Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 37(2), 2010, Pages 7–17
ISSN: 1223-6934

Algebraic properties of ω-trees (II)

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Abstract. In [16] we defined the concept of ω-labeled tree as a binary, ordered and labeled tree with several features concerning the labels and order between the direct descendants of a node. This paper includes several further results concerning these structures. The main results presented in this paper are the following: we introduce an equivalence relation $\simeq$ on the set $OBT(\omega)$ of ω-trees and a partial order on the factor set $OBT(\omega)/_{\simeq}$.

2010 Mathematics Subject Classification. Primary 05C20; Secondary 18A10.
Key words and phrases. directed ordered graph, tree, ordered tree, partial order, embedding mapping.

1. Introduction

Various implications of new algebraic structures into computer science area were established in the recent years. The Peano algebras and graph theory were applied successfully in knowledge representation ([2], [3], [7], [9], [10], [11], [12], [13]). Several properties of pseudo-BCK algebras show their connection with fuzzy structures and the class of pseudo-BCK algebras with pseudo-double negation generalizes some particular structures with applications in mathematical logic ([4]). The labeled ordered trees were implied successfully in theoretical and applied computer science. A Tree Algebra for XML, named TAX, was developed as a natural extension of relational algebra for manipulating XML data, modeled as forests of labeled ordered trees([6]). An algebra for manipulating collections with ordering specifications was developed in [8]. The Peano Count Tree (P-tree) gives a tree representation of spatial data. The algebra and properties of P-tree structure as well as fast algorithms for P-tree generation and P-tree operations are treated in [5].
The concept of ω-tree was introduced in [16]. This structure is a binary tree whose nodes are labeled by means of a mapping ω that specifies the labeling process. There are two kinds of labels: terminal and non-terminal labels. Only the nodes labeled by non-terminal labels may contain direct descendants. On the set $OBT(\omega)$ of ω-trees we introduced a binary relation, which is not a partial order. In the present paper we develop the idea introduced in [16].

This paper is organized as follows: Section 2 contains the basic notions and results obtained in [16]; in Section 3 we define and study an equivalence relation $\simeq$ on the set $OBT(\omega)$; in Section 4 a partial order on the factor set $OBT(\omega)/\simeq$ is defined and studied. The last section contains conclusions and future work.
The purpose of this research is to apply these results to obtain algebraic structures useful to describe the computations in cooperating structures based on semantic schemas ([14], [15]).

Received March 17, 2010. Revision received April 26, 2010.

2. Basic notions and notations

A *directed ordered graph* ([1]) is a pair $G = (A, D)$, where A is a finite set of elements called *nodes*, D is a finite set of elements of the form $[(i, i_1), \ldots, (i, i_n)]$, where $n \geq 1$ and $i, i_1, \ldots, i_n \in A$ and D satisfies the following condition: if $[(i, i_1), \ldots, (i, i_n)] \in D$ and $[(j, j_1), \ldots, (j, j_s)] \in D$ then $i \neq j$. We observe that for an element $[(i, i_1), \ldots, (i, i_n)] \in D$ we may have $i_j = i_k$ for some $j \neq k$. On the other hand, an element of D is a list and the *order* of its elements are taken into consideration. An element of a list is a *directed arc* and simply is named *arc*.
We can represent a directed ordered graph as follows. We represent, as usual, a node of the graph by a point. If $[(i, i_1), \ldots, (i, i_n)] \in D$ then we draw *an arc* from node i to node i_j for every $j \in \{1, \ldots n\}$. The elements $i_1, \ldots, i_n$ are called *the direct descendants* of i. We shall consider that all direct descendants of i are ordered linearly and the order is given by the place of i_j in the element $[(i, i_1), \ldots, (i, i_n)]$.

If $G = (A, D)$ is a directed ordered graph then we can associate to G a *directed graph* $G' = (A, D')$, where

$$D' = \{(i, j) \mid \exists[(i, i_1), \ldots, (i, i_n)] \in D, \exists r \in \{1, \ldots, n\} : j = i_r\}$$

An *ordered tree* is a directed ordered graph $G = (A, D)$ such that D' is a tree and the following property is satisfied:

$$[(i, i_1), \ldots, (i, i_n)] \in D, j, r \in \{1, \ldots, n\}, j \neq r \Rightarrow i_j \neq i_r \tag{1}$$

A *path* in a directed ordered graph is a sequence $d = (n_0, n_1, \ldots, n_k)$ of nodes such that for every $i \in \{0, \ldots, k-1\}$ we have an arc from n_i to n_{i+1}. The number k is the *length* of d. We denote by $Path(G)$ the set of all paths in G.

A *binary tree* is a tree such that every node has exactly zero or two direct descendants. The *root* is a node that is not a direct descendant of any other node. A tree has a single root. Every node that is not the root in the binary tree is reachable from the root node by a unique path. A node with neither a left descendant nor a right descendant is called a *leaf*. By an abuse of language we shall use the concepts of arc and path in an ordered tree t and in this case we suppose that these concepts are applied to the graph associated to t. Moreover, for an ordered tree t we denote by $Path(t)$ the set of all paths of t.

We consider a finite set L and a decomposition $L = L_N \cup L_T$, where $L_N \cap L_T = \emptyset$. The elements of L_N are called *nonterminal labels* and those of L_T are called *terminal labels*. The elements of L are called *labels*. A **split mapping** on L ([16]) is a function $\omega : L_N \longrightarrow L \times L$. For each $x \in L_N$ we denote $\omega(x) = (\omega_1(x), \omega_2(x))$. The entity $\omega_1(x)$ is named the **left component** and $\omega_2(x)$ is the **right component** of $\omega(x)$.

Let $\omega : L_N \longrightarrow L \times L$ be a split mapping on L. An **ω-tree** ([16]) is a tuple $t = (A, D, h)$, where

- (A, D) is an ordered tree and every element of D is of the form $[(i, i_1), (i, i_2)]$;
- $h : A \longrightarrow L$ is a mapping such that

$$[(i, i_1), (i, i_2)] \in D \Rightarrow h(i) \in L_N \ \&\ \omega(h(i)) = (h(i_1), h(i_2)) \tag{2}$$

For each $i \in A$ the element $h(i)$ is called the **label** of the node i. The mapping h is named the **labeling mapping** of t. By $OBT(\omega)$ we denote the set of all ω-trees.

Let $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$ be two elements of $OBT(\omega)$ and an arbitrary mapping $\alpha : A_1 \longrightarrow A_2$. For every $u = [(i, i_1), (i, i_2)] \in D_1$ we denote

$$\overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))]$$

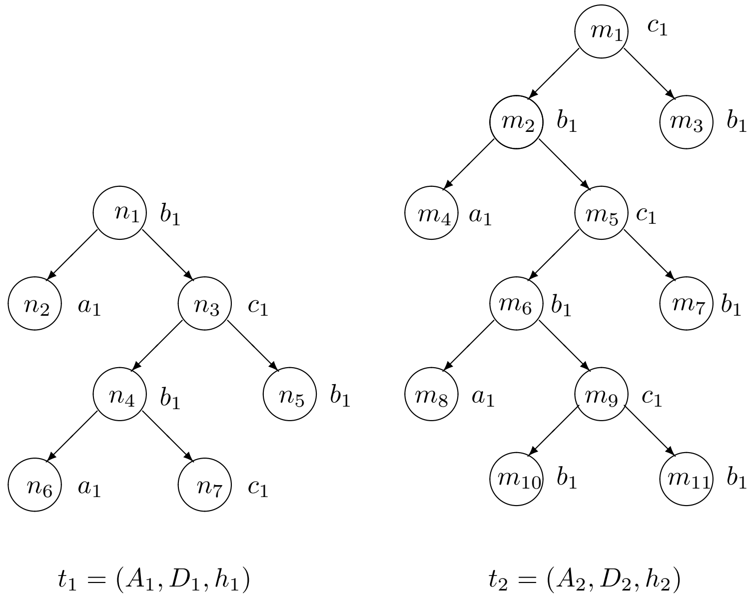

FIGURE 1. Two ω-trees

If $t = (A, D, h)$ is an ω-tree then we denote by $root(t)$ the element of A designated by the root of t.

Definition 2.1. ([16]) *If $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ then we define the relation $t_1 \preceq t_2$ if there is a mapping $\alpha : A_1 \longrightarrow A_2$ such that:*

$$u \in D_1 \Rightarrow \overline{\alpha}(u) \in D_2 \tag{3}$$

$$h_1(root(t_1)) = h_2(\alpha(root(t_1))) \tag{4}$$

*Such a mapping α is an **embedding mapping** of t_1 into t_2.*

Remark 2.1. *In an ω-tree we have the following property: if a node n has direct descendants then the label $h(n)$ of n is an element of L_N. Moreover, in this case the left (right) descendant of n is labeled by the left (right) component of $\omega(h(n))$. A leaf of an ω-labeled tree may be labeled by an element of L_N, but a node labeled by an element of L_T is a leaf.*

In order to exemplify these concepts we consider the following case:
- $L = \{a_i, b_i, c_i\}_{i \geq 1}$, $L_N = \{b_i, c_i\}_{i \geq 1}$, $L_T = \{a_i\}_{i \geq 1}$
- $\omega(b_i) = (a_i, c_i)$ and $\omega(c_i) = (b_i, b_i)$ for $i \geq 1$

In Figure 1 we represented the following two ω-trees $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$, where:
- $A_1 = \{n_j\}_{j=1,\ldots,7}$; $A_2 = \{m_j\}_{j=1,\ldots,11}$
- $h_1(n_i) = b_1$ for $i \in \{1, 4, 5\}$; $h_1(n_2) = h(n_6) = a_1$; $h_1(n_3) = h_1(n_7) = c_1$;
- $h_2(m_4) = h_2(m_8) = a_1$; $h_2(m_i) = b_1$ for $i \in \{2, 3, 6, 7, 10, 11\}$; $h_2(m_1) = h_2(m_5) = h_2(m_9) = c_1$;
- $D_1 = \{[(n_1, n_2), (n_1, n_3)], [(n_3, n_4), (n_3, n_5)], [(n_4, n_6), (n_4, n_7)]\}$

- $D_2 = \{[(m_1, m_2), (m_1, m_3)], [(m_2, m_4), (m_2, m_5)], [(m_5, m_6), (m_5, m_7)],$
$[(m_6, m_8), (m_6, m_9)], [(m_9, m_{10}), (m_9, m_{11})]\}$

Remark 2.2. *We observe the following property of t_1 and t_2 from Figure 1: if we make abstraction of node names and translate t_1 such that n_1 overlaps m_2 then t_1 becomes a part of t_2. This part can be viewed as an "image" of t_1 into t_2. We see that the image of t_1 is not a subtree of t_2 because the nodes m_{10} and m_{11} do not belong to this image. Obviously a tree t_1 can have multiple images into t_2.*

The following properties are proved in [16]:
P1: Let be $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. If $t_1 \preceq t_2$ then $h_1(i) = h_2(\alpha(i))$ for every $i \in A_1$, where α is an embedding mapping.
P2: Let us denote by $\alpha : A_1 \longrightarrow A_2$ an embedding mapping of t_1 into t_2. If (m, n) is an arc in t_1 then $(\alpha(m), \alpha(n))$ is an arc in t_2. If $d = (n_0, n_1, \ldots, n_k) \in Path(t_1)$ then $\alpha(d) = (\alpha(n_0), \alpha(n_1), \ldots, \alpha(n_k)) \in Path(t_2)$.
P3: An embedding mapping is injective.
P4: The relation $\preceq$ is reflexive and transitive, but is not antisymmetric.

3. An equivalence relation on $OBT(\omega)$

Based on the relation $\preceq$ we introduce an equivalence relation on the set $OBT(\omega)$. In this section we study this relation.

Definition 3.1. *We consider $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. We define $t_1 \simeq t_2$ if $t_1 \preceq t_2$ and $t_2 \preceq t_1$.*

Proposition 3.1. *The relation $\simeq$ is an equivalence relation.*

Proof. The reflexivity and transitivity of the relation $\simeq$ are obtained from property P4. If $t_1 \simeq t_2$ then obviously $t_2 \simeq t_1$ and thus $\simeq$ is a symmetric relation. □

Remark 3.1.
- *If $t \in OBT(\omega)$ then we denote by $[t]$ the equivalence class of t with respect to $\simeq$.*
- *For a finite set A we denote by $Card(A)$ the cardinal number of A.*

Proposition 3.2. *If $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ and $t_1 \simeq t_2$ then every embedding mapping $\alpha : A_1 \longrightarrow A_2$ of t_1 into t_2 satisfies the following conditions:*
(1) *α is a bijective mapping and α^{-1} is an embedding mapping of t_2 into t_1;*
(2) *The image by α of the root of t_1 is the root of t_2:*

$$\alpha(root(t_1)) = root(t_2) \tag{5}$$

Proof. We suppose that $t_1 \simeq t_2$. We consider an embedding mapping $\alpha : A_1 \longrightarrow A_2$ of t_1 into t_2 and an embedding mapping $\beta : A_2 \longrightarrow A_1$ of t_2 into t_1. From property P3 we deduce that $\alpha : A_1 \longrightarrow A_2$ and $\beta : A_2 \longrightarrow A_1$ are injective mappings. Moreover, the following conditions are satisfied:

$$u \in D_1 \Rightarrow \overline{\alpha}(u) \in D_2 \tag{6}$$

$$u \in D_2 \Rightarrow \overline{\beta}(u) \in D_1 \tag{7}$$

We have $Card(A_1) \leq Card(A_2)$ because $\alpha : A_1 \longrightarrow A_2$ is an injective mapping. Similarly by means of the mapping β we deduce that $Card(A_2) \leq Card(A_1)$. It

follows that $Card(A_1) = Card(A_2)$ and thus α and β are bijective mappings.
Let us prove first that the following properties are fulfilled:

$$\alpha(root(t_1)) = root(t_2), \beta(root(t_2)) = root(t_1), \alpha(\beta(root(t_2))) = root(t_2) \tag{8}$$

Suppose that

$$\beta(root(t_2)) \neq root(t_1) \tag{9}$$

We denote $\beta(root(t_2)) = i$ and suppose that $[(root(t_2), j_1), (root(t_2), j_2)] \in D_2$. If we denote $\beta(j_1) = i_1$ and $\beta(j_2) = i_2$ then by (7) we have

$$[(i, i_1), (i, i_2)] \in D_1 \tag{10}$$

There is a direct predecessor and only one r of i in t_1 because $i \neq root(t_1)$. Without loss of generality we can suppose that i is a right direct descendant of r. In other words we find an element

$$[(r, p), (r, i)] \in D_1 \tag{11}$$

But β is a surjective mapping, therefore there is $q \in A_2$ such that $\beta(q) = r$. We have $q \neq root(t_2)$. Really, if we suppose $q = root(t_2)$ then $r = \beta(q) = \beta(root(t_2)) = i$, which is not true by (11). We have $q \neq j_1$ and $q \neq j_2$. Really, if $q = j_1$ then $r = \beta(q) = \beta(j_1) = i_1$. From (11) and (10) we deduce now that (r, i, i_1) is a circuit in the tree t_1, which is not true because a tree does not contain any circuit. In a similar manner we deduce that $q \neq j_2$. It follows that there is $(root(t_2), j_2, \ldots, j_k, q) \in Path(t_2)$ or $(root(t_2), j_1, \ldots, j_k, q) \in Path(t_2)$. Without loss of generality we can suppose that $(root(t_2), j_2, \ldots, j_k, q) \in Path(t_2)$. We have in this case $(j_2, \ldots, j_k, q) \in Path(t_2)$ and by property P2 we deduce that $(\beta(j_2), \ldots, \beta(j_k), \beta(q)) \in Path(t_1)$. But $\beta(j_2) = i_2$ and $\beta(q) = r$ therefore

$$(i_2, \ldots, \beta(j_k), r) \in Path(t_1) \tag{12}$$

On the other hand from (11) and (10) we deduce that

$$(r, i, i_2) \in Path(t_1) \tag{13}$$

From (12) and (13) we deduce that $(i_2, \ldots, \beta(j_k), r, i, i_2) \in Path(t_1)$ and this property is not true because t_1 is a tree.
It follows that the assumption (9) is not true, therefore

$$\beta(root(t_2)) = root(t_1) \tag{14}$$

By symmetry we have also

$$\alpha(root(t_1)) = root(t_2) \tag{15}$$

therefore (5) is true. Now from (14) and (15) we obtain

$$\alpha(\beta(toot(t_2))) = root(t_2) \tag{16}$$

and (8) is proved.
Let us prove that (16) can be extended to all nodes of t_2:

$$\alpha(\beta(j)) = j \text{ for every } j \in A_2 \tag{17}$$

In order to do this we denote $M_0 = \{root(t_2)\}$ and for $k \geq 1$ we denote by M_k the set of all nodes $j \in A_2$ such that there is a path of length k from $root(t_2)$ to j. We prove the following property for every $k \geq 0$:

$$j \in M_k \Rightarrow \alpha(\beta(j)) = j \tag{18}$$

From this property and the relation $A_2 = \bigcup_{k \geq 0} M_k$ we deduce (17).
For $k = 0$ we have to verify that $\alpha(\beta(root(t_2))) = root(t_2)$ because M_0 contains only one element, namely $root(t_2)$. But this property is true in virtue of (16), therefore (18) is true for $k = 0$.

We suppose that (18) is true for some k and we verify the same relation for $k+1$. Let $j \in M_{k+1}$. From the definition of M_{k+1} we deduce that there is a path $(r_0, r_1, \ldots, r_k, j)$ in t_2, where $r_0 = root(t_2)$. The sequence $(r_0, r_1, \ldots, r_k)$ is also a path in t_2, therefore $r_k \in M_k$. By the inductive assumption we have

$$\alpha(\beta(r_k)) = r_k \tag{19}$$

The node r_k is a direct predecessor of the node j and we may suppose that j is the left direct descendant of r_k. This means that there is a pair

$$[(r_k, j), (r_k, p)] \in D_2 \tag{20}$$

From (7) we deduce that

$$[(\beta(r_k), \beta(j)), (\beta(r_k), \beta(p))] \in D_1$$

Using (6) we can say that

$$[(\alpha(\beta(r_k)), \alpha(\beta(j))), (\alpha(\beta(r_k)), \alpha(\beta(p)))] \in D_2 \tag{21}$$

Taking into account the relation (19), the condition (21) becomes

$$[(r_k, \alpha(\beta(j))), (r_k, \alpha(\beta(p)))] \in D_2 \tag{22}$$

If we compare now (20) and (22) then we obtain $\alpha(\beta(j)) = j$. Thus (18) is true for $k+1$.
We can conclude that (17) is true. This relation shows that $\beta = \alpha^{-1}$ because α and β are bijective mappings. It follows that α^{-1} is an embedding mapping of t_2 into t_1 because β was also such a mapping and $\beta = \alpha^{-1}$. □

Proposition 3.3. *Suppose that $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ and $t_1 \simeq t_2$. If $\alpha_1 : A_1 \longrightarrow A_2$ and $\alpha_2 : A_1 \longrightarrow A_2$ are two embedding mappings of t_1 into t_2 then $\alpha_1 = \alpha_2$.*

Proof. From Proposition 3.2 we obtain $\alpha_1(root(t_1)) = root(t_2)$ and $\alpha_2(root(t_1)) = root(t_2)$. It follows that

$$\alpha_1(root(t_1)) = \alpha_2(root(t_1)) \tag{23}$$

As we have done in other cases we denote $M_0 = \{root(t_2)\}$ and for $k \geq 1$ we denote by M_k the set of all nodes $j \in A_1$ such that there is a path of length k from $root(t_1)$ to j. We prove the following property $P(k)$ by induction on $k \geq 0$:

$$i \in M_k \Rightarrow \alpha_1(i) = \alpha_2(i) \tag{24}$$

Using this property and the relation $A_1 = \bigcup_{k \geq 0} M_k$ we deduce that $\alpha_1 = \alpha_2$.
From (23) we see that (24) is true for $k = 0$. Suppose this relation is true for k and we verify (24) for $k+1$. Let us consider an element $i \in M_{k+1}$. There is a sequence $(root(t_1), p_1, \ldots, p_k, i) \in Path(t_1)$. It follows that $p_k \in M_k$ therefore from the inductive assumption we have

$$\alpha_1(p_k) = \alpha_2(p_k) \tag{25}$$

Without loss of generality we can suppose that i is the left direct descendant of p_k. This means that there is $j \in A_1$ such that $[(p_k, i), (p_k, j)] \in D_1$. But α_1 and α_2 are two embedding mappings of t_1 into t_2 therefore

$$[(\alpha_1(p_k), \alpha_1(i)), (\alpha_1(p_k), \alpha_1(j))] \in D_2 \tag{26}$$

$$[(\alpha_2(p_k), \alpha_2(i)), (\alpha_2(p_k), \alpha_2(j))] \in D_2 \tag{27}$$

Taking into account (25) from (26) and (27) we obtain $\alpha_1(i) = \alpha_2(i)$. Thus (24) is true for $k+1$. In conclusion we have $\alpha_1 = \alpha_2$. □

Corollary 3.1. *Suppose that $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ and $t_1 \simeq t_2$. There is one and only one embedding mapping α of t_1 into t_2, α is bijective and α^{-1} is the unique embedding mapping of t_2 into t_1. Moreover, the mapping α satisfies the following conditions:*

$$\alpha(root(t_1)) = root(t_2)$$

$$u \in D_1 \Leftrightarrow \overline{\alpha}(u) \in D_2$$

$$h_1(root(t_1)) = h_2(\alpha(root(t_1)))$$

Proof. Immediate from Proposition 3.2 and Proposition 3.3. □

Proposition 3.4. *Suppose that $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. If there is a bijective mapping $\alpha : A_1 \longrightarrow A_2$ such that the following three conditions are satisfied:*

(1) *The image by α of the root of t_1 is the root of t_2:*

$$\alpha(root(t_1)) = root(t_2) \tag{28}$$

(2) *For every $i, i_1, i_2 \in A_1$ we have*

$$[(i, i_1), (i, i_2)] \in D_1 \Leftrightarrow [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2 \tag{29}$$

(3) *The roots of t_1 and t_2 have the same label:*

$$h_1(root(t_1)) = h_2(\alpha(root(t_1))) \tag{30}$$

then $t_1 \simeq t_2$.

Proof. We suppose that there is a bijective mapping $\alpha : A_1 \longrightarrow A_2$ such that the conditions (28), (29) and (30) are satisfied. Directly from (29) and (30) we deduce that

$$t_1 \preceq t_2 \tag{31}$$

We take $\beta = \alpha^{-1}$. From (28) we deduce

$$\beta(root(t_2)) = root(t_1) \tag{32}$$

Let us prove the implication

$$v \in D_2 \Rightarrow \overline{\beta}(v) \in D_1 \tag{33}$$

We suppose that $v = [(j, j_1), (j, j_2)] \in D_2$, where $j, j_1, j_2 \in A_2$. The mapping α is a surjective mapping, therefore there are $i, i_1, i_2 \in A_1$ such that $\alpha(i) = j$, $\alpha(i_1) = j_1$ and $\alpha(i_2) = j_2$. Using these notations, the condition $v \in D_2$ can be written equivalently $[(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2$. From (29) we deduce that $[(i, i_1), (i, i_2)] \in D_1$. But $\beta(j) = i$, $\beta(j_1) = i_1$ and $\beta(j_2) = i_2$, therefore $[(\beta(j), \beta(j_1)), (\beta(j), \beta(j_2))] \in D_1$. But $[(\beta(j), \beta(j_1)), (\beta(j), \beta(j_2))] = \overline{\beta}(v)$ and thus (33) is proved.
Now we prove that the following relation is true:

$$h_2(root(t_2)) = h_1(\beta(root(t_2))) \tag{34}$$

Using (32) we obtain from (30) the following sequence of computations:

$$h_1(\beta(root(t_2))) = h_2(\alpha(\beta(root(t_2)))) = h_2(root(t_2))$$

because $\beta = \alpha^{-1}$. Thus (34) is true. From (33) and (34) we deduce that

$$t_2 \preceq t_1 \tag{35}$$

From (31) and (35) we have $t_1 \simeq t_2$. □

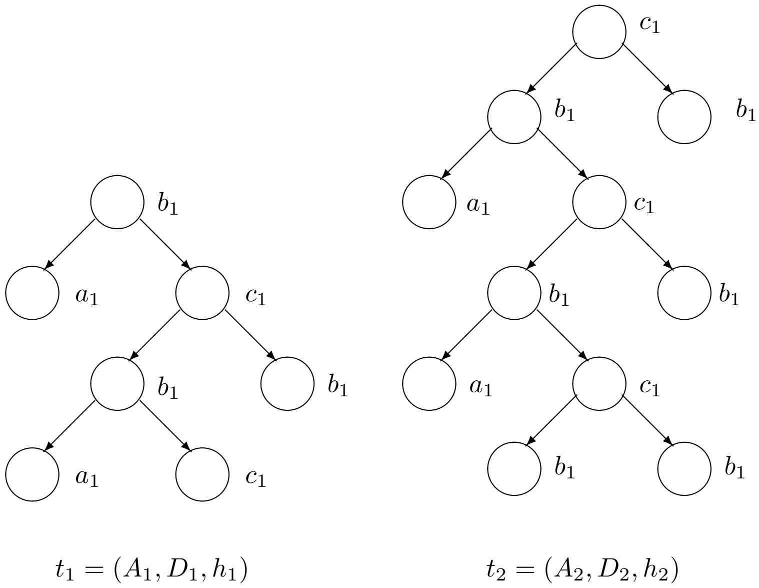

FIGURE 2. The intuitive visualization of $t_1 \preceq t_2$

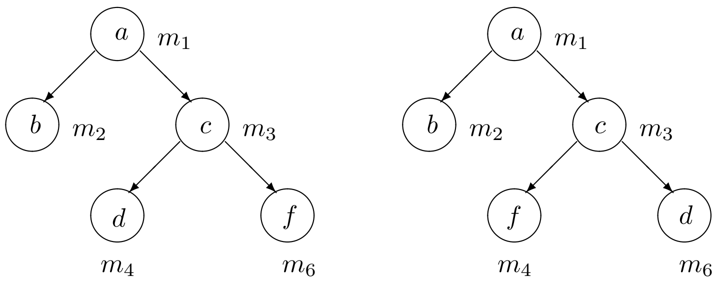

FIGURE 3. Equivalent ordered trees

The relation $\preceq$ can be visualized in an intuitive manner as in Figure 1. If we ignore the nodes name and we translate the tree t_1 so as we overlap the root of t_1 with some node of t_2 then the structure of t_1 overlaps with some part of t_2. We see this image in Figure 2.

In Figure 3 we represented two equivalent trees. If we denote by $t_1 = (A_1, D_1, h_1)$ the left tree and $t_2 = (A_2, D_2, h_2)$ the right tree then these trees are equivalent if we take into consideration the following relations:

$A_1 = A_2 = \{a, b, c, d, f\}$

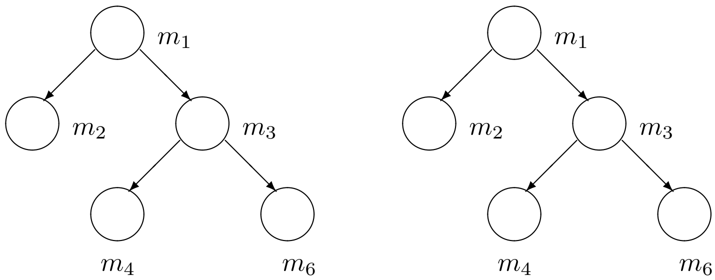

FIGURE 4. Images by eliminating the node names

$\alpha(a) = a, \alpha(b) = b, \alpha(c) = c, \alpha(d) = f, \alpha(f) = d$
$h_1(a) = m_1, h_1(b) = m_2, h_1(c) = m_3, h_1(d) = m_4, h_1(f) = m_6$
$h_2(a) = m_1, h_2(b) = m_2, h_2(c) = m_3, h_2(d) = m_6, h_2(f) = m_4$
We observe that two trees are equivalent if they have the same structure less nodes name. If we ignore these names then we obtain the same structure as we can view in Figure 4.

4. A partial order on $OBT(\omega)/\simeq$

In this section we introduce a partial order on the factor set $OBT(\omega)/_{\simeq}$. We recall that an element of the set $OBT(\omega)/_{\simeq}$ is denoted by $[t]$.

Definition 4.1. *Let us consider $[t_1] \in OBT(\omega)/_{\simeq}$ and $[t_2] \in OBT(\omega)/_{\simeq}$. We define the relation $[t_1] \lesssim [t_2]$ if and only if $t_1 \preceq t_2$.*

Proposition 4.1. *The relation $\lesssim$ does not depend on representatives.*

Proof. Let us consider the elements $t_i = (A_i, D_i, h_i) \in OBT(\omega)$ for $i \in \{1, 2, 3, 4\}$ such that $t_1 \preceq t_2$, $t_3 \simeq t_1$ and $t_4 \simeq t_2$. We have to prove that $t_3 \preceq t_4$. Taking the embedding mapping α of t_3 into t_1, the embedding mapping β of t_2 into t_4 and applying Corollary 3.1 we obtain:

$$\alpha(root(t_3)) = root(t_1) \tag{36}$$

$$u \in D_3 \Leftrightarrow \overline{\alpha}(u) \in D_1 \tag{37}$$

$$h_3(root(t_3)) = h_1(\alpha(root(t_3))) \tag{38}$$

$$u \in D_2 \Leftrightarrow \overline{\beta}(u) \in D_4 \tag{39}$$

and from Definition 2.1 we deduce that there is a mapping $\gamma : A_1 \longrightarrow A_2$ such that

$$u \in D_1 \Rightarrow \overline{\gamma}(u) \in D_2 \tag{40}$$

Let us verify the following properties:

$$u \in D_3 \Rightarrow \overline{\alpha \circ \gamma \circ \beta}(u) \in D_4 \tag{41}$$

$$h_3(root(t_3)) = h_4((\alpha \circ \gamma \circ \beta)(root(t_3))) \tag{42}$$

Relation (41) is obtained if we use (37), (40) and (39):

$$u \in D_3 \Rightarrow \overline{\alpha}(u) \in D_1 \Rightarrow \overline{\gamma}(\overline{\alpha}(u)) \in D_2 \Rightarrow \overline{\beta}(\overline{\gamma}(\overline{\alpha}(u))) = \overline{\alpha \circ \gamma \circ \beta}(u) \in D_4$$

In what concerns the relation (42) we observe that:

- if we use the relations (38) and (36) then we obtain

$$h_3(root(t_3)) = h_1(\alpha(root(t_3))) = h_1(root(t_1))$$

- if we apply property P1 then we have $h_4(\beta(i)) = h_2(i)$ for every $i \in A_2$, $h_1(i) = h_2(\gamma(i))$ for every $i \in A_1$; as a consequence we obtain

$$h_4(\alpha \circ \gamma \circ \beta(root(t_3))) = h_4(\beta(\gamma(\alpha(root(t_3))))) = h_2(\gamma(\alpha(root(t_3)))) =$$
$$h_1(\alpha(root(t_3))) = h_1(root(t_1))$$

from which we obtain (42). □

Proposition 4.2. *The pair* $(OBT(\omega)/_{\simeq}, \lesssim)$ *is a partial ordered set.*

Proof. Obviously the relation $\lesssim$ is reflexive. The transitivity of $\lesssim$ is obtained from the transitivity of $\preceq$. If $[t_1] \lesssim [t_2]$ and $[t_2] \lesssim [t_1]$ then $t_1 \preceq t_2$ and $t_2 \preceq t_1$. This shows that $[t_1] = [t_2]$. Thus the relation $\lesssim$ is antisymmetric. □

5. Conclusions

In this paper the research work initiated in [16] is continued and new results are presented. An equivalence relation is defined on the set $OBT(\omega)$ of the ω-trees and a partial order on the factor set of equivalence classes is studied. The results presented in this paper allow to introduce the concept of ω-templates by means of which we can characterize the formal computations in a master-slave system based on semantic schemas. This aspect is treated in a forthcoming paper.

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