. Let us take $V \in L^q(\Omega)$ so that

$$\|V\|_{L^q} \leq \min_{p\in[p_0,p_1]} \left\{ \frac{1}{c^{p-1}} \frac{(p_1-1)^{p-1}}{e^{p_1-1}} \right\} \tag{35}$$

where c is the uniform constant in the interval $[p_0, p_1]$ in Proposition(3.1) and the remark that follows. And consider the problem

$$\begin{cases} -\Delta_p u &= V(x)e^u \\ u &= 0 \end{cases} \tag{36}$$

This problem was deeply studied in [11] where we can find theorems about existence and non-existence of solutions but here we will see the link between them, in fact we will show that there are curves of solutions and that the curve of minimal solutions has some regularity.

Let $T : [p_0, p_1] \times C_0 \longrightarrow C_0$, be the operator defined by $T(p, v) = u$ if and only if

$$\begin{cases} -\Delta_p u &= V(x)e^v \text{ in } \Omega \\ u &= 0 \text{ in } \partial\Omega \end{cases} \tag{37}$$

Claim : $B(0, p_1 - 1)$ is stabilized by T,

In fact if $u \in B(0, p_1 - 1)$ is a solution of (36) then according to Proposition (3.1)

$$\|u\|_\infty \leq c\,\|V\|_{L^q}^{\frac{1}{p-1}}\, e^{\frac{p_1-1}{p-1}} \leq p_1 - 1 \tag{38}$$

because of the assumption maid on the norm of V.

Now using Lemma(4.1) we know that the operator is continuous with respect to p therefore we can take a homotopy from p to 2 and recall that the case $p = 2$ was treated by [10], where we can easily compute the degree, (using linearisation or by a homotopy to Id) we get

$$\deg(id - T_{p_0}, 0, B(0, p_1 - 1)) = \deg(id - T_2, 0, B(0, p_1 - 1)) = 1. \tag{39}$$

This proves the existence.

Now to prove the corollary one can simply again use Lemma(4.1), and after noticing that the operator $R(u) = e^u$ is continuous from $C^0(\Omega)$ to $C(\Omega)$ and by the regularity in Proposition(3.1), one can see that if we take a sequence $u(p)$ of solution of problem (34) in $B(0, p_1 - 1)$ then $\lim_{p\longrightarrow p_0} u(p)$ is a solution of the problem at p_0, Now if we start with the sequence of minimal solutions that we know already its existence according to [11] then after passing to the limit one finds that

$$\lim_{p\longrightarrow p_0} \underline{u}(p) = u(p_0) \geq \underline{u}(p_0) \tag{40}$$

therefore the curve of minimal solutions is upper semi-continuous.

Remark 5.1. *The same procedure can be done for problems having a monotone increasing right hand side and by slightly modifying the assumption on V. Also one can see that it works also for equation of this form*

$$\left\{\begin{array}{rcl} -\Delta_p u & = & V(x)f(p,u) \text{ in } \Omega \\ u & = & 0 \text{ in } \partial\Omega \end{array}\right. \tag{41}$$

where f is continuous in p and if we adjust the assumption on V, Then the previous results hold.

6. Proof of Theorem 3

Here we will use the notation of [11], so first take

$$\mathcal{H}_\psi(\Omega') = \{v \in L^q(\Omega);\ v > \psi \text{ on } \Omega' \subset \Omega\}. \tag{42}$$

Let f be a continuous Lipschitz function such that,

$$\lim_{\substack{p \longrightarrow p_0 \\ s \longrightarrow \infty}} \frac{f(x,st)}{|s|^{p-1}} = V_{p_0}(x)g(t) \text{ uniformly in } x. \tag{43}$$

Where V_p is in L^q and g is monotone increasing.
Assume that

$$\|V_{p_0}\|_{L^q} \geq \lambda_{1,p}(\Omega', \psi) \sup_{\alpha>0} \frac{\alpha^{p-1}}{g(\alpha)}. \tag{44}$$

Where $\lambda_{1,p}(\Omega', \psi)$ is the first eigenvalue of the weighted resonance problem

$$\left\{\begin{array}{rcl} -\Delta_p u & = & \lambda\psi |u|^{p-2} u \text{ in } \Omega' \\ u & = & 0 \text{ in } \partial\Omega' \end{array}\right. \tag{45}$$

Let us consider the following homotopy

$$H(p,u) = u - (-\Delta_p)^{-1}(f(x,u)) \tag{46}$$

Lemma 6.1. *There exist $R > 0$ large enough so that $B(0,R)$ is admissible.*

Proof of lemma. Assume that there exist a blowing-up sequence $u_k \in C^0(\Omega)$ so that

$$\left\{\begin{array}{rcl} -\Delta_{p_k} u_k & = & f(x,u_k) \text{ in } \Omega \\ u_k & = & 0 \text{ in } \partial\Omega \end{array}\right. \tag{47}$$

and take $v_k = \dfrac{u_k}{\|u_k\|_\infty^{p_k-1}}$, then we have v_k satisfies

$$\left\{\begin{array}{rcl} -\Delta_{p_k} v_k & = & \dfrac{f(x, \|u_k\|_\infty^{p_k-1} v_k)}{\|u_k\|_\infty^{p_k-1}} \text{ in } \Omega \\ u_k & = & 0 \text{ in } \Omega \end{array}\right. \tag{48}$$

taking p_k to $\widetilde{p}$ we have, after passing to a subsequence that,

$$\left\{\begin{array}{rcl} -\Delta_{\widetilde{p}} v & = & V_{\widetilde{p}}(x)g(u) \text{ in } \Omega \\ v & = & 0 \text{ in } \partial\Omega \end{array}\right. \tag{49}$$

where $V_{\widetilde{p}} \in \mathcal{H}_\psi(\Omega')$ and satisfies inequality (44), which is impossible, and this completes the proof of the Lemma. $\square$

Therefore using the continuity proved in Lemma(4.1) and Proposition(3.1) we have

$$\deg\left(id - T_2, 0, B(0,R)\right) = \deg\left(id - T_p, 0, B(0,R)\right) \tag{50}$$

For the proof of the Corollary let us assume that f satisfies in addition

$$b = \sup\Big(\liminf_{s\longrightarrow-\infty}\frac{f(x,s)}{s}, \limsup_{s\longrightarrow+\infty}\frac{f(x,s)}{s}\Big) < \lambda_{1,2} \tag{51}$$

Let us give a lemma that will be used in the proof of the previous corollary

Lemma 6.2. *Let $\mathcal{M}$ be a bounded open set which contains* 0, *of a Banach space and let $T : \overline{\mathcal{M}} \longmapsto E$ be a compact operator, If*

$$Tu \neq \lambda u,\ \forall u \in \partial\mathcal{M},\ \forall\lambda \geq 1 \tag{52}$$

Then $deg(I - T, \mathcal{M}, 0) = 1$.

Proof of Lemma. Let H be the homotopy defined by $H(t,u) = u - tTu$; by the assumption imposed on $\partial\mathcal{M}$ we have the compatibility hypothesis of H so we have

$$deg(H(1,.), \mathcal{M}, 0) = deg(H(0,.), \mathcal{M}, 0) = deg(id, \mathcal{M}, 0) = 1. \tag{53}$$

□

Proof of Corollary. If we consider the partial order induced by $\mathbb{R}$ in $C^0(\Omega)$ we know that the operator $(-\Delta)^{-1}$ is positive linear, and thus we have

$$|T_2 u| \leq B\,|u| + u_0,\ \forall u \in C^0(\Omega) \tag{54}$$

Where $B = b\,(-\Delta)^{-1}$, and remark that $id - B$ is invertible, also if we take $K = \left\{u \in C^0(\Omega)\,; T_2 u = \lambda u \text{ for } \lambda \geq 1\right\}$, then K is bounded, in fact,

$$T_2 u = \lambda u \iff \lambda\,|u| \leq B\,|u| + u_0 \tag{55}$$

Therefore

$$(id - B)\,|u| \leq u_0 \tag{56}$$

so we have $|u| \leq (id - B)^{-1}\,u_0$.

Now take R large enough so that $K \subset B(0,R)$, with that we have $T_2 u \neq \lambda u$ for every $\lambda \geq 1$ and $u \in \partial B(0,R)$, Using Lemma(6.2) we have

$$\deg\left(id - T_2, 0, B(0,R)\right) = 1. \tag{57}$$

□

Remark 6.1. *One can see that the same result hold if the function g in the assumption made on f, depends continuously on p.*

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