Algebraic properties of $\omega$-trees (II)

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Abstract. In [16] we defined the concept of $\omega$-labeled tree as a binary, ordered and labeled tree with several features concerning the labels and order between the direct descendants of a node. This paper includes several further results concerning these structures. The main results presented in this paper are the following: we introduce an equivalence relation $\simeq$ on the set $OBT(\omega)$ of $\omega$-trees and a partial order on the factor set $OBT(\omega)/\simeq$.

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1. Introduction

Various implications of new algebraic structures into computer science area were established in the recent years. The Peano algebras and graph theory were applied successfully in knowledge representation ([2], [3], [7], [9], [10], [11], [12], [13]). Several properties of pseudo-BCK algebras show their connection with fuzzy structures and the class of pseudo-BCK algebras with pseudo-double negation generalizes some particular structures with applications in mathematical logic ([4]). The labeled ordered trees were implied successfully in theoretical and applied computer science. A Tree Algebra for XML, named TAX, was developed as a natural extension of relational algebra for manipulating XML data, modeled as forests of labeled ordered trees([6]). An algebra for manipulating collections with ordering specifications was developed in [8]. The Peano Count Tree (P-tree) gives a tree representation of spatial data. The algebra and properties of P-tree structure as well as fast algorithms for P-tree generation and P-tree operations are treated in [5].

The concept of $\omega$-tree was introduced in [16]. This structure is a binary tree whose nodes are labeled by means of a mapping $\omega$ that specifies the labeling process. There are two kinds of labels: terminal and non-terminal labels. Only the nodes labeled by non-terminal labels may contain direct descendants. On the set $OBT(\omega)$ of $\omega$-trees we introduced a binary relation, which is not a partial order. In the present paper we develop the idea introduced in [16].

This paper is organized as follows: Section 2 contains the basic notions and results obtained in [16]; in Section 3 we define and study an equivalence relation $\simeq$ on the set $OBT(\omega)$; in Section 4 a partial order on the factor set $OBT(\omega)/\simeq$ is defined and studied. The last section contains conclusions and future work.

The purpose of this research is to apply these results to obtain algebraic structures useful to describe the computations in cooperating structures based on semantic schemas ([14], [15]).
2. Basic notions and notations

A directed ordered graph ([1]) is a pair \( G = (A, D) \), where \( A \) is a finite set of elements called nodes, \( D \) is a finite set of elements of the form \([(i, i_1), \ldots, (i, i_n)]\), where \( n \geq 1 \) and \( i, i_1, \ldots, i_n \in A \) and \( D \) satisfies the following condition: if \([(i, i_1), \ldots, (i, i_n)] \in D \) and \([(j, j_1), \ldots, (j, j_s)] \in D \) then \( i \neq j \). We observe that for an element \([(i, i_1), \ldots, (i, i_n)] \in D \) we may have \( i_j = i_k \) for some \( j \neq k \). On the other hand, an element of \( D \) is a list and the order of its elements are taken into consideration. An element of a list is a directed arc and simply is named arc.

We can represent a directed ordered graph as follows. We represent, as usual, a directed arc \( \rightarrow \) as \( (i, j) \) and a list \( \{i, i_1, \ldots, i_n\} \) is named the path following property is satisfied:

\[ \{i, i_1, \ldots, i_n\} \in D, j, r \in \{1, \ldots, n\}, j \neq r \Rightarrow i_j = i_r \]  

An ordered tree is a directed ordered graph \( G = (A, D) \) such that \( D' \) is a tree and the following property is satisfied:

\[ [(i, i_1), \ldots, (i, i_n)] \in D, j, r \in \{1, \ldots, n\}, j \neq r \Rightarrow i_j = i_r \]  

A path in a directed ordered graph is a sequence \( d = (n_0, n_1, \ldots, n_k) \) of nodes such that for every \( i \in \{0, \ldots, k - 1\} \), we have an arc from \( n_i \) to \( n_{i+1} \). The number \( k \) is the length of \( d \). We denote by \( \text{Path}(G) \) the set of all paths in \( G \).

A binary tree is a tree such that every node has exactly zero or two direct descendants. The root is a node that is not a direct descendant of any other node. A tree has a single root. Every node that is not the root in the binary tree is reachable from the root node by a unique path. A node with neither a left descendant nor a right descendant is called a leaf. By an abuse of language we shall use the concepts of arc and path in an ordered tree \( t \) and in this case we suppose that these concepts are applied to the graph associated to \( t \). Moreover, for an ordered tree \( t \) we denote by \( \text{Path}(t) \) the set of all paths of \( t \).

We consider a finite set \( L \) and a decomposition \( L = L_N \cup L_T \), where \( L_N \cap L_T = \emptyset \). The elements of \( L_N \) are called nonterminal labels and those of \( L_T \) are called terminal labels. The elements of \( L \) are called labels. A split mapping on \( L \) ([16]) is a function \( \omega : L_N \rightarrow L \times L \). For each \( x \in L_N \) we denote \( \omega(x) = (\omega_1(x), \omega_2(x)) \). The entity \( \omega_1(x) \) is named the left component and \( \omega_2(x) \) is the right component of \( \omega(x) \).

Let \( \omega : L_N \rightarrow L \times L \) be a split mapping on \( L \). An \( \omega \)-tree ([16]) is a tuple \( t = (A, D, h) \), where

- \((A, D)\) is an ordered tree and every element of \( D \) is of the form \([(i, i_1), (i, i_2)]\);
- \( h : A \rightarrow L \) is a mapping such that

\[ [(i, i_1), (i, i_2)] \in D \Rightarrow h(i) \in L_N \& \omega(h(i)) = (h(i_1), h(i_2)) \]  

For each \( i \in A \) the element \( h(i) \) is called the label of the node \( i \). The mapping \( h \) is named the labeling mapping of \( t \). By \( \text{OBT}(\omega) \) we denote the set of all \( \omega \)-trees.

Let \( t_1 = (A_1, D_1, h_1) \) and \( t_2 = (A_2, D_2, h_2) \) be two elements of \( \text{OBT}(\omega) \) and an arbitrary mapping \( \alpha : A_1 \rightarrow A_2 \). For every \( u = [(i, i_1), (i, i_2)] \in D_1 \) we denote

\[ \overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \]
If \( t = (A, D, h) \) is an \( \omega \)-tree then we denote by \( \text{root}(t) \) the element of \( A \) designated by the root of \( t \).

**Definition 2.1.** ([16]) If \( t_1 = (A_1, D_1, h_1) \in OBT(\omega) \) and \( t_2 = (A_2, D_2, h_2) \in OBT(\omega) \) then we define the relation \( t_1 \preceq t_2 \) if there is a mapping \( \alpha : A_1 \to A_2 \) such that:

\[
u \in D_1 \Rightarrow \overline{\pi}(u) \in D_2 \tag{3}\]

\[
h_1(\text{root}(t_1)) = h_2(\alpha(\text{root}(t_1))) \tag{4}\]

Such a mapping \( \alpha \) is an embedding mapping of \( t_1 \) into \( t_2 \).

**Remark 2.1.** In an \( \omega \)-tree we have the following property: if a node \( n \) has direct descendants then the label \( h(n) \) of \( n \) is an element of \( L_N \). Moreover, in this case the left (right) descendant of \( n \) is labeled by the left (right) component of \( \omega(h(n)) \). A leaf of an \( \omega \)-labeled tree may be labeled by an element of \( L_N \), but a node labeled by an element of \( L_T \) is a leaf.

In order to exemplify these concepts we consider the following case:

- \( L = \{a_i, b_i, c_i\}_{i \geq 1}, L_N = \{b_i, c_i\}_{i \geq 1}, L_T = \{a_i\}_{i \geq 1} \)
- \( \omega(b_i) = (a_i, c_i) \) and \( \omega(c_i) = (b_i, b_i) \) for \( i \geq 1 \)

In Figure 1 we represented the following two \( \omega \)-trees \( t_1 = (A_1, D_1, h_1) \) and \( t_2 = (A_2, D_2, h_2) \), where:

- \( A_1 = \{n_j\}_{j=1,\ldots,\gamma}; A_2 = \{m_j\}_{j=1,\ldots,11} \)
- \( h_1(n_i) = b_1 \) for \( i \in \{1, 4, 5\}; h_2(n_2) = h(n_6) = a_1; h_1(n_3) = h_1(n_7) = c_1 \)
- \( h_2(m_4) = h_2(m_8) = a_1; h_2(m_5) = b_1 \) for \( i \in \{2, 3, 6, 7, 10, 11\}; h_2(m_1) = h_2(m_5) = h_2(m_9) = c_1 \)
- \( D_1 = \{[(n_1, n_2), (n_1, n_3)], [(n_3, n_4), (n_5, n_5)], [(n_4, n_6), (n_4, n_7)]\} \)
We observe the following property of $\alpha \beta$ We suppose that

If the reflexivity and transitivity of the relation $\alpha \beta$

Remark 3.1. Card

3. An equivalence relation on $OBT(\omega)$

Based on the relation $\succeq$ we introduce an equivalence relation on the set $OBT(\omega)$. In this section we study this relation.

Definition 3.1. We consider $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. We define $t_1 \simeq t_2$ if $t_1 \succeq t_2$ and $t_2 \succeq t_1$.

Proposition 3.1. The relation $\simeq$ is an equivalence relation.

Proof. The reflexivity and transitivity of the relation $\simeq$ are obtained from property P4. If $t_1 \simeq t_2$ then obviously $t_2 \simeq t_1$ and thus $\simeq$ is a symmetric relation.

Remark 3.1.

• If $t \in OBT(\omega)$ then we denote by $[t]$ the equivalence class of $t$ with respect to $\simeq$.

• For a finite set $A$ we denote by $\text{Card}(A)$ the cardinal number of $A$.

Proposition 3.2. If $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ and $t_1 \simeq t_2$ then every embedding mapping $\alpha : A_1 \rightarrow A_2$ of $t_1$ into $t_2$ satisfies the following conditions:

1. $\alpha$ is a bijective mapping and $\alpha^{-1}$ is an embedding mapping of $t_2$ into $t_1$;

2. The image by $\alpha$ of the root of $t_1$ is the root of $t_2$:

$$\alpha(\text{root}(t_1)) = \text{root}(t_2)$$

(5)

Proof. We suppose that $t_1 \simeq t_2$. We consider an embedding mapping $\alpha : A_1 \rightarrow A_2$ of $t_1$ into $t_2$ and an embedding mapping $\beta : A_2 \rightarrow A_1$ of $t_2$ into $t_1$. From property P3 we deduce that $\alpha : A_1 \rightarrow A_2$ and $\beta : A_2 \rightarrow A_1$ are injective mappings. Moreover, the following conditions are satisfied:

$$u \in D_1 \Rightarrow \overline{\alpha}(u) \in D_2$$

(6)

$$u \in D_2 \Rightarrow \overline{\beta}(u) \in D_1$$

(7)

We have $\text{Card}(A_1) \leq \text{Card}(A_2)$ because $\alpha : A_1 \rightarrow A_2$ is an injective mapping. Similarly by means of the mapping $\beta$ we deduce that $\text{Card}(A_2) \leq \text{Card}(A_1)$. It
follows that \( \text{Card}(A_1) = \text{Card}(A_2) \) and thus \( \alpha \) and \( \beta \) are bijective mappings.

Let us prove first that the following properties are fulfilled:

\[
\alpha(\text{root}(t_1)) = \text{root}(t_2), \beta(\text{root}(t_2)) = \text{root}(t_1), \alpha(\beta(\text{root}(t_2))) = \text{root}(t_2)
\] (8)

Suppose that

\[
\beta(\text{root}(t_2)) \neq \text{root}(t_1)
\] (9)

We denote \( \beta(\text{root}(t_2)) = i \) and suppose that \( [(\text{root}(t_2), j_1), (\text{root}(t_2), j_2)] \in D_2 \). If we denote \( \beta(j_1) = i_1 \) and \( \beta(j_2) = i_2 \) then by (7) we have

\[
[i, i_1], (i, i_2)] \in D_1 \quad \text{(10)}
\]

There is a direct predecessor and only one \( r \) of \( i \) in \( t_1 \) because \( i \neq \text{root}(t_1) \). Without loss of generality we can suppose that \( i \) is a right direct descendant of \( r \). In other words we find an element

\[
[(r, p), (r, i)] \in D_1 \quad \text{(11)}
\]

But \( \beta \) is a surjective mapping, therefore there is \( q \in A_2 \) such that \( \beta(q) = r \). We have \( q \neq \text{root}(t_2) \). Really, if we suppose \( q = \text{root}(t_2) \) then \( r = \beta(q) = \beta(\text{root}(t_2)) = i \), which is not true by (11). We have \( q \neq j_1 \) and \( q \neq j_2 \). Really, if \( q = j_1 \) then \( r = \beta(q) = \beta(j_1) = i_1 \). From (11) and (10) we deduce now that \( (r, i, i_1) \) is a circuit in the tree \( t_1 \), which is not true because a tree does not contain any circuit. In a similar manner we deduce that \( q \neq j_2 \). It follows that there is \( (\text{root}(t_2), j_2, . . . , j_k, q) \in \text{Path}(t_2) \) or \( (\text{root}(t_2), j_1, . . . , j_k, q) \in \text{Path}(t_2) \). Without loss of generality we can suppose that \( (\text{root}(t_2), j_2, . . . , j_k, q) \in \text{Path}(t_2) \). We have in this case \( (j_2, . . . , j_k, q) \in \text{Path}(t_2) \) and by property P2 we deduce that \( (\beta(j_2), . . . , \beta(j_k), \beta(q)) \in \text{Path}(t_1) \). But \( \beta(j_2) = i_2 \) and \( \beta(q) = r \) therefore

\[
(i_2, . . . , \beta(j_k), r) \in \text{Path}(t_1) \quad \text{(12)}
\]

On the other hand from (11) and (10) we deduce that

\[
(r, i, i_2) \in \text{Path}(t_1) \quad \text{(13)}
\]

From (12) and (13) we deduce that \( (i_2, . . . , \beta(j_k), r, i, i_2) \in \text{Path}(t_1) \) and this property is not true because \( t_1 \) is a tree.

It follows that the assumption (9) is not true, therefore

\[
\beta(\text{root}(t_2)) = \text{root}(t_1) \quad \text{(14)}
\]

By symmetry we have also

\[
\alpha(\text{root}(t_1)) = \text{root}(t_2) \quad \text{(15)}
\]

therefore (5) is true. Now from (14) and (15) we obtain

\[
\alpha(\beta(\text{root}(t_2))) = \text{root}(t_2) \quad \text{(16)}
\]

and (8) is proved.

Let us prove that (16) can be extended to all nodes of \( t_2 \):

\[
\alpha(\beta(j)) = j \quad \text{for } \text{every } j \in A_2
\]

(17)

In order to do this we denote \( M_0 = \{\text{root}(t_2)\} \) and for \( k \geq 1 \) we denote by \( M_k \) the set of all nodes \( j \in A_2 \) such that there is a path of length \( k \) from \( \text{root}(t_2) \) to \( j \). We prove the following property for every \( k \geq 0 \):

\[
j \in M_k \Rightarrow \alpha(\beta(j)) = j
\]

(18)

From this property and the relation \( A_2 = \bigcup_{k \geq 0} M_k \) we deduce (17).

For \( k = 0 \) we have to verify that \( \alpha(\beta(\text{root}(t_2))) = \text{root}(t_2) \) because \( M_0 \) contains only one element, namely \( \text{root}(t_2) \). But this property is true in virtue of (16), therefore (18) is true for \( k = 0 \).
We suppose that (18) is true for some \( k \) and we verify the same relation for \( k + 1 \). Let \( j \in M_{k+1} \). From the definition of \( M_{k+1} \) we deduce that there is a path \((r_0, r_1, \ldots, r_k, j)\) in \( t_2 \), where \( r_0 = \text{root}(t_2) \). The sequence \((r_0, r_1, \ldots, r_k)\) is also a path in \( t_2 \), therefore \( r_k \in M_k \). By the inductive assumption we have
\[
\alpha(\beta(r_k)) = r_k
\] (19)
The node \( r_k \) is a direct predecessor of the node \( j \) and we may suppose that \( j \) is the left direct descendant of \( r_k \). This means that there is a pair
\[
[(r_k, j), (r_k, p)] \in D_2
\] (20)
From (7) we deduce that
\[
[(\beta(r_k), \beta(j)), (\beta(r_k), \beta(p))] \in D_1
\]
Using (6) we can say that
\[
[(\alpha(\beta(r_k)), \alpha(\beta(j))), (\alpha(\beta(r_k)), \alpha(\beta(p)))] \in D_2
\] (21)
Taking into account the relation (19), the condition (21) becomes
\[
[(r_k, \alpha(\beta(j))), (r_k, \alpha(\beta(p)))] \in D_2
\] (22)
If we compare now (20) and (22) then we obtain \( \alpha(\beta(j)) = j \). Thus (18) is true for \( k + 1 \).

We can conclude that (17) is true. This relation shows that \( \beta = \alpha^{-1} \) because \( \alpha \) and \( \beta \) are bijective mappings. It follows that \( \alpha^{-1} \) is an embedding mapping of \( t_2 \) into \( t_1 \) because \( \beta \) was also such a mapping and \( \beta = \alpha^{-1} \).

**Proposition 3.3.** Suppose that \( t_1 = (A_1, D_1, h_1) \in \text{OBT}(\omega) \), \( t_2 = (A_2, D_2, h_2) \in \text{OBT}(\omega) \) and \( t_1 \simeq t_2 \). If \( \alpha_1 : A_1 \longrightarrow A_2 \) and \( \alpha_2 : A_1 \longrightarrow A_2 \) are two embedding mappings of \( t_1 \) into \( t_2 \) then \( \alpha_1 = \alpha_2 \).

**Proof.** From Proposition 3.2 we obtain \( \alpha_1(\text{root}(t_1)) = \text{root}(t_2) \) and \( \alpha_2(\text{root}(t_1)) = \text{root}(t_2) \). It follows that
\[
\alpha_1(\text{root}(t_1)) = \alpha_2(\text{root}(t_1))
\] (23)
As we have done in other cases we denote \( M_0 = \{\text{root}(t_2)\} \) and for \( k \geq 1 \) we denote by \( M_k \) the set of all nodes \( j \in A_1 \) such that there is a path of length \( k \) from \( \text{root}(t_1) \) to \( j \). We prove the following property \( P(k) \) by induction on \( k \geq 0 \):
\[
i \in M_k \Rightarrow \alpha_1(i) = \alpha_2(i)
\] (24)
Using this property and the relation \( A_1 = \bigcup_{k \geq 0} M_k \) we deduce that \( \alpha_1 = \alpha_2 \).

From (23) we see that (24) is true for \( k = 0 \). Suppose this relation is true for \( k \) and we verify (24) for \( k + 1 \). Let us consider an element \( i \in M_{k+1} \). There is a sequence \((\text{root}(t_1), p_1, \ldots, p_k, i) \in \text{Path}(t_1) \). It follows that \( p_k \in M_k \) therefore from the inductive assumption we have
\[
\alpha_1(p_k) = \alpha_2(p_k)
\] (25)
Without loss of generality we can suppose that \( i \) is the left direct descendant of \( p_k \). This means that there is \( j \in A_1 \) such that \([p_k, i], (p_k, j) \in D_1 \). But \( \alpha_1 \) and \( \alpha_2 \) are two embedding mappings of \( t_1 \) into \( t_2 \) therefore
\[
[(\alpha_1(p_k), \alpha_1(i)), (\alpha_1(p_k), \alpha_1(j))] \in D_2
\] (26)
\[
[(\alpha_2(p_k), \alpha_2(i)), (\alpha_2(p_k), \alpha_2(j))] \in D_2
\] (27)
Taking into account (25) from (26) and (27) we obtain \( \alpha_1(i) = \alpha_2(i) \). Thus (24) is true for \( k + 1 \). In conclusion we have \( \alpha_1 = \alpha_2 \).
Suppose that \( t_1 = (A_1, D_1, h_1) \in \text{OBT}(\omega) \), \( t_2 = (A_2, D_2, h_2) \in \text{OBT}(\omega) \) and \( t_1 \simeq t_2 \). There is one and only one embedding mapping \( \alpha \) of \( t_1 \) into \( t_2 \), \( \alpha \) is bijective and \( \alpha^{-1} \) is the unique embedding mapping of \( t_2 \) into \( t_1 \). Moreover, the mapping \( \alpha \) satisfies the following conditions:

\[
\alpha(\text{root}(t_1)) = \text{root}(t_2)
\]

\[
u \in D_1 \iff \overline{\alpha}(u) \in D_2
\]

\[
h_1(\text{root}(t_1)) = h_2(\alpha(\text{root}(t_1)))
\]

**Proof.** Immediate from Proposition 3.2 and Proposition 3.3. \( \square \)

**Proposition 3.4.** Suppose that \( t_1 = (A_1, D_1, h_1) \in \text{OBT}(\omega) \) and \( t_2 = (A_2, D_2, h_2) \in \text{OBT}(\omega) \). If there is a bijective mapping \( \alpha : A_1 \longrightarrow A_2 \) such that the following three conditions are satisfied:

1. The image by \( \alpha \) of the root of \( t_1 \) is the root of \( t_2 \):

\[
\alpha(\text{root}(t_1)) = \text{root}(t_2)
\]

2. For every \( i, i_1, i_2 \in A_1 \) we have:

\[
[(i, i_1), (i, i_2)] \in D_1 \iff [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2
\]

3. The roots of \( t_1 \) and \( t_2 \) have the same label:

\[
h_1(\text{root}(t_1)) = h_2(\alpha(\text{root}(t_1)))
\]

then \( t_1 \simeq t_2 \).

**Proof.** We suppose that there is a bijective mapping \( \alpha : A_1 \longrightarrow A_2 \) such that the conditions (28), (29) and (30) are satisfied. Directly from (29) and (30) we deduce that

\[
t_1 \preceq t_2
\]

We take \( \beta = \alpha^{-1} \). From (28) we deduce

\[
\beta(\text{root}(t_2)) = \text{root}(t_1)
\]

Let us prove the implication

\[
v \in D_2 \Rightarrow \overline{\beta}(v) \in D_1
\]

We suppose that \( v = [(j, j_1), (j, j_2)] \in D_2 \), where \( j, j_1, j_2 \in A_2 \). The mapping \( \alpha \) is a surjective mapping, therefore there are \( i, i_1, i_2 \in A_1 \) such that \( \alpha(i) = j, \alpha(i_1) = j_1 \) and \( \alpha(i_2) = j_2 \). Using these notations, the condition \( v \in D_2 \) can be written equivalently:

\[
[(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2
\]

From (29) we deduce that \( [(i, i_1), (i, i_2)] \in D_1 \). But \( \beta(j) = i, \beta(j_1) = i_1 \) and \( \beta(j_2) = i_2 \) therefore \( [(\beta(j), \beta(j_1)), (\beta(j), \beta(j_2))] \in D_1 \). But \( [(\beta(j), \beta(j_1)), (\beta(j), \beta(j_2))] = \overline{\beta}(v) \) and thus (33) is proved.

Now we prove that the following relation is true:

\[
h_2(\text{root}(t_2)) = h_1(\beta(\text{root}(t_2)))
\]

Using (32) we obtain from (30) the following sequence of computations:

\[
h_1(\beta(\text{root}(t_2))) = h_2(\alpha(\beta(\text{root}(t_2)))) = h_2(\text{root}(t_2))
\]

because \( \beta = \alpha^{-1} \). Thus (34) is true. From (33) and (34) we deduce that

\[
t_2 \preceq t_1
\]

From (31) and (35) we have \( t_1 \simeq t_2 \). \( \square \)
The relation $\preceq$ can be visualized in an intuitive manner as in Figure 1. If we ignore the nodes name and we translate the tree $t_1$ so as we overlap the root of $t_1$ with some node of $t_2$ then the structure of $t_1$ overlaps with some part of $t_2$. We see this image in Figure 2.

In Figure 3 we represented two equivalent trees. If we denote by $t_1 = (A_1, D_1, h_1)$ the left tree and $t_2 = (A_2, D_2, h_2)$ the right tree then these trees are equivalent if we take into consideration the following relations:

$$A_1 = A_2 = \{a, b, c, d, f\}$$
Figure 4. Images by eliminating the node names

\[
\alpha(a) = a, \alpha(b) = b, \alpha(c) = c, \alpha(d) = f, \alpha(f) = d
\]
\[
h_1(a) = m_1, h_1(b) = m_2, h_1(c) = m_3, h_1(d) = m_4, h_1(f) = m_6
\]
\[
h_2(a) = m_1, h_2(b) = m_2, h_2(c) = m_3, h_2(d) = m_6, h_2(f) = m_4
\]

We observe that two trees are equivalent if they have the same structure less nodes name. If we ignore these names then we obtain the same structure as we can view in Figure 4.

4. A partial order on \(OBT(\omega)/\sim\)

In this section we introduce a partial order on the factor set \(OBT(\omega)/\sim\). We recall that an element of the set \(OBT(\omega)/\sim\) is denoted by \([t]\).

**Definition 4.1.** Let us consider \([t_1] \in OBT(\omega)/\sim\) and \([t_2] \in OBT(\omega)/\sim\). We define the relation \([t_1] \preceq [t_2]\) if and only if \(t_1 \preceq t_2\).

**Proposition 4.1.** The relation \(\preceq\) does not depend on representatives.

**Proof.** Let us consider the elements \(t_i = (A_i, D_i, h_i) \in OBT(\omega)\) for \(i \in \{1, 2, 3, 4\}\) such that \(t_1 \preceq t_2, t_3 \preceq t_1\) and \(t_4 \preceq t_2\). We have to prove that \(t_3 \preceq t_4\). Taking the embedding mapping \(\alpha\) of \(t_3\) into \(t_1\), the embedding mapping \(\beta\) of \(t_2\) into \(t_4\) and applying Corollary 3.1 we obtain:

\[
\alpha(\text{root}(t_3)) = \text{root}(t_1)
\]
\[
u \in D_3 \Leftrightarrow \overline{\alpha}(u) \in D_1
\]
\[
h_3(\text{root}(t_3)) = h_1(\alpha(\text{root}(t_3)))
\]
\[
u \in D_2 \Leftrightarrow \overline{\beta}(u) \in D_4
\]

and from Definition 2.1 we deduce that there is a mapping \(\gamma : A_1 \longrightarrow A_2\) such that

\[
u \in D_1 \Rightarrow \overline{\gamma}(u) \in D_2
\]
Let us verify the following properties:

\[ u \in D_3 \Rightarrow \alpha \circ \gamma \circ \beta (u) \in D_4 \]  

(41)

\[ h_3(\text{root}(t_3)) = h_4((\alpha \circ \gamma \circ \beta)(\text{root}(t_3))) \]  

(42)

Relation (41) is obtained if we use (37), (40) and (39):

\[ u \in D_3 \Rightarrow \alpha (u) \in D_1 \Rightarrow \gamma (\alpha (u)) \in D_2 \Rightarrow \beta (\gamma (\alpha (u))) = \alpha \circ \gamma \circ \beta (u) \in D_4 \]

In what concerns the relation (42) we observe that:

- if we use the relations (38) and (36) then we obtain
  \[ h_3(\text{root}(t_3)) = h_1(\alpha (\text{root}(t_3))) = h_1(\text{root}(t_1)) \]

- if we apply property P1 then we have \( h_4(\beta (i)) = h_2(i) \) for every \( i \in A_2 \); hence, we obtain
  \[ h_4(\alpha \circ \gamma \circ \beta (\text{root}(t_3))) = h_4(\beta (\gamma (\alpha (\text{root}(t_3)))))) = h_2(\gamma (\alpha (\text{root}(t_3)))) = h_1(\alpha (\text{root}(t_3))) = h_1(\text{root}(t_1)) \]

from which we obtain (42).

\[ \Box \]

Proposition 4.2. The pair \( (\text{OBT}(\omega)/\sim, \preceq) \) is a partial ordered set.

Proof. Obviously the relation \( \preceq \) is reflexive. The transitivity of \( \preceq \) is obtained from the transitivity of \( \preceq \). If \([t_1] \preceq [t_2]\) and \([t_2] \preceq [t_1]\) then \( t_1 \preceq t_2 \) and \( t_2 \preceq t_1 \). This shows that \([t_1] = [t_2]\). Thus the relation \( \preceq \) is antisymmetric.

5. Conclusions

In this paper the research work initiated in [16] is continued and new results are presented. An equivalence relation is defined on the set \( \text{OBT}(\omega) \) of the \( \omega \)-trees and a partial order on the factor set of equivalence classes is studied. The results presented in this paper allow to introduce the concept of \( \omega \)-templates by means of which we can characterize the formal computations in a master-slave system based on semantic schemas. This aspect is treated in a forthcoming paper.

References


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