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Folding theory applied to *Rl-monoid*

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Abstract. In this paper we define n-fold (positive) implicative *Rl-monoid.* Also we introduce n-fold (positive) implicative filter in *Rl-monoid* and we prove some relations between these filters and construct quotient algebras via these filters.

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1. Introduction

BL-algebras have been introduced by P. Hajek as an algebraic counterpart of the basic fuzzy logic BL [2]. Omitting the requirement of pre-linearity in the definition of a BL-algebra, one obtains the definition of a bounded commutative residuated lattice ordered monoid (*Rl-monoid*). Nevertheless, bounded commutative *Rl-monoid* are a generalization not only of BL-algebra but also of Heyting algebras which are an algebraic counterpart of the intuitionistic propositional logic.Theorefore, bounded commutative *Rl-monoid* could be taken as an algebraic semantics of a more general logic than Hajek's fuzzy logic. In both BL-algebra and bounded commutative *Rl-monoid*, filters coincide with deductive systems of those algebras and are exactly the kernel of their congruences. Various types of filters of BL-algebras were studied in [3]. In this paper we further develop the theory of filters of bounded commutative *Rl-monoids* and among others, we generalize some results of [4].

2. Preliminiaries

Definition 2.1. [4] *A bounded commutative Rl-monoid is an algebra* $A = (A, \wedge, \vee, *, \rightarrow 0, 1)$ *with four binary operations* $\wedge$, $\vee$, $*$, $\rightarrow$ *and two constant 0,1 such that:*

(Rl1) $A = (A, \vee, \wedge, 0, 1)$ *is a bounded lattice,*

(Rl2) $A = (A, *, 1)$ *is a commutative monoid,*

(Rl3) $*$ *and* $\rightarrow$ *form a adjoint pair, i.e,*$a * c \leq b$ *if and only if* $c \leq a \rightarrow b$*, for all* $a, b, c \in A$*,*

(Rl4) $a \wedge b = a * (a \rightarrow b)$*, for all* $a, b \in A$*.*

In the sequel, by a *Rl-monoid* we will mean a bounded commutative *Rl-monoid*. Bounded commutative *Rl-monoids* are special cases of residuated lattices, more precisely (see for instance [1]).

An *Rl-monoids* A is a BL-algebra iff A satisfies the identity of pre-linearly $(x \rightarrow y) \vee (y \rightarrow x) = 1$ an MV-algebra iff A fulfills the doube negation $(x^-)^- = x$ where $x^- = x \rightarrow 0$ a Heyting algebra iff the operation $*$ is idempotent.

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Lemma 2.1. [4, 5] *In any Rl-monoid A, the following relations hold for all* $x, y, z \in A$:

(1) $x * (x \to y) \le y$,
(2)$x \le (y \to (x * y))$,
(3)$x \le y$ *if and only if* $x \to y = 1$,
(4)$x \to (y \to z) = y \to (x \to z)$,
(5)*If* $x \le y$, *then* $y \to z \le x \to z$ *and* $z \to x \le z \to y$,
(6)$y \le (y \to x) \to x$,
(7)$x \to y \le (y \to z) \to (x \to z)$,
(8)$x \to y \le (z \to x) \to (z \to y)$,
(9) $(x * y) \to z = x \to (y \to z)$,
(10) $1 \to x = x$,$x \to x = 1$

Definition 2.2. [1, 4] *A nonempty subset F of Rl-monoid A is called a filter of A if:*

(1) $a * b \in F$, *for all* $a, b \in F$,
(2) $a \le b$ *and* $a \in F$ *imply* $b \in F$.

Definition 2.3. [1, 4] *A nonempty subset D of Rl-monoid A is called a deductive system of A if:*

(1) $1 \in D$,
(2) *If* $x \in D$ *and* $x \to y \in D$, *then* $y \in D$.

Proposition 2.1. [1] *A nonempty subset F of Rl-monoid A is a deductive system if and only if is a filter of Rl-monoid A.*

By [6], filters of commutative *Rl-monoid* are exactly the kernels of their congruences. If F is a filter of A, then F is the kernel of the unique congruence $\theta(F)$ such that $(x, y) \in \theta(F)$ iff $(x \to y) \wedge (y \to x) \in F$ for any $x, y \in A$. Hence we will consider quotient *Rl-monoid* A/F of *Rl-monoid* A by their filters.

Definition 2.4. [4] *A non-empty subset F of Rl-monoid A is called an implicative filter of A if it satisfies:*

(1) $1 \in F$,
(2) $x \to (y \to z) \in F$ *and* $x \to y \in F$ *imply* $x \to z \in F$, *for all* $x, y, z \in A$

Theorem 2.1. [4] *Let F be a filter of Rl-monoid A. Then F is an implicative filter if and only if A/F is a Heyting algebra.*

Definition 2.5. [4] *A non-empty subset F of Rl-monoid A is called an positive implicative filter of A if it satisfies:*

(1) $1 \in F$,
(2) $x \to ((y \to z) \to y) \in F$ *and* $x \in F$ *imply* $y \in F$, *for all* $x, y, z \in A$.

Theorem 2.2. [4] *Let F be a filter of Rl-monoid A. Then F is a positive implicative filter if and only if* $(x \to y) \to x \in F$ *implies* $x \in F$, *for all* $x, y \in A$.

Theorem 2.3. [4] *In any Rl-monoid A, the following conditions are equivalent:*

(a) $\{1\}$ *is a positive implicative filter,*
(b) *Every filter of A is a positive implicative filter,*
(c) $A(a) = \{x \in A \mid x \ge a\}$ *is a positive implicative filter,*
(d) $(x \to y) \to x = x$ *for all* $x, y \in A$,
(e) *A is Boolean algebra.*

3. n-fold implicative Rl-monoid

Definition 3.1. *An n-fold implicative Rl-monoid is a Rl-monoid* $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ *if it satisfies:* $x^{n+1} = x^n$*, for all* $x \in A$ *where* $x^n = x * x * ... * x$*(n-times).*

Theorem 3.1. *If A is an n-fold implicative Rl-monoid then A is an* $(n+1)$*-fold implicative Rl-monoid.*

Proof. Since A is n-fold implicative *Rl-monoid*, $x^{n+1} = x^n$, for all $x \in A$. But $x^{n+2} = x^{n+1} * x = x^n * x = x^{n+1}$ and hence A is $(n+1)$-fold implicative *Rl-monoid.* □

By the following example we show that the converse is not true:

Example 3.1. *Let* $B = \{0, a, b, 1\}$*. Define* $*$ *and* $\rightarrow$ *as follows:*

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Then $(B, \wedge, \vee, *, \rightarrow, 0, 1)$ *is a Rl-monoid and it is clear that B is a 3-implicative Rl-monoid but since* $b^3 \neq b^2$*, B is not 2-fold implicative Rl-monoid.*

4. n-fold implicative filters

Definition 4.1. *A non-empty subset F of a Rl-monoid A is called an n-fold implicative filter of A if it satisfies:*
(1) $1 \in F$
(2) $x^n \rightarrow (y \rightarrow z) \in F$*,* $x^n \rightarrow y \in F$ *imply* $x^n \rightarrow z \in F$*, for all* $x, y, z \in A$

Theorem 4.1. *Any n-fold implicative filter of A is a filter of A.*

Proof. Let $x, x \rightarrow y \in F$. Hence $1 \rightarrow x \in F$ and $1 \rightarrow (x \rightarrow y) \in F$. But $1 = 1^n$, thus $y = 1 \rightarrow y \in F$, that is, F is a filter of A. □

The following example shows that the converse of theorem 4.1 is not true.

Example 4.1. *Let* $B = \{0, a, b, 1\}$*. Define* $*$ *and* $\rightarrow$ *as follows:*

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(B, \wedge, \vee, *, \rightarrow, 0, 1)$ is a Rl-monoid and it is clear that $F = \{b, 1\}$ is a filter, while it is not a 1-implicative filter since $a \rightarrow (a \rightarrow 0) \in F$ and $a \rightarrow a \in F$ but $a \rightarrow 0 \notin F$.

Theorem 4.2. *For any $a \in A$, $A(a) = \{x \in A | a \leq x\}$ is a filter if and only if $a \leq x$ wherever $a \leq y \rightarrow x$ and $a \leq y$ for all $x, y \in A$.*

Proof. Let $A(a)$ be a filter and $a \leq y \rightarrow x$ and $a \leq y$ then $y \rightarrow x \in A(a)$ and $y \in A(a)$. Since $A(a)$ is a filter, $x \in A(a)$, that is, $a \leq x$. Conversely, since $a \leq 1$, $1 \in A(a)$. If $x, x \rightarrow y \in A(a)$, then $a \leq x$ and $a \leq y \rightarrow x$. By assumption $a \leq y$ and hence $y \in A(a)$. Hence $A(a)$ is a filter. □

Theorem 4.3. *Let a be an element of A. If $A(a)$ is an n-fold implicative filter of A, then for all $x, y \in A$, $a^{n+1} \rightarrow (x \rightarrow y) = 1$, $a^{n+1} \rightarrow x = 1$ imply $a^{n+1} \rightarrow y = 1$.*

Proof. Let $A(a)$ be an n-fold implicative filter and $a^{n+1} \rightarrow (x \rightarrow y) = 1$, $a^{n+1} \rightarrow x = 1$. Since $a \rightarrow (a^n \rightarrow (x \rightarrow y)) = a^{n+1} \rightarrow (x \rightarrow y) = 1$, $a^n \rightarrow (x \rightarrow y) \in A(a)$. Similarly $a^n \rightarrow x \in A(a)$. Since $A(a)$ is n-fold implicative filter, $a^n \rightarrow y \in A(a)$, that is, $a \leq a^n \rightarrow y$. Thus $a^{n+1} \rightarrow y = 1$. □

Theorem 4.4. *Let F be a filter of A. Then for all the following conditions are equivalent:*

(a) F is an n-fold implicative filter of A,

(b) $x^n \rightarrow x^{2n} \in F$, for all $x \in A$,

(c) $x^{n+1} \rightarrow y \in F$ implies $x^n \rightarrow y \in F$,

(d) $x^n \rightarrow (y \rightarrow z) \in F$ implies $(x^n \rightarrow y) \rightarrow (x^n \rightarrow z) \in F$.

Proof. (a $\Rightarrow$ b): Let $x \in A$ hence by lemma 2.1, $x^n \rightarrow (x^n \rightarrow x^{2n}) = x^{2n} \rightarrow x^{2n} = 1 \in F$ and $x^n \rightarrow x^n = 1 \in F$. Since F is n-fold implicative filter, we get $x^n \rightarrow x^{2n} \in F$.

(b $\Rightarrow$ a): Let $x, y, z \in A$ be such that $x^n \rightarrow (y \rightarrow z) \in F$, $x^n \rightarrow y \in F$. Since $(x^n \rightarrow (y \rightarrow z)) * (x^n \rightarrow y) * x^n * x^n = (x^n \wedge (y \rightarrow z)) * (x^n \wedge y) \leq (y \rightarrow z) * y = y \wedge z \leq z$ then $(x^n \rightarrow (y \rightarrow z)) * (x^n \rightarrow y) \leq x^{2n} \rightarrow z$. Since $x^n \rightarrow (y \rightarrow z) \in F$, $x^n \rightarrow y \in F$ we get $(x^n \rightarrow (y \rightarrow z)) * (x^n \rightarrow y) \in F$ and so $x^{2n} \rightarrow z \in F$. By lemma 2.1 $x^n \rightarrow x^{2n} \leq (x^{2n} \rightarrow z) \rightarrow (x^n \rightarrow z)$. On the other hand $x^{2n} \rightarrow z \in F$ and $x^n \rightarrow x^{2n} \in F$, then $x^n \rightarrow z \in F$. Hence F is an n-fold implicative filter of A.

(b $\Rightarrow$ c): Since (b) holds, F is an n-fold implicative filter of A. On the other hand $x^{n+1} \rightarrow y = x^n \rightarrow (x \rightarrow y) \in F$ and $x^n \rightarrow x = 1 \in F$ hence $x^n \rightarrow y \in F$.

(c $\Rightarrow$ b): We have $x^{n+1} \rightarrow (x^{n-1} \rightarrow x^{2n}) = x^{2n} \rightarrow x^{2n} = 1 \in F$ hence by (c) $x^n \rightarrow (x^{n-1} \rightarrow x^{2n}) \in F$. But $x^{n+1} \rightarrow (x^{n-2} \rightarrow x^{2n}) = x^{n-2} \rightarrow x^{2n} = x^n \rightarrow (x^{n-1} \rightarrow x^{2n}) \in F$, that is, $x^{n+1} \rightarrow (x^{n-2} \rightarrow x^{2n}) \in F$ and so $x^n \rightarrow (x^{n-2} \rightarrow x^{2n}) \in F$. By repeating the process n times we get $x^n \rightarrow (x^0 \rightarrow x) = x^n \rightarrow (1 \rightarrow x) = x^n \rightarrow x^{2n} \in F$.

(b $\Rightarrow$ d): Let $x^n \rightarrow (y \rightarrow z) \in F$, by lemma 2.1, $x^n \rightarrow (y \rightarrow z) \leq x^n \rightarrow ((x^n \rightarrow y) \rightarrow (x^n \rightarrow z)) = x^n \rightarrow (x^n \rightarrow ((x^n \rightarrow y) \rightarrow z) = x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z)$.

Hence $x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z) \in F$. By (b), we have $x^n \rightarrow x^{2n} \in F$, now by lemma 2.1, $x^{2n} \rightarrow ((x^n \rightarrow y) \rightarrow z) \leq (x^n \rightarrow x^{2n}) \rightarrow (x^n \rightarrow ((x^n \rightarrow y) \rightarrow z))$. Then we get $(x^n \rightarrow y) \rightarrow (x^n \rightarrow z) = x^n \rightarrow ((x^n \rightarrow y) \rightarrow z)) \in F$.

(d $\Rightarrow$ b): Since $x^n \to (x^n \to x^{2n}) = x^{2n} \to x^{2n} = 1 \in F$, by (d) we have $x^n \to x^{2n} = (x^n \to x^n) \to (x^n \to x^{2n}) \in F$ □

Theorem 4.5. *et F be a filter of a Rl-monoid A. If F is an n-fold implicative filter, then F is an $(n+1)$-fold implicative filter.*

Proof. Let $x, y \in A$ be such that $x^{n+2} \to y \in F$. By lemma 2.1, $x^{n+1} \to (x \to y) = x^{n+2} \to y$ and since F is n-fold implicative filter by theorem 4.4, $x^n \to (x \to y) \in F$. Hence $x^{n+1} \to y \in F$, that is, F is $(n+1)$-fold implicative filter. □

By the following example we show that the converse is not true.

Example 4.2. *In example 3.1, $\{1\}$ is a 3-fold implicative filter but since $b^3 \to 0 = 1 \in \{1\}$ and $b^2 \to 0 = b \neq 1$, $\{1\}$ is not a 2-fold implicative filter.*

Corollary 4.1. *In an n-fold implicative Rl-monoid, the concept of filters and n-fold implicative filters coincide.*

Proof. It follows from theorem 4.4 and the definition of an n-fold implicative *Rl-monoid.* □

Theorem 4.6. *A is an n-fold implicative Rl-monoid if and only if $\{1\}$ is an n-fold implicative filter of A.*

Proof. If A is an n-fold implicative *Rl-monoid*, then $x^{n+1} = x^n$, for all $x \in A$ and so $x^{n+2} = x^{n+1} * x = x^n * x = x^{n+1} = x^n$. By the similar way $x^{2n} = x^n$, for all $x \in A$, that is, $x^n \to x^{2n} = 1 \in \{1\}$, for all $x \in A$. By condition (b) of theorem 4.4, $\{1\}$ is an n-fold implicative filter of A. Conversely, let the filter $\{1\}$ of A be an n-fold implicative filter. Since $x^n \to (x^n \to x^{n+1}) = x^{2n} \to x^{n+1} = 1 \in \{1\}$ and $x^n \to x^n = 1 \in \{1\}$ we get $x^n \to x^{n+1} \in \{1\}$, that is, $x^{n+1} = x^n$. Hence A is an n-fold implicative *Rl-monoid.* □

Theorem 4.7. *Let F and G be filters of A such that $F \subseteq G$. If F is an n-fold implicative filter then G is also an n-fold implicative filter.*

Proof. Let$F$ be an n-fold implicative filter of A. Then by theorem 4.4, $x^n \to x^{2n} \in F$, for all $x \in A$, and so $x^n \to x^{2n} \in G$, for all $x \in A$. Hence G is an n-fold implicative filter. □

Theorem 4.8. *In any Rl-monoid A, the following conditions are equivalent:*
(a) A is an n-fold implicative Rl-monoid,
(b) Every filter of A is an n-fold implicative filter,
(c) $\{1\}$ is an n-fold implicative filter,
(d) $x^n = x^{2n}$, for all $x \in A$.

Proof. $(a \Rightarrow b)$: It is clear by the Definition of an n-fold implicative *Rl-monoid* and theorem 4.4.

$(b \Rightarrow c)$: is clear.

$(c \Rightarrow a)$: By theorem 4.6 is clear.

$(a \Rightarrow d)$: Let A is an n-fold implicative *Rl-monoid*, hence $x^{n+1} = x^n$, for all $x \in A$. We have $x^{n+2} = x^{n+1} * x = x^n * x = x^{n+1} = x^n$. By repeating the process n times, we get $x^n = x^{2n}$, for all $x \in A$.

$(d \Rightarrow a)$: If $x^n = x^{2n}$, for all $x \in A$.Then $x^n \to x^{2n} = 1 \in \{1\}$, for all $x \in A$. By theorem 4.4, $\{1\}$ is an n-fold implicative filter. Since (a) and (c) are equivalent, A is an n-fold implicative *Rl-monoid.* □

Theorem 4.9. *Let F be a filter of A. Then F is an n-fold implicative filter if and only if A/F is an n-fold implicative Rl-monoid.*

Proof. Let F be an n-fold implicative filter, by theorem 4.4, $x^n \to x^{2n} \in F$, for all $x \in A$. Then $[x]^n \to [x]^{2n} = [x^n \to x^{2n}] = [1]$ and so $[x]^n \leq [x]^{2n}$, that is, $[x]^n = [x]^{2n}$, for all $x \in A$. Hence by theorem 4.8, A/F is an n-fold implicative *Rl-monoid*. Conversely, let A/F be an n-fold implicative *Rl-monoid*, then $[x]^n = [x]^{2n}$, for all $x \in A$. Hence $[x^n \to x^{2n}] = [x]^n \to [x]^{2n} = [1]$, that is, $x^n \to x^{2n} \in F$, for all $x \in A$. Therefore by theorem 4.4, F is an n-fold implicative filter. □

Corollary 4.2. *Let F be a filter of A. Then F is a 1-fold implicative filter if and only if A/F is a Heyting algebra.*

Proof. By theorem 2.1 is clear. □

5. n-fold positive implicative *Rl-monoid*

Definition 5.1. *An n-fold positive implicative Rl-monoid is a Rl-monoid $A = (A, \wedge, \vee, *, \to 0, 1)$ if it satisfies $(x^n \to 0) \to x = x$, for all $x \in A$.*

Theorem 5.1. *Every n-fold positive implicative Rl-monoid is an $(n+1)$-fold positive implicative Rl-monoid.*

Proof. Let A be an n-fold positive implicative *Rl-monoid*. Then $(x^n \to 0) \to x = x$, for all $x \in A$. Since $x^{n+1} \leq x^n$ then $x^n \to 0 \leq x^{n+1} \to 0$ and so $(x^{n+1} \to 0) \to 0 \leq (x^n \to 0) \to 0 = x$. But by lemma 2.1 $x \leq (x^{n+1} \to 0) \to 0$. Hence $x = (x^{n+1} \to 0) \to 0$, that is, A is an $(n+1)$-fold positive implicative *Rl-monoid*. □

The following example shows that the converse of theorem 5.1 is not true.

Example 5.1. *In example 3.1, B is a 3-fold positive Rl-monoid but since $(b^2 \to 0) \to b \neq b$, B is not a 2-fold positive implicative Rl-monoid.*

6. n-fold positive implicative filters

Definition 6.1. *A non-empty subset F of A is called a n-fold positive implicative filter if it satisfies:*
(1) $1 \in F$,
(2) $x \to ((y^n \to z) \to y) \in F$ *and* $x \in F$ *imply* $y \in F$, *foe all* $x, y, z \in A$

Theorem 6.1. *Every n-fold positive implicative filter of A is a filter of A.*

Proof. Let F be a n-fold positive implicative filter and $x, y \in A$ be such that $x, x \to y \in F$. But $x \to ((y^n \to 1) \to y) = x \to y$ and so $x \to ((y^n \to 1) \to y) \in F$. Since $x \in F$ and F is n-fold positive implicative filter of A we get $y \in F$. Hence F is a filter. □

Theorem 6.2. *Let F be a filter of A. Then the following conditions are equivalent:*
(a) F is an n-fold positive implicative filter,
(b) $(x^n \to 0) \to x \in F$ *implies* $x \in F$, *for all* $x \in A$,
(c) $(x^n \to y) \to x \in F$ *implies* $x \in F$, *for all* $x \in A$

Proof. (a $\Rightarrow$ c): Let F be a n-fold positive implicative filter of A and $(x^n \to y) \to x \in F$. Since $1 \to ((x^n \to y) \to x) = (x^n \to y) \to x$, we get $1 \to ((x^n \to y) \to x) \in F$. We have $1 \in F$, thus $x \in F$.

(c $\Rightarrow$ b): is clear.

(b $\Rightarrow$ a): Let $x \to ((y^n \to z) \to y) \in F$ and $x \in F$. Since F is filter, we get $(y^n \to z) \to y \in F$. Since $0 \le z$, by lemma 2.1, $y^n \to 0 \le y^n \to z$ and so $(y^n \to z) \to y \le (y^n \to 0) \to y$. Hence $(y^n \to 0) \to y \in F$ and so by assumption we get $y \in F$. Therefore F is an n-fold positive implicative filter. □

Theorem 6.3. *Let F be a filter of Rl-monoid A. If F is an n-fold positive implicative filter then F is an $(n+1)$-fold positive implicative filter*

Proof. Let F be an n-fold positive implicative filter and $x \in A$ be such that $(x^{n+1} \to 0) \to x \in F$. Since $x^{n+1} \le x^n$, we have $x^n \to 0 \le x^{n+1} \to 0$ and so $(x^{n+1} \to 0) \to 0 \le (x^n \to 0) \to 0$. Since F is a filter, $(x^n \to 0) \to 0 \in F$. Since F is an n-fold positive implicative filter, we have $x \in F$ and so F is an $(n+1)$-fold positive implicative filter. □

By the following example we show that the converse is not true.

Example 6.1. *In example 3.1, $\{1\}$ is a 3-fold positive implicative filter, but $(b^2 \to 0) \to b = 1$ and $b \neq 1$, hence $\{1\}$ is not a 2-fold positive implicative filter.*

Theorem 6.4. *Every n-fold positive implicative filter is an n-fold implicative filter.*

Proof. Let F be an n-fold positive implicative filter of A. By theorem 6.1, F is a filter of A. Let $x, y \in A$ be such that $x^{n+1} \to y \in F$. Then by lemma 2.1:

$(x^{n+1} \to y)^n \to (x^n \to y) = (x^{n+1} \to y)^{n-1}(x^{n+1} \to y) \to (x^n \to y)$
$= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^n \to y))$
$= (x^{n+1} \to y)^{n-1} \to ((x^{n+1} \to y) \to (x^{n-1} \to (x \to y)))$
$= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x^{n+1} \to y) \to (x \to y)))$
$= (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to ((x \to (x^n \to y)) \to (x \to y)))$
$\ge (x^{n+1} \to y)^{n-1} \to (x^{n-1} \to (x^n \to y)) \to y)$
$= (x^{n+1} \to y)^{n-1} \to ((x^n \to y) \to (x^{n-1} \to y))$
$= (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$

We show that

$$(x^{n+1} \to y)^n \to (x^n \to y) \ge (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))$$

Now consider

$(x^n \to y)(x^{n+1} \to y)^{n-1}x^{n-1} = (x^n \to y)(x^{n+1} \to y)^{n-2}(x^{n+1} \to y)xx^{n-2}$
$= (x^n \to y)(x^{n+1} \to y)^{n-2}x^{n-2}x(x^{n+1} \to y)$

Since $x^{n+1} \to y \le x^{n+1} \to y = x \to (x^n \to y)$ then $x(x^{n+1} \to y) \le x^n \to y$, we get $(x^n \to y)(x^{n+1} \to y)^{n-1}x^{n-1} \le (x^n \to y)^2(x^{n+1} \to y)^{n-2}x^{n-2}$. Hence
$((x^n \to y)^2(x^{n+1} \to y)^{n-2}x^{n-2}) \to y \le ((x^n \to y)(x^{n+1} \to y)^{n-1}x^{n-1}) \to y$ and so
$(x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \le (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$

But we had
$(x^{n+1} \to y)^n \to (x^n \to y) \ge (x^n \to y) \to ((x^{n+1} \to y)^{n-1} \to (x^{n-1} \to y))$
Then
$(x^{n+1} \to y)^n \to (x^n \to y) \ge (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y))$
Hence by repeating the process n times we get
$(x^{n+1} \to y)^n \to (x^n \to y) \ge (x^n \to y)^2 \to ((x^{n+1} \to y)^{n-2} \to (x^{n-2} \to y)) \ge \dots \ge$

$(x^n \to y)^n \to ((x^{n+1} \to y)^0 \to (x^0 \to y))$
$= (x^n \to y)^n \to (1 \to (1 \to y)) = (x^n \to y)^n \to y$

Hence $((x^n \to y)^n \to y) \to ((x^{n+1} \to y)^n \to (x^n \to y)) = 1$ and so $(x^{n+1} \to y)^n \to (((x^n \to y)^n \to y) \to (x^n \to y)) = 1$. Since F is a filter and $x^{n+1} \to y \in F$, we get $(x^{n+1} \to y)^n \in F$ and so $((x^n \to y)^n \to y) \to (x^n \to y) \in F$. Since F is an n-fold positive implicative filter by theorem 6.2 we have $x^n \to y \in F$.

Hence by theorem 4.4, F is an n-fold implicative filter. □

The following example shows that the converse of theorem 6.4 is not true.

Example 6.2. *Let $B = \{0, a, b, c, 1\}$. Define $*$ and $\to$ as follow:*

$*$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then $(B, \wedge, \vee, *, \to, 0, 1)$ is a Rl-monoid and it is clear that $F = \{b, 1\}$ is a 2-fold implicative filter but it is not a 2-fold positive implicative filter, since $(a^2 \to 0) \to a = 1 \in F$ and $a \notin F$.

Lemma 6.1. *In the n-fold positive implicative Rl-monoid, the notion of an n-fold positive implicative filter and a filter is coincide.*

Proof. By definition of an n-fold positive implicative *Rl-monoid* and theorem 6.2 is clear. □

Theorem 6.5. *Let A be a Rl-monoid. Then the following conditions are equivalent:*
(a) A is an n-fold positive implicative Rl-monoid
(b) Every filter of A is an n-fold positive implicative filter,
(c)$\{1\}$ is an n-fold positive implicative filter.

Proof. (a $\Rightarrow$ b): By lemma 6.1, is clear.

(b $\Rightarrow$ c): is clear.

(c $\Rightarrow$ d): Let $\{1\}$ be an n-fold positive implicative filter of A. Consider $x \in A$ and let $t = ((x^n \to 0) \to x) \to x$. Then by lemma 2.1 we have
$(t^n \to 0) \to t = (t^n \to 0) \to (((x^n \to 0) \to x) \to x)$
$= ((x^n \to 0) \to x) \to ((t^n \to 0) \to x \geq (t^n \to 0) \to (x^n \to 0) \geq x^n \to t^n = 1$

The last equality follows from $x \leq ((x^n \to 0) \to x) \to x = t$. Then $x^n \leq t^n$, that is, $x^n \to t^n = 1$. Hence $(t^n \to 0) \to t = 1 \in \{1\}$ and since $\{1\}$ is an n-fold positive implicative filter, $t = ((x^n \to 0) \to x) \to x = 1$, that is, $(x^n \to 0) \to x \leq x$. On the other hand by lemma 2.1 we have $(x^n \to 0) \to x \geq x$. Hence we get $(x^n \to 0) \to x = x$, for all $x \in A$, that is, A is an n-fold positive implicative *Rl-monoid*. □

Theorem 6.6. *Let F be a filter of A. Then A/F is an n-fold positive implicative Rl-monoid if and only if F is an n-fold positive implicative filter.*

Proof. Let F be an n-fold positive implicative filter and $x \in A$ be such that $([x]^n \to [0]) \to [x] = [1]$. Then $[(x^n \to 0) \to x] = ([x]^n \to [0]) \to [x] = [1]$ and so $(x^n \to 0) \to x \in F$. Since F is n-fold positive implicative filter by theorem 6.2, $x \in F$, hence $[x] = [1]$ and so $\{[1]\}$ is a n-fold positive implicative filter of A/F. Then by theorem 6.5, A/F is an n-fold positive implicative *Rl-monoid*

Conversely, let A/F be an n-fold positive implicative *Rl-monoid* and $x \in A$ be such that $(x^n \to 0) \to x \in F$. Then $[x] = ([x]^n \to [0]) \to [x] = [(x^n \to 0) \to x] = [1]$ and so $[x] = [1]$, that is, $x \in F$. It follows from theorem 6.2 that F is an n-fold positive implicative filter. □

Corollary 6.1. *Let F be a filter of a Rl-monoid A. Then A/F is a Boolean algebra if and only if F is a 1-fold positive implicative filter.*

Proof. Assume that F is a 1-fold positive implicative filter and $x, y \in A$ be such that $([x] \to [y]) \to [x] = [1]$, then $(x \to y) \to x \in F$. By theorem 2.2, $x \in F$. Hence $[x] = [1]$ which proves that $\{[1]\}$ is a positive implicative filter. By theorem 2.3, A/F is a Boolean algebra.

Conversely, let $(x \to y) \to x \in F$, for $x, y \in F$. Then $([x] \to [y]) \to [x] = [(x \to y) \to x] = [1]$. Since A/F is a Boolean algebra, by theorem 2.3 $\{[1]\}$ is a positive implicative filter, then $[x] = [1]$, i.e, $x \in F$. Hence F is a 1-fold positive implicative filter. □

Theorem 6.7. *Let F and G be filters of a Rl-monoid A such that $F \subseteq G$. If F is an n-fold positive implicative filter, then G is an n-fold positive implicative filter.*

Proof. Let A be such that $(x^n \to 0) \to x \in G$. Since F is an n-fold positive implicative filter, by theorem 6.6, A/F is an n-fold positive implicative *Rl-monoid*. Then $[(x^n \to 0) \to x] = (([x]^n \to [0]) \to [x] = [x]$ and so $((x^n \to 0) \to x) \to x \in F \subseteq G$. Since G is filter and $(x^n \to 0) \to x \in G$, $x \in G$. Hence by theorem 6.2, G is an n-fold positive implicative filter. □

References

[1] A. Dvurecenenskij and J. Rachunek, Probabilistic averaging in bounded commutative residuated l-monoids, *Discrete Mathematics* **306** (2006), 1317–1326.

[2] P. Hajek, *Metamathematics of Fuzzy Logic, Kluwer Academic Publishers*, Dordrecht, 1988.

[3] M. Haveshki and E. Eslami, n-fold filters in BL-algebras, *Math. Logic. Quart.* **54** (2008), no. 2, 176–186.

[4] J. Rachunek and D. Salounova, Classes of filter in generalizations of commutative fuzzy structure, *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica* **48** (2009), no. 1, 93–107.

[5] J. Rachunek and D. Slezak, Negation in bounded commutative DRl-monoids, *Czechoslovak Math. J.* **56** (2006), 755–763.

[6] K.L.N. Swamy, Dually residuated lattice ordered semigroup III, *Math. Ann.* **167** (1966), 71–74.

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