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Two examples of weighted majorization

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Abstract. In this paper we discuss two examples of weighted majorization. The first example relates the spectrum of a normal matrix and the spectrum of its principal submatrix. We apply Sherman's theorem in order to deduce inequalities involving the elements in the two spectra. The second example relates the diagonal elements of the Grassman product of a normal matrix and the Grassman product of its principal submatrix.

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1. Introduction

The classical concept of majorization for pairs of n real numbers was studied by many mathematicians. Initially it was used by Hardy, Littlewood and Pólya to outline a class of convex inequalities and by Schur to relate the spectra of a Hermite matrix to its diagonal. In recent years a number of remarkable applications were found in combinatorics, statistics, information theory, quantum mechanics, wireless communication, etc. Weighted majorization is a useful generalization of classical majorization which can be traced back to Sherman. Borcea [3] considered a very interesting illustration of this concept relating the critical points and the poles of the potential of the electrostatic force generated by a finite configuration of coplanar positive charges. In this paper we will describe two other examples of weighted majorization and comment on the inequalities derived via Sherman's theorem. The first example relates the spectrum of a normal matrix and the spectrum of its principal submatrix. The second example is the extension of the previous one via the Grassman product.

We will use the following notation.

Let $k \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define

$$\begin{aligned} K_n^k &= \left\{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}^k, 1 \le i \le n\right\} \\ A_n &= \left\{(a_1, a_2, ..., a_n) : a_i \in (0,1)\,,\ 1 \le i \le n,\ \sum_{i=1}^{n} a_i = 1\right\} \\ \mathbb{k}_n^k &= K_n^k \times A_n \end{aligned} \tag{1}$$

and put

$$\mathbb{k}^n = \cup_{m=1}^{\infty} \mathbb{k}_m^n. \tag{2}$$

For $x = (x_1, x_2, ..., x_n) \in K_n^k$ we have

$$x^T = (x_1, x_2, ..., x_n)^T,$$

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which means the ordered pair of n vectors written in a row. We will identify x^T with a $n \times k$-matrix in which the row of order i consists of the coordinates of vector x_i written in standard basis of $\mathbb{R}^n$.

We will denote the set of row stochastic $n \times k$-matrices by $\Omega^{rs}_{n\times k}$.

The notion of weighted majorization has as starting point the following theorem of Sherman[2]:

Theorem 1.1. *Let $x := \{x_i\}_1^l$ be an l-couple and $y := \{y_j\}_1^m$ an m-couple of vectors from $\mathbb{R}^k$. Let $\{a_i\}_1^l$ and $\{b_j\}_1^m$ two collections of positive weights. The following statements are equivalent:*

1) *There is a matrix $S = (s_{ij}) \in \mathbb{R}^{l\times m}$ with positive elements such that $(x_i)_{i=1}^l = S(y)_{j=1}^m$ and*

$$\sum_{j=1}^{m} s_{ij} = 1, \quad i \in \{1,2,...,l\}, \quad \sum_{i=1}^{l} s_{ij} a_i = b_j, \quad j \in \{1,2,...,m\}. \tag{3}$$

2) *The weights $\{a_i\}_1^l$,$\{b_j\}_1^m$ satisfy $\sum\limits_{i=1}^{l} a_i = \sum\limits_{j=1}^{m} b_j$ and the following inequality occurs*

$$\sum_{i=1}^{l} a_i f(x_i) \leq \sum_{j=1}^{m} b_j f(y_j) \tag{4}$$

for any function $f \in CV(\mathbb{R}^k)$.

See [3] for a proof.

Theorem 1.1 leads to the following concept of majorization:

Definition 1.1. *We say that the pair $(x,a) \in \Bbbk_l^k$ is weightily majorized by $(y,b) \in \Bbbk_m^k$ (and we denote this by $(x,a) \prec (y,b)$), if the conditions of Theorem 1.1 are satisfied.*

In the following section we will see that the spectrum of the principal submatrix of a normal matrix is weightily majorized by the spectrum of the given matrix.

2. A relation between the spectrum of a normal matrix and the spectrum of its principal submatrix

In what follows A will denote a normal matrix (of order $n \times n$) with the spectrum

$$Z = (z_1, z_2, ..., z_n)$$

and the orthonormal basis of its eigenvectors $(u_1, u_2, ..., u_n)$.

Our first goal is to comment on the notion of principal submatrix of A.

$\mathbb{C}^{n\times n}$ will denote the set of all $n \times n$-matrices with complex numbers (seen as operators on the complex Euclidean $\mathbb{C}^n$).

Let $(p_1, p_2, ..., p_n)$ be a collection of n positive weights with $\sum\limits_{k=1}^{n} p_k = 1$ and the unitary vector

$$v_n = \sum_{k=1}^{n} \sqrt{p_k} u_k. \tag{5}$$

We take the orthogonal projection P on $v_n^{\perp}$ and call the matrix

$$B = PAP|_{P(\mathbb{C}^{n\times n})}$$

the *compression* of A to $P\left(\mathbb{C}^{n\times n}\right)$; this is the upper left-hand of the $(n-1)\times(n-1)$-principal submatrix of A when we consider a base in which the last vector is v_n. See [5]. By the adjoint formula for the inverse of a matrix, if $A-zI_n$ is invertible, then the element of index (n,n) from $\left(A-zI_n\right)^{-1}$ is given by

$$\left\langle \left(A-zI_n\right)^{-1} v_n, v_n\right\rangle = \frac{\det\left(B-zI_{n-1}\right)}{\det\left(A-zI_n\right)}. \tag{6}$$

Lemma 2.1. *Let A be a normal $n\times n$-matrix and let B be the compression of A to $P\left(\mathbb{C}^{n\times n}\right)$, where P is the orthogonal projection on $v_n^{\perp}$ as above. Then the eigenvalues of B are the zeroes of the function*

$$f(z)=\sum_{k=1}^{n}\frac{p_k}{z_k-z}.$$

Proof. For a normal $n\times n$-matrix A we have the orthonormal basis of its eigenvectors $(u_1,u_2,...,u_n)$.Using the relationship (5) and taking into account the expression (6) for v_n we have (for $z\neq z_1,...,z_n$) :

$$\begin{aligned}\frac{\det\left(B-zI_{n-1}\right)}{\det\left(A-zI_n\right)} &= \left\langle \left(A-zI_n\right)^{-1} v_n, v_n\right\rangle \\ &= \left\langle \sum_{j=1}^{n}\frac{1}{z_j-z}\sqrt{p_j}u_j, \sum_{k=1}^{n}\sqrt{p_k}u_k\right\rangle = \sum_{k=1}^{n}\frac{p_k}{z_k-z}.\end{aligned}$$

□

Following this lemma and the fact that the function that represents the electrostatic force for a finite configuration of coplanar charges $(z_k)_1^n$ with positive weights $(p_k)_1^n$ has the expression $-f(z)$, the eigenvalues of B coincide with the zeroes of f. Then we can state the following theorem.

Theorem 2.1. *Between the spectra $Z=(z_k)_1^n$ and $W=(w_j)_1^{n-1}$ of the two matrices A and B described above we have the following relationship of weighted majorization*

$$\left(W^T,a\right)\prec\left(Z^T,b\right), \tag{7}$$

where

$$a=\left(\frac{1}{n-1},\frac{1}{n-1},...,\frac{1}{n-1}\right)\in A_{n-1} \tag{8}$$

and

$$b=\left(\frac{1-p_1}{n-1},\frac{1-p_2}{n-1},...,\frac{1-p_n}{n-1}\right)\in A_n. \tag{9}$$

Proof. The argument is based on the theorem of Sherman. Let $(v_j)_1^{n-1}$ be an orthonormal base which makes B. Any vector v_j with $1\leq j\leq n-1$ is written in the base $(u_k)_1^n$ as

$$v_j=\sum_{k=1}^{n}\langle v_j,u_k\rangle u_k,$$

and

$$w_j=\langle Bv_j,v_j\rangle=\langle Av_j,v_j\rangle=\sum_{k=1}^{n}z_k\left|\langle v_j,u_k\rangle\right|^2.$$

We take the matrix $S=(s_{jk})\in\mathbb{R}^{(n-1)\times n}$ where $s_{jk}=|\langle v_j,u_k\rangle|^2$ and we deduce that $S\in\Omega^{rs}_{(n-1)\times k}$ because for any $1\le j\le n-1$ and for any $1\le k\le n$ we have

$$\sum_{k=1}^{n}s_{jk}=\sum_{k=1}^{n}|\langle v_j,u_k\rangle|^2=\|v_j\|^2=1$$

and

$$\sum_{j=1}^{n-1}s_{jk}=\sum_{j=1}^{n}|\langle v_j,u_k\rangle|^2-|\langle v_n,u_k\rangle|^2=\|u_k\|^2-p_k=1-p_k.$$

Then the statement 1) in Sherman's theorem is verified:

$$\begin{aligned}W^T &= SZ^T,\ \sum_{i=1}^{l}s_{jk}a_j=b_k,\quad k\in\{1,2,...,n\},\\ a &= \left(\frac{1}{n-1},\frac{1}{n-1},...,\frac{1}{n-1}\right)\in A_{n-1}\ ,\\ b &= \left(\frac{1-p_1}{n-1},\frac{1-p_2}{n-1},...,\frac{1-p_n}{n-1}\right)\in A_n.\end{aligned}$$

□

Corollary 2.1. *For any convex function Φ defined on $\mathbb{C}$ the eigenvalues $Z=(z_k)_1^n$, $W=(w_j)_1^{n-1}$ of a normal $n\times n$-dimensional matrix A, and respectively of its principal submatrix B verify the following inequalities*

$$\sum_{j=1}^{n-1}\Phi(w_j)\le\sum_{k=1}^{n}(1-p_k)\Phi(z_k)\,. \tag{10}$$

If we consider the case of a Hermite matrix with its own real and positive values and the spectre of the submatrix also formed of positive numbers and take for example $\Phi(z)=-\log z$ we obtain an inequality for the determinant of the submatrix B:

$$\prod_{k=1}^{n}z_k^{1-p_k}\le\det B.$$

In the next section we will describe other relation of weighted majorization which can be ascribed to normal matrices (and extends the above discussion).

3. Weighted majorization and the Grassman product

We start with a short preparation about the Grassman product.

For $m\in\mathbb{N}$ and $1\le k\le m$ we consider the set of multi-indices

$$Q_{k,m}:=\{i=(i_1,...,i_k)/1\le i_1<...<i_k\le m\}$$

endowed with the lexicographic ordering:

$$i\ge j\text{ for }i,j\in Q_{k,m}$$

if the first term in the row

$$i_1-j_1,...,i_k-j_k$$

is positive.

If B is a $m \times m$-matrix and $i, j \in Q_{k,m}$, then the $k \times k$-submatrix situated at the intersection of the rows $i_1, ..., i_k$ and the columns $j_1, ..., j_k$ will be denoted

$$B(i,j) = B\begin{pmatrix} i_1...i_k \\ j_1...j_k \end{pmatrix}.$$

Given a normal $n \times n$-matrix A, for each $1 \le k \le n-1$ we can attach to it a $\binom{n}{k} \times \binom{n}{k}$-matrix $A^{(k)}$, the Grassman k power of A:

$$A^{(k)} = (\det A(i,j))_{i,j \in Q_{k,n}}.$$

For the principal submatrix B of matrix A the Grassman k power is a matrix $\binom{n-1}{k} \times \binom{n-1}{k}$-dimensional.

In the following

$$diag(z_1, z_2, ..., z_n)$$

will denote the vector from $\mathbb{C}^n$ formed by the diagonal elements of a square $n \times n$-matrix, while

$$Diag(z_1, z_2, ..., z_n),$$

will denote a diagonal $n \times n$-matrix with $z_1, z_2, ..., z_n$ (in this order) on the main diagonal.

When A is a normal matrix, the vectors $diag\left(B^{(k)}\right)$ and $diag\left(A^{(k)}\right)$ are related by weighted majorization. This is based on the Cauchy-Binet formula,

$$(AB)^{(k)} = A^{(k)} B^{(k)},$$

and the following properties of the Grassman k power:

Remark 3.1. $(A^*)^{(k)} = A^{*(k)}$.

Remark 3.2. *If* $D = Diag(a_1, ..., a_n)$, *then* $D^{(k)}$ *is also a diagonal matrix* ($\binom{n}{k} \times \binom{n}{k}$ *dimensional*) *and its elements on the diagonal are products* $a_{i_i}...a_{i_k}$ *and*

$$traceD^{(k)} = S_k(a_1, ..., a_n),$$

where S_k *is the* $k-$ *symmetric function of* n *symbols,*

$$S_k(a_1, ..., a_n) = \sum_{1 \le i_1 < ... < i_k \le n} a_{i_i} \cdots a_{i_k}.$$

Remark 3.3. *If* B *is a upper triangular* $n \times n$ *matrix, then* $B^{(k)}$ *is also diagonal matrix* $\binom{n}{k} \times \binom{n}{k}$ *and denoting by* $w_{j_1}, ..., w_{j_n}$ *the characteristic values of* B, *the characteristic values of* $B^{(k)}$ *are products of the type* $w_{j_1} \cdots w_{j_k}$, *where* $(j_1, ..., j_k) \in Q_{k,n}$.

Remark 3.4. *If* U *is a unitary* $n \times n$*-matrix, then* $U^{(k)}$ *is also unitary.*

Let us consider the matrix B the orthogonal projection of the normal $n \times n$-matrix A on $v_n^{\perp}$, where the unitary vector v_n is given by n positive weights $(p_1, p_2, ..., p_n)$ with $\sum_{k=1}^{n} p_k = 1$, $v_n = \sum_{k=1}^{n} \sqrt{p_k} u_k$,and $(u_k)_1^n$ is the orthonormal basis of the eigenvectors of the matrix A. We say that B represents the principal submatrix of A and we have $(v_j)_1^{n-1}$ orthonormal bases which is triangular with the matrix B. Malamud [4] gave a theorem in which he demonstrates the majorization between the vectors formed with products of k eigenvalues of a normal matrix and the products of k eigenvalues of the principal submatrix and we will see that between such vectors we have in fact a weighted majorization. Moreover, these products represent the diagonal elements of the Grassman k power of the normal matrix A and respectively the diagonal elements of the Grassman k power for its main submatrix B.

Theorem 3.1. *For any k smaller than n a weighted majorization takes place between the vectors given by the diagonal elements of the Grassman k powers of a normal matrix A and its principal submatrix B:*

$$\left(diag\left(B^{(k)}\right);a^{[k]}\right) \prec \left(diag\left(A^{(k)}\right);b^{[k]}\right),$$

where

$$a^{[k]} = \left(\frac{1}{\binom{n-1}{k}},...,\frac{1}{\binom{n-1}{k}}\right) \in (0;1)^{\binom{n-1}{k}},$$

and

$$b^{[k]} = \left(\frac{1-\sum\limits_{l=1}^{k} p_{j_l}}{\binom{n-1}{k}},...,\frac{1-\sum\limits_{l=1}^{k} p_{j_l}}{\binom{n-1}{k}}\right) \in (0;1)^{\binom{n}{k}}.$$

Remark 3.5. *For $k=1$ we have the particular case of the previous theorem. The proof is the same as the one given by Borcea for the case of the equilibrium points of a finite configuration of electric charges , and the weights $a^{[k]}$,$b^{[k]}$ will be justified.*

Proof. The idea is to determine a stochastic matrix on R_k rows such as to have a condition in Sherman's theorem verified, namely

$$\begin{aligned} b^{[k]} &= a^{[k]}R_k, \\ diag\left(B^{(k)}\right)^{\mathrm{T}} &= R_k diag\left(A^{(k)}\right)^{\mathrm{T}}. \end{aligned} \tag{11}$$

If U is the unitary matrix of that maps $(v_1,...v_n)$ into $(u_1,...u_n)$, then

$$A = UDiag\left(z_1,...,z_n\right)U^* = \begin{pmatrix} B & * \\ 0 & \zeta \end{pmatrix},$$

and based on the Cauchy-Binet formula and the previous remarks we have

$$A^{(k)} = U^{(k)}Diag\left(z_1,...,z_n\right)^{(k)}U^{*(k)} = \begin{pmatrix} B & * \\ 0 & \zeta \end{pmatrix}^{(k)}.$$

As $U^{(k)}$ is unitary, $S_k = U^{(k)} \circ U^{*(k)}$ is unitary double stochastic. The diagonal elements of $A^{(k)}$ are the main $k\times k$ minors of B and they have the form $\det B\left(i,i\right), i \in Q_{k,n-1}$. In total we have $\binom{n-1}{k}$ diagonal elements and they must coincide with the coordinates of the vector $diag\left(B^{(k)}\right)^{\mathrm{T}}$, because B is upper triangular. Let R_k be the stochastic matrix on rows $\binom{n-1}{k}\times\binom{n}{k}$ obtained from S_k by erasing all the elements $S_k\left(i,j\right)$ where $i \in Q_{k,n} - Q_{k,n-1}$. To do this we show that for any $j=(j_1,...,j_k) \in Q_{k,n}$ the following identity takes place

$$\sum_{1\leq i_1<...<i_{k-1}\leq n-1} S_k\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix} = \sum_{l=1}^{k} p_{j_l},$$

which is equivalent with

$$\sum_{1\leq i_1<...<i_{k-1}\leq n-1} \left|\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix}\right|^2 = \sum_{l=1}^{k} p_{j_l}, \tag{12}$$

or

$$\sum_{i\in Q_{k,n-1}} \left|\det U\left(i,j\right)\right|^2 = 1-\sum_{l=1}^{k} p_{j_l}. \tag{13}$$

But $u_{nj}=\sqrt{p_j}$, $1\leq j\leq n$, therefore developing any determinant which appears in the last line we have

$$\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix} = \sum_{m=1}^{k} (-1)^{n+j_m}\sqrt{p_{j_m}}\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} \\ j_1 & ,..\widehat{j_m},.. & j_k \end{pmatrix},$$

and thus

$$\left|\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix}\right|^2$$

$$= \sum_{r,s=1}^{k} (-1)^{r+s}\sqrt{p_{j_r}p_{j_s}}\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} \\ j_1 & ,..\widehat{j_r},.. & j_k \end{pmatrix}\det\overline{U}\begin{pmatrix} i_1 & ,..., & i_{k-1} \\ j_1 & ,..\widehat{j_s},.. & j_k \end{pmatrix}$$

for $1\leq i_1<...<i_{k-1}\leq n-1$. Then

$$\sum_{1\leq i_1<...<i_{k-1}\leq n-1}\left|\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix}\right|^2 = \sum_{r,s=1}^{k}(-1)^{r+s}\sqrt{p_{j_r}p_{j_s}}\alpha_{r,s}(j),$$

where

$$\alpha_{r,s}(j) = \sum_{1\leq i_1<...<i_{k-1}\leq n-1}\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} \\ j_1 & ,..\widehat{j_r},.. & j_k \end{pmatrix}\det\overline{U}\begin{pmatrix} i_1 & ,..., & i_{k-1} \\ j_1 & ,..\widehat{j_s},.. & j_k \end{pmatrix}.$$

Let us consider the matrix $k\times(n-1)$

$$M = U\begin{pmatrix} 1 & ,..., & n-1 \\ j_1 & ,..., & j_k \end{pmatrix}^T$$

and the matrix $k\times k$ $,C=(c_{ij})=MM^*$ is also a $k\times k$ matrix given by the $k-1$ Grassman power $C^{(k-1)}=\left(c_{ij}^{(k-1)}\right)$. From

$$C^{(k-1)} = M^{(k-1)}M^{*(k-1)}$$

we deduce that

$$c_{k+1-r,k+1-s}^{(k-1)} = \alpha_{r,s}(j),\ 1\leq r,s\leq k.$$

So the left member from (12) is

$$\sum_{1\leq i_1<...<i_{k-1}\leq n-1}\left|\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix}\right|^2 = \sum_{r,s=1}^{k}(-1)^{r+s}\sqrt{p_{j_r}p_{j_s}}c_{k+1-r,k+1-s}^{(k-1)}.$$

From this expression we obtain

$$\sum_{1\leq i_1<...<i_{k-1}\leq n-1}\left|\det U\begin{pmatrix} i_1 & ,..., & i_{k-1} & n \\ j_1 & ,.. & .., & j_k \end{pmatrix}\right|^2 = \sum_{i\in Q_{k-1,k}}\det\ C(i,i) - k\det C$$

and

$$\sum_{i\in Q_{k-1,k}}\left|\det\ U(i,j)\right|^2 = 1-\sum_{i\in Q_{k-1,k}}\det\ C(i,i)+k\det C.$$

In order to show that the right member of the identity is equal with the left one, we define the vector $u=\left(\sqrt{p_{j_1}},...,\sqrt{p_{j_k}}\right)$. Because $C=I_k-u^Tu$, it results that C depends only on numbers p_{j_l},$1\leq l\leq k$. We deduce that the expression

$$\sum_{1\leq i_1<...<i_k\leq n-1}\left|\det U\begin{pmatrix} i_1 & ,..., & i_k \\ j_1 & ,.. & ..j_k \end{pmatrix}\right|^2 = \sum_{i\in Q_{k-1,k}}\left|\det\ U(i,j)\right|^2$$

depends exclusively on the numbers c_{j_l},$1 \leq l \leq k$ and therefore is independent of z_i,$1 \leq i \leq n$. On the other hand adding to each term all the coordinates corresponding to $i \in Q_{k,n-1}$ we have

$$\sum_{1\leq i_1<\ldots<i_k\leq n-1} \prod_{r=1}^{k} w_{i_r} \tag{14}$$

$$= \sum_{1\leq s_1<\ldots<s_k\leq n} \left(\sum_{1\leq i_1<\ldots<i_k\leq n-1} \left| \det U \begin{pmatrix} i_1 & ,..., & ...i_k \\ s_1 & ,..., & ...s_k \end{pmatrix} \right|^2 \right) \prod_{l=1}^{k} z_{s_l}$$

The vector $diag\left(B^{(k)}\right)^{\mathrm{T}}$ has $\binom{n-1}{k}$ elements equal to $\prod\limits_{r=1}^{k} w_{i_r}$, where $i = (i_1, ..., i_k)$ $\in Q_{k,n-1}$ and $diag\left(A^{(k)}\right)^{\mathrm{T}}$ has $\binom{n}{k}$ elements equal to $\prod\limits_{i=1}^{k} z_{j_i}$, where $j = (j_1, ..., j_k)$ $\in Q_{k,n-1}$. In fact we have

$$\sum_{1\leq i_1<\ldots<i_k\leq n-1} \frac{1}{\binom{n}{k} - \binom{n-1}{k-1}} \prod_{r=1}^{k} w_{i_r} = \sum_{1\leq s_1<\ldots<s_k\leq n} \frac{1 - \sum\limits_{l=1}^{k} p_{s_l}}{\binom{n}{k} - \binom{n-1}{k-1}} \prod_{l=1}^{k} z_{s_l}$$

which for

$$a^{[k]} = \left(\frac{1}{\binom{n-1}{k}}, ..., \frac{1}{\binom{n-1}{k}} \right)$$

and

$$b^{[k]} = \left(\frac{1 - \sum\limits_{l=1}^{k} p_{j_l}}{\binom{n-1}{k}}, ..., \frac{1 - \sum\limits_{l=1}^{k} p_{j_l}}{\binom{n-1}{k}} \right)$$

represents the first condition in Sherman's theorem. □

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