

On some groups related to the Braid Groups of type A

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ABSTRACT. We prove that a family of groups $R(n)$ forms the algebraic structure of an operad and that they admit a presentation similar to that of the Braid groups of type A. This result provides a new proof that the Braid Groups form an operad, a topic emphasized in [19] [18]. These groups proved to be useful in several problems which belong to different areas of Mathematics. Representations of $R(n)$ came from a system of mixed Yang-Baxter type equations. We define the Hopf equation in braided monoidal categories and we prove that representations for our groups came from any braided Hopf algebra with invertible antipode. Using this result, we prove that there is a morphism from $R(n)$ to the mapping class group $\Gamma_{n,1}$, using some results from 3-dimensional topology.

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1. Introduction

In [3], we associated a permutation, to every binary rooted tree and to every edge of the associahedron, which is a special convex polytope studied by Stasheff, May, Loday and other researchers. We generalized that procedure and we defined a sequence of groups $R(n)$ which decorate the associahedron. In the present paper we continue the study of these groups. In [2], there is a different way to associate a permutation to the vertices and to the edges of the associahedron. In the last section of the present paper, we link the groups $R(n)$ to the mapping class groups. These groups play a central role in TQFT (topological quantum field theories). Certainly until a point, the study of these new groups $R(n)$ (classifying spaces, representations, simply-connected spaces acted on by them and a quantum algebra approach) is fruitfully connected with the above mentioned structures. In the present paper, we prove that these group form an operad, using a cabling procedure (an analog of comultiplication in bialgebras, which suggest a relation between these groups and Hopf algebras). Wahl(2001), Salvatore(2006),Tillmann(2006), Blanchet and Marin(2006) pointed cabling phenomenae in Braid theory. We hope that the present paper contribute to a better understanding of these topics.

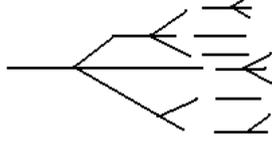
1.1. Operads. Several decades ago, the concept of operad appears in the study of loop spaces, from the point of view of algebraic topology.

An *Operad* is a sequence of *sets*, called sets of operations, or sets of multi-variable functions in analogy with $\text{Map}(X^n, X)$. Each set has an action of the symmetric group

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$S(n)$ (an action on variables or on certain labels). If we ignore the action of the symmetric groups on each set of the sequence, we talk about non- Σ operads [15] [19] [12].



Definition 1.1. A non- Σ operad O is a collection of sets $O(n)$, $n \geq 1$ such that:
 There is a composition law $f: O(m) \otimes O(n_1) \otimes \dots \otimes O(n_m) \longrightarrow O(n_1 + \dots + n_m)$
 There is a unit $e \in O(1)$. $f(g; e, e, \dots, e) = g$ for any $g \in O(k)$.
 The composition law f is associative:

$$f[f(g; g_1, g_2, \dots, g_n); r_1^1, r_2^1, \dots, r_{x_1}^1, r_1^2, r_2^2, \dots, r_{x_2}^2, \dots, r_1^n, r_2^n, \dots, r_{x_n}^n] = f(g; f(g_1; r_1^1, r_2^1, \dots, r_{x_1}^1), f(g_2; r_1^2, r_2^2, \dots, r_{x_2}^2) \dots f(g_n; r_1^n, r_2^n, \dots, r_{x_n}^n))$$

The associativity of the composition law described by the figure above in the case of the operad of binary rooted trees, graded by the number of internal vertices, (the glueing of some trees to the univalent vertices of a bigger tree - defn 1.4.7. pag 49 [15], where several examples of operads are given), shows that it does not matter the order of this operation if we apply it two times:

$Graft(Graft(T; t_j); a_k) = Graft(T; Graft(t_j; a_k))$ Tree operads are involved in the construction of free operads generated by a set V . An element of the free non- Σ operad generated by V is a tree with internal vertices labelled by elements of the vector space V .

2. The groups $R(n)$

Let $R(n)$ be the following group, given by generators and relations:

- generators: $R_{x,y}$, where $1 \leq x < y \leq n$
- relations: $R_{x,y}R_{x,z} = R_{y,z}R_{x,y}R_{x+1,y+1}$ if $x < y < z$
- $R_{x,y}R_{z,t} = R_{z,t}R_{x,y}$ if $x < y < z < t$ and
- $R_{a,y}R_{x,z} = R_{x,z}R_{a+1,y+1}$, where $x < a < y < z$.

The relations above are the relations satisfied by the "left to the right" insertions in the symmetric group.

An insertion in a permutation $a := a(1)a(2)\dots a(n)$ is the following transformation applied to a : insert the element $a(y)$ between two consecutive elements in a : $a(x-1)$ and $a(x)$. Under this transformation, we get the permutation $ar(x,y)$, where $r(x,y)$ is the insertion:

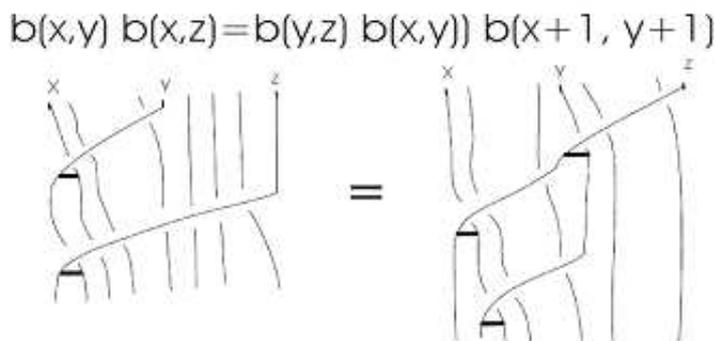
$$r(x,y) = \{ 1, 2, \dots, x, x+1, \dots, y, \dots, n \quad 1, 2, \dots, y, x, \dots, y-1, \dots, n \}.$$

Lemma 2.1. [3] We have a morphism b from $R(n)$ to $B(n)$ which associate to $R_{x,y}$ the insertion braid $b(x,y)$.

The insertion braid $b(x,y)$ is defined as $b(x,y) = \prod_{k=x}^{y-1} s_{x+y-1-k}$, where s_i are the Artin generators of the Braid group. The Braid group $B(n)$ is generated by $n-1$ generators s_i ; non-consecutive generators commute, and two consecutive generators satisfy the braid relation: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

The insertion braids satisfy the defining relations of $R(n)$. The figure below shows the first relation from $R(n)$ satisfied in the Braid Group: the strings from the left

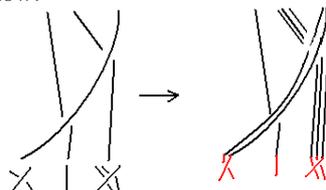
hand side of the figure can be deformed in the 3-dimensional space to the strings from the right hand side.



Theorem 2.1. *The groups $R(n)$, $1 \leq n$, form an operad.*

Proof. To prove the statement, we have to define the "brace map" f , whose outcome is denoted $g[g_1, g_2, \dots, g_n]$, from the definition of the non- Σ operad above, and to check that this composition law satisfy the axioms. The composition law resembles the composition law of a very important operad: the Braid operad. Isolated, the Braid operad was not really studied. We mention several instances or statements where it appears: a braided monoidal category is acted by the Braid operad; the automorphism group of a certain completion of the Braid operad is the famous Grothendieck-Teichmuller group; the braid operad is used to define an operad of Lie algebras associated with the pure braid group. In any case, the cabling procedure, used by Blanchet, Marin, Papadima and Wahl, suggest that any "significant" property or object for the braid group $B(3)$ can be in some way extended to all braid groups $B(n)$.

The Braid groups form an operad using the cabling operation described the figure below:

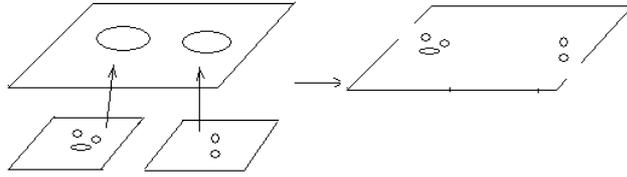


The strings of the principal braid become multiple strings where we insert the secondary braids. The associativity from the definition of an operad is a consequence of this definition. If we have a second level of braids which have to be inserted, we can cable the secondary strings which were not already attached to accomodate the last level of braids, and after that we perform the cabling operation with the principal braid. Or, we can multiply the strings of the principal braid to glue the first level of secondary braids, and repeat the operation with the last level.

The Little Disk Operad, studied by Boardman, May, Stasheff, Salvatore, Wahl ([15] sec 1.17) ([19]), has as spaces $O(n) :=$ the set of embeddings of n disjoint disks in the plane, modulo the action of the symmetric groups on the labelling of the disks. $\pi_1(O(n)) = B(n)$.

The topological spaces involved are $K(\pi, 1)$ and their homology is given by the homology of their fundamental groups.

The composition law of the little disk operad is described by the figure below:



For the groups $R(n)$, given $g \in R(n)$ and $g_i \in R(m_i)$, $1 \leq i \leq n$, we define the following composition $f(g; g_1 \dots g_n)$:

We represent g as a marked braid, as in Lemma 1, using a particular presentation of g as product of generators; we mark the last crossing from the involved insertion braids $b(x,y)$.



Any braid, so any element of $R(n)$ define a permutation (of the tip of each string) σ . The string indexed by i from g will be transformed, as in the Braid Group case, in $m_{\sigma(i)}$ strings, to be composed with $g_{\sigma(i)}$. This cabling of strings generates a cabling of all $R(n)$ -generators involved in that particular representation of g .

- a marked crossing between two strings, after this cabling operation applied to g , will be a crossing between two fascicles of strings, which will be marked as in the figure below. All the strings above receive markers which unite them to only one string (the left most one). So the string fascicle from an under-crossing has only one string united by a marker with all strings above.



The cabling of the first string of $R_{1,2}$ by m and of the second string by n will transform this generator in $R_{1,m+1}R_{2,m+2} \dots R_{n,m+n}$.

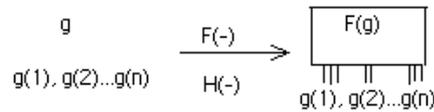
The cabling of the first string of $R_{x,y}$ by m and of the second string by n will transform this generator in $R_{x,y+m-1}R_{x+1,y+m} \dots R_{x+n-1,y+m+n-2}$.

If g is a specific product of generators from $R(n)$, then $F(g)$ is the product of *cabled* generators, where the cabling was described above.

$$f(g; g_1 \dots g_n) = F(g) \prod_{k=1}^n H(g_k) \quad g_k \in R(m_k).$$

$H(g_i)$ is the translation of the element g_i , to be able to be composed with $F(g)$ on the designated position, where the i -th string was cabled. If $g_i = \prod R_{uv}$, then $H(g_i) = \prod R_{u+L, v+L}$; all the indices involved in the $R(n)$ generators are shifted by L ,

where $L = \sum_{k=1}^{i-1} m_k$.

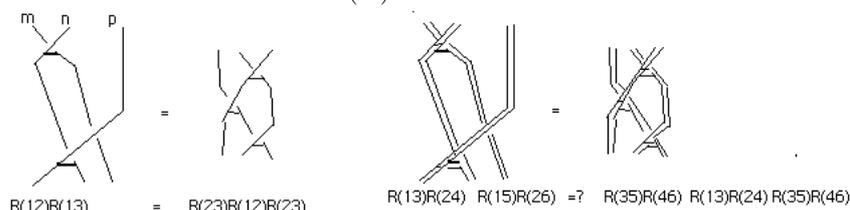


The cabling of g , denoted $F(g)$ is in $R(\sum_{i=1}^n m_i)$. Given a particular representation of g as product of generators, we can build the element $F(g)$: the cabling and the rule above which multiply the marked crossings give us a way to read from top to the bottom of the cabled marked braid diagram, the product $F(g)$ of cabled generators.

For the groups $R(n)$, we algebraically formalized what happens also in the Braid operad, where until a point the strings and cabling have an intuitive geometric description. Algebraic topology definitions as fundamental groups, homology, strengthen this intuition. For the new groups $R(n)$, specific computations are need to be done. We need to prove the following lemma:

Lemma 2.2. $F(g)$ does not depend on the particular representation of g as product of generators.

Proof. Any two representations D_1 and D_2 of g as products of $R(n)$ -generators, are united by a sequence of elementary steps where we apply a relation from $R(n)$. The same cabling applied to build $F(g)$ will unite $F(D_1)$ and $F(D_2)$ by a sequence of elementary steps where we apply a *cabled* relation from $R(n)$. So, we have to prove that string multiplication will transform "primary" equalities given by relations in equalities of two elements from $R(m)$.



We give several exemples where the relation $R(12)R(13) = R(23)R(12)R(23)$ was cabled.

We cable every string of the first relation by $m=n=p=2$ and the right figure above has to represent an equality in $R(n)$.

(A) $R_{1,3}R_{2,4}R_{1,5}R_{2,6} = R_{3,5}R_{4,6}R_{1,3}R_{2,4}R_{3,5}R_{4,6}$ (B)

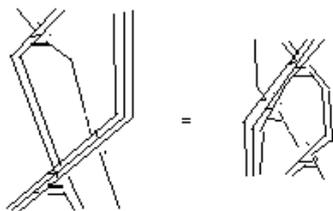
Apply the commutativity relation $R_{a,y}R_{x,z} = R_{x,z}R_{a+1,y+1}$:

$R_{2,4}R_{1,5} = R_{1,5}R_{3,5}$ and $R_{3,5}R_{2,6} = R_{2,6}R_{4,6}$ in (A), and simplify $R_{4,6}$ from both sides.

(A)=(B) $\Leftrightarrow R_{1,3}R_{1,5}R_{2,6} = R_{3,5}R_{4,6}R_{1,3}R_{2,4}R_{3,5} = R_{3,5}R_{1,3}R_{4,6}R_{2,4}R_{3,5}$

$R_{1,3}R_{1,5} = R_{3,5}R_{1,3}R_{2,4}$.

$R_{3,5}R_{1,3}R_{2,4}R_{2,6} = R_{3,5}R_{1,3}R_{4,6}R_{2,4}R_{3,5} \Leftrightarrow R_{2,4}R_{2,6} = R_{4,6}R_{2,4}R_{3,5}$, equality which is a relation in $R(n)$.



In the figure above, $m=1; n=2; p=3$, and we have to prove that:

$R_{12}R_{23}R_{14}R_{25}R_{36} = R_{24}R_{35}R_{46}R_{12}R_{23}R_{34}R_{45}R_{56}$.

The equality is a consequence of the following relations applied in the right hand side:

$R_{46}R_{12}R_{23} = R_{12}R_{23}R_{46}$

$R_{46}R_{34}R_{45} = R_{34}R_{36}$

$R_{35}R_{12} = R_{12}R_{35}$

$R_{35}R_{23}R_{34} = R_{23}R_{25}$

$R_{36}R_{56} = R_{45}R_{36}; R_{25}R_{45} = R_{34}R_{25}$

If in a relation we cable a string which doesn't have any marked line, we get another relation (see the marked insertion braids from Lemma 1). The first string, cabled m times is an undercrossing string, it does not play a fundamental role. We can consider

$m=1$. The equality under cabling of the relation $R(1,2)R(1,3)=R(2,3)R(1,2)R(2,3)$ for $m=1$, any n and p is enough to prove Lemma 2.

For a given (m,n,p) -cabling of the relation $R_{12}R_{13} = R_{23}R_{12}R_{23}$, the left hand side become $A_{m,n,p}=R(1,m+1)R(2,m+2)\dots R(n,m+n)R(1,m+n+1)R(2,m+n+2)\dots R(p,m+n+p)$ the right hand side is $B_{m,n,p}=R(m+1,m+n+1)R(m+2,m+n+2)\dots R(m+p,m+n+p)R(1,m+1)R(2,m+2)\dots R(p+n,p+n+m)$. The cabling of the first string by m is involved in the length of the generators. We just add m to all indices of the generators involved. $\text{length}(R(x,y))=y-x$. As we saw in the examples above, $B_{m,n,p}$ is transformed into $A_{m,n,p}$ applying a sequence of commutativity and pentagonal relations. We do not even need to use the inverse of generators.

$A_{m,n,p+1} = A_{m,n,p}R_{p+1,m+n+p+1}$
 If $B_{m,n,p}=XY$, then $B_{m,n,p+1}=XR_{m+p+1,m+n+p+1}YR_{p+n+1,p+n+m+1}$, where $X=R(m+1,m+n+1)R(m+2,m+n+2)\dots R(m+p,m+n+p)$ and $Y=R(1,m+1)R(2,m+2)\dots R(p+n,p+n+m)$
 If $A_{m,n,p}=B_{m,n,p}$ then $A_{m,n,p+1}=B_{m,n,p+1} \Leftrightarrow$
 $YR_{p+1,p+n+m+1} = R_{m+p+1,m+n+p+1}YR_{p+n+1,p+n+m+1}$
 $\Leftrightarrow Y_{p-1}R_{p+1,p+n+m+1} = R_{m+p+1,m+n+p+1}Y_p$.
 $R_{m+p+1,m+n+p+1}$ commutes with the first p terms of $Y_p=R(1,m+1)R(2,m+2)\dots R(p+n,p+n+m)$, which are also the first p terms from Y_{p-1} . For the next two terms, we apply the following pentagon relation:
 $R_{p+m+1,p+n+m+1}R_{p+1,m+1+p}R_{p+2,m+2+p} = R_{p+1,m+1+p}R_{p+1,p+n+m+1}$ and the result follows by applying commutativity relations.

In conclusion, it is enough to prove the equality $A_{m,n,p} = B_{m,n,p}$ for $m=p=1$ and for any n ; for other cases we apply induction over p .

$$A_{1,n,1} = R(1,2)R(2,3)\dots R(n,n+1)R(1,n+2).$$

$$B_{1,n,1} = R(2,n+2)R(1,2)R(2,3)\dots R(n+1,n+2)$$

We apply $R(x,x+1)R(1,n+2)=R(1,n+2)R(x+1,x+2)$ for $x=n,n-1,n-2\dots 3,2$, and we simplify from the right on both sides (A) and (B).

$$A_{1,n,1} = B_{1,n,1} \Leftrightarrow R(1,2)R(1,n+2)=R(2,n+2)R(1,2)R(2,3) \text{ which is a relation. } \square$$

The groups $R(n)$ form an operad using the well defined F as above if the associativity axiom is satisfied:

$$\begin{aligned} & f[f(g; g_1, g_2, \dots, g_n); r_1^1, r_2^1, \dots, r_{x_1}^1, r_1^2, r_2^2, \dots, r_{x_2}^2, \dots, r_1^n, r_2^n, \dots, r_{x_n}^n] = \\ & = f(g; f(g_1; r_1^1, r_2^1, \dots, r_{x_1}^1), f(g_2; r_1^2, r_2^2, \dots, r_{x_2}^2) \dots f(g_n; r_1^n, r_2^n, \dots, r_{x_n}^n)) \end{aligned}$$

In our case, both terms are equal to $F^2(g) \prod H(F(g_i)) \prod H(r_\beta^\alpha)$.

$F^2(g) = F(F(g))$ is the element obtained from g by applying the cabling operation two times. $H(-)$ is the shifting operator applied to the generators (or product of generators/relations/elements from $R(m)$) of the secondary braids p_y to be able to build $f(w; p_1, p_2, \dots, p_n)$, which is the product between the cabling of w and all $H(p_y)$.

The operad has an unit, the identity of the group $R(1)$. We have the following corollary:

Corollary 2.1. *We can extend the morphism b from Lemma 1 to a morphism of non-symmetric group operads $R(n) \rightarrow B(n)$*

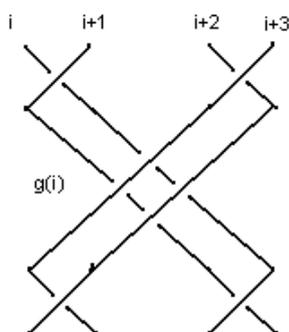
A consequence of the Operad structure of the family of groups $R(n)$ is a *new* proof that the Braid Groups form an operad under cabling, where the cabling operation and the composition law are defined using the insertion braids (which also generate $B(n)$), which satisfy the defining relations of $R(n)$. \square

3. A braid-type presentation of the group $R(n)$

In this section, we will prove the following theorem:

Theorem 3.1. $R(n)$ has the following presentation:
 $R(n)$ is generated by $R(i, i+1)$, where $1 \leq i \leq n-1$; $R(i, i+1)$ satisfy the relations:
 1) $[R(i, i+1), R(j, j+1)] = 0$ if $|i - j|$ is not equal to 1.
 2) For any i , $R(i, i+1)$, $R(i+1, i+2)$ and $R(i+2, i+3)$ satisfy the relation:
 $g(i) R(i+2, i+3) = R(i, i+1) B(i) g(i)$, where: $B(i)$ is the "braid element"
 $(R_{i+1, i+2} R_{i, i+1} R_{i+1, i+2}) (R_{i, i+1} R_{i+1, i+2} R_{i, i+1})^{-1}$
 $g(i) = R(i+2, i+3) v^2 R(i, i+1)$, where $v = R(i, i+1) R(i+1, i+2) R(i+2, i+3)$

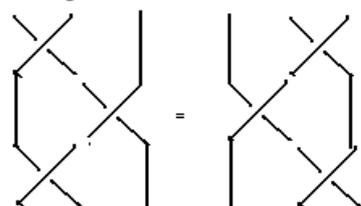
The element $g(i)$ has the following graphical braid -type representation :



The product of the elements corresponding to each crossing, from top to bottom, is equal to $g(i)$.

Because of the pentagonal relation in the initial presentation of $R(n)$, which gives a formula for $R(x, z)$ using generators of smaller length, the group $R(n)$ is generated by $R(i, i+1)$, where $1 \leq i \leq n-1$.

The presentation above is similar to the presentation of the braid group $B(n)$. $B(n)$ is generated by $S(i, i+1)$, where $1 \leq i \leq n-1$. The relation $S(i, i+1)S(i+1, i+2)S(i, i+1) = S(i+1, i+2)S(i, i+1)S(i+1, i+2)$ admits a graphical representation given by the equality of two braids. The diagrams below show two equal elements of $B(n)$, if we replace the crossings by the generators whose indices are on the strings.



Note: During this proof we also use notations $R(xy)$ or $R(x, y)$ for generators of $R(n)$, besides the first defining notation $R_{x, y}$, in order to emphasize a specific relation among them. The proof consists in two main steps. First we show how the prescribed relation above appears; then we will use Lemma 2 to provide an isomorphism between $R(n)$ and the group given by generators and relations as in Theorem 2.

Proof. $R(4)$ is generated by $R(12)$, $R(23)$ and $R(34)$.

$$R_{13} = R(12)^{-1} R(23) R(12) R(23)$$

$$R_{24} = R(23)^{-1} R(34) R(23) R(34)$$

The first relation among $R(12)$, $R(23)$ and $R(34)$ is given by:

$R_{12}R_{14} = R_{24}R_{12}R_{23}$ and $R_{13}R_{14} = R_{34}R_{13}R_{24}$, if we equalize the two expressions for $R(14)$ derived from these.

This relation among $R(12)$, $R(23)$ and $R(34)$ is given by:

$R(14)=R(12)^{-1}R(24)R(12)R(23)=R(13)^{-1}R(34)R(13)R(24) \iff$
 (X) $R_{12}^{-1}R_{23}^{-1}R_{34}R_{23}R_{34}R_{12}R_{23}=R_{23}^{-1}R_{12}^{-1}R_{23}^{-1}R_{12}R_{34}R_{12}^{-1}R_{23}R_{12}R_{23}R_{23}^{-1}R_{34}R_{23}R_{34}$
 The relation $R(23)R(14)=R(14)R(34)$, for $R(14)=R(12)^{-1}R(24)R(12)R(23)$, has the same form (we don't obtain a new relation among $R(i,i+1)$).

The relation (X) is exactly the relation prescribed by the theorem above. More exactly, the relation between $R(12)$, $R(23)$ and $R(34)$ which appears in the theorem 2 above can be transformed into relation (X) using the commutativity between $R(12)$ and $R(34)$ and passing several terms to/from left and right hand sides.

Any diagram D with m strings which represent a trivial braid (or just an equality between two elements of $B(n)$), can be used to define a group $R(D)$, with generators $R(x, x+1)$ and a relation among m consecutive generators prescribed by the diagram D . D represents a trivial braid, so it is a product of conjugates of "Braid diagrams": $(R_{i+1,i+2}R_{i,i+1}R_{i+1,i+2})(R_{i,i+1}R_{i+1,i+2}R_{i,i+1})^{-1}$. For the groups $R(n)$, it is remarkable its operadic structure and the fact that commutativity relations does not bring other new relations.

For any indices x,y,z,t the generators R_{xy}, R_{yz}, R_{zt} verifies the same relation as the relation obtained above, the calculations being the same (we change 1,2,3,4 by x,y,z,t).

Let $C(n)$ be the group generated by $R(i,i+1)$, with the commutativity and generalized braid type relation among 3 consecutive generators as above. We can define elements $R(x,y)$ by induction over $y-x$, using the pentagonal relation from $R(n)$.

$R(k,k+2)$ are defined as $R(13), R(24)$ above, using the basic generators $R(i,i+1)$. $R(k,k+3)$ can be defined in two ways using generators of length 1 and 2. The relation between 3 consecutive generators assure us that the result will be the same element of $R(n)$ - the defining relation of $C(n)$ appears here.

$R(k,k+p+1)$ is involved in p pentagonal relations from where we can define $R(k,k+p+1)$ using generators of smaller length, according to the defining relations of $R(n)$.

Suppose we already build $R(x,x+p)$ which satisfy the relations from $R(n)$, together with the other elements $R(j,j+p-h)$.

There are p ways to define $R(k,k+p+1)$ according to the pentagonal relation. We want to prove that all are equal to the same element, denoted $R(k,k+p+1)$.

We use induction over p applied to the following statement: any two ways to define $R(k,k+p)$ using pentagonal relations are equal being united by a sequence of steps where we apply pentagonal relations of smaller length where the already defined $R(j,j+p-h)$ are involved.

Let $F(1)$ and $F(2)$ be two expressions obtained from pentagonal relations which define $R(k,k+p)$ in the group $C(n)$ and which are equal by induction hypothesis. There are united by a sequence of steps where we apply the defining relations of $R(n)$, where generators of smaller length are involved.

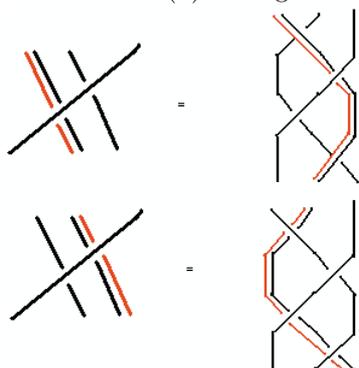
We cable in $F(i)$, as well in the relations involved in the path above, the same string α . The cabled relations are equalities of elements from $R(n)$ (Lemma 2), which unite $\text{Cab}(F(1))$ and $\text{Cab}(F(2))$. They represent the same element denoted $R_\alpha(k, k+p+1)$, which depends on the cabled string α .

$Cab(F(i))_\alpha$ are two pentagonal relations which theoretically can be used to define $R(k,k+p+1)$.

We want to prove that $R_\alpha(k,k+p+1)=R_\beta(k,k+p+1)$.

If $\beta=\alpha+1$ and $2 \leq \alpha$, then $R_\alpha(k,k+p+1)$ and $R_\beta(k,k+p+1)$ are equal as words, not only as elements from the same group.

If $\alpha=1$ and $\beta=2$, a direct calculus shows that the expressions are equal, by applying relations from $R(n)$ of length at most p .



The doubling of the first string from $R(13)$ generates $R(23)^{-1}R(12)^{-1}R(24)R(12)R(23)R(34)$. The doubling of the second string from $R(13)$ generates $R(13)^{-1}R(34)R(13)R(24)$.

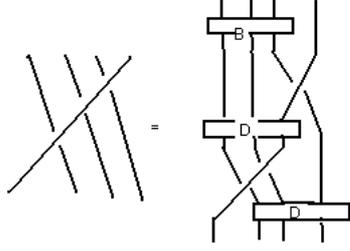
The two elements are equal by applying the pentagonal relations for $R(13)$ and $R(24)$. We don't apply the generalized braid relation (which was already used to define $R(14)$, $R(25)$..) The general case, for $R(1,x+1)$ is similar: we define $R(1,x+2)$ as the cabling of a product of generators which define $R(1,x+1)$ according to a pentagonal relation. The cabling does not depend on the pentagonal relation used, nor on the cabled string.

There is a surjection f onto $R(n)$, from the group $C(n)$, generated by $R(i,i+1)$, where $1 \leq i \leq n-1$; $R(i,i+1)$ satisfy the relations:

- 1) $[R(i,i+1), R(j,j+1)]=0$ if $|i-j|$ is not equal to 1.
 - 2) For any i , $R(i,i+1)$, $R(i+1,i+2)$ and $R(i+2,i+3)$ satisfy the relation: $g(i) R(i+2,i+3)=R(i,i+1) B(i) g(i)$, where: $B(i)$ is the "braid element" $(R_{i+1,i+2}R_{i,i+1}R_{i+1,i+2})(R_{i,i+1}R_{i+1,i+2}R_{i,i+1})^{-1}$
- $g(i)=R(i+2,i+3) v^2 R(i,i+1)$, where $v=R(i,i+1)R(i+1,i+2)R(i+2,i+3)$

In $C(n)$ we can define the elements $R(x,y)$ as above, which satisfy the same relations as $R_{x,y}$ from $R(n)$. f has an inverse g , which sends the generators $R_{x,y}$ into the elements $R(x,y)$. □

The fundamental relation between 3 consecutive generators given by Theorem 2 above can be written in many ways. A second way is given by the diagram below. There are other forms which are equivalent as words; we use only xx^{-1} cancellation, commutativity between non-consecutive generators and we transfer several terms from the right to the left hand side of the generalized braid relation prescribed by the theorem above. Any sequence of Reidemeister moves which transforms one into the other give rise to a system of mixed Yang-Baxter type equations on multiple tensor product of a vector space M .



4. Hopf equation in braided monoidal categories

In this section, we prove that a solution of a certain system provides representations for $R(n)$

A particular case of the system, (when $c =$ regular flip $(x,y) \dashrightarrow (y,x)$) is the much studied Hopf equation .

We diagrammatically prove that the system make sense in any braided monoidal category. We assume the reader is familiar with basic notions of Hopf algebras in the category of Vector Spaces, with braided monoidal categories and with braided Hopf algebras, in the sense of Majid.

Let M be a vector space.

Let R and c be invertible operators which satisfy the system:

$$R(23)R(12)R(23) = R(12)c(23)R(12)$$

$$R(23)c(12)c(23) = c(12)c(23)R(12)$$

$$R, c: M \otimes M \rightarrow M \otimes M.$$

The indices are the positions on $M \otimes M \otimes M$ where we apply R or c , on the other positions we apply Id_M .

We use group-like notation for invertible maps , so fgh means $h(g(f(x)))$, x in the domain of f .

Theorem 4.1. : *If (R,c) is a solution of the system above, then R satisfies the defining relation for the group $R(n)$, so R gives a representation for the groups $R(n)$ on $(M^{\otimes n})$, by assigning to $R(j,j+1)$ the operator $id \otimes R_{i,i+1} \otimes id$.*

Proof. Let $R: M \otimes M \rightarrow M \otimes M$ be an invertible linear operator.

We consider the following equation:

$$B_{123}R_{34}D_{123}R_{23}R_{12}D_{234} = R_{34}R_{23}R_{12}, \text{ where } D \text{ is the inverse of the "braid operator" } B: H \otimes H \otimes H \rightarrow H \otimes H \otimes H. B = (R_{23}R_{12}R_{23})(R_{12}R_{23}R_{12})^{-1}$$

The equation represents the equality of two operators, built using R and defined from $M \otimes M \otimes M \otimes M \rightarrow M \otimes M \otimes M \otimes M$. The indices attached to R, D, B show the positions from the tensor product where these operators act, on the remaining positions the action is given by identity.

We expand the equation above, which is the defining equation for the groups $R(n)$. Let S the inverse of the operator R .

$$\begin{aligned} R_{23}R_{12}R_{23}S_{12}S_{23}S_{12}R_{34}R_{12}R_{23}R_{12}S_{23}S_{12}S_{23}R_{23}R_{12}R_{23}R_{34}R_{23}S_{34}S_{23}S_{34} &= \\ R_{23}R_{12}R_{23}S_{12}S_{23}R_{34}R_{23}R_{12}R_{34}R_{23}S_{34}S_{23}S_{34} &= \\ R_{12}c_{23}R_{12}S_{12}S_{23}R_{34}R_{23}R_{12}R_{34}R_{23}S_{34}S_{23}S_{34} &= \\ R_{12}c_{23}S_{23}R_{34}R_{23}R_{12}R_{34}R_{23}S_{34}S_{23}S_{34} &= \\ R_{12}c_{23}S_{23}R_{34}R_{23}R_{34}R_{12}R_{23}S_{34}S_{23}S_{34} &= \\ R_{12}c_{23}S_{23}R_{23}c_{34}R_{23}R_{12}R_{23}S_{34}S_{23}S_{34} &= \\ R_{12}c_{23}c_{34}R_{23}R_{12}R_{23}S_{34}S_{23}S_{34} &= \end{aligned}$$

$$\begin{aligned}
 R_{12}R_{34}c_{23}c_{34}R_{12}R_{23}S_{34}S_{23}S_{34} &= \\
 R_{34}R_{12}c_{23}R_{12}c_{34}R_{23}S_{34}S_{23}S_{34} &= \\
 R_{34}R_{23}R_{12}R_{23}c_{34}R_{23}S_{34}S_{23}S_{34} &= R_{34}R_{23}R_{12}
 \end{aligned}$$

□

We sketch a second, non-rigorous proof, based on the representation of $R(x,y)$ as marked braids:

Define $r(x,y) = \prod_{k=x}^{y-2} c_{x+y-1-k} R_x$, where the indices show the first position in the tensor product of M 's where we apply c or R . We represent the generators $R(x,y)$ as marked braids and we "color" the last (marked) crossing by R , and the remaining crossing by c (as in Lemma 1). The second equation of the system shows that we can push the R -marking of the last crossing of $R(x,y)$ above; the first equation implies, for $R(1,3) = c(23)R(12)$, $r(x,y)r(x,z) = r(y,z)r(x,y)r(x+1,y+1)$ for $x=1, y=2$ and $z=3$. For any y and z , using the second equality of the system, by induction over $y+z$, the equality is equivalent to the first one.

An example of solution for the system above is given by the Takesaki operator (as Militaru mentioned in his series of papers dedicated to the Hopf equation) for a Hopf algebra A , $T: (a,b) \rightarrow (a_1, a_2, b) \rightarrow (a_1b, a_2)$ and the regular flip. T is also called fusion operator, in a language close to physicists. We will use Sweedler notation and we omit the sum for elements of tensor products.

$T(12)\text{flip}(23)T(12): (a,b,c) \rightarrow (a_1b, a_2, c) \rightarrow (a_1b, c, a_2) \rightarrow ((a_1b)_1c, (a_1b)_2, a_2)$
 $T(23)T(12)T(23): (a,b,c) \rightarrow (a, b_1c, b_2) \rightarrow (a_1(b_1c), a_2, b_2) \rightarrow (a_1(b_1c), (a_2)_1b_2, (a_2)_2)$
 both terms are equal to (a_1b_1c, a_2b_2, a_3) . It is a consequence of the axioms of the bialgebra A .

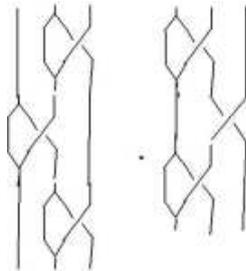
Let C be a strict braided monoidal category, with braiding c (Defn 4.6 [19], [17]). Let B be a braided Hopf algebra in C . So, B is an object in this category, together with a comultiplication

$\delta: B \rightarrow B \otimes B$, multiplication m , unit and counit ϵ which are morphisms in C , which satisfy the usual axioms for a Hopf algebra. δ is a braided algebra morphism:

$$\delta(xy) = (m \otimes m)(id \otimes c \otimes id)(\delta(x) \otimes \delta(y))$$

We can define a fusion operator $T(b \otimes c) = (m \otimes id) \circ (id \otimes c) \circ (\delta \otimes id)$

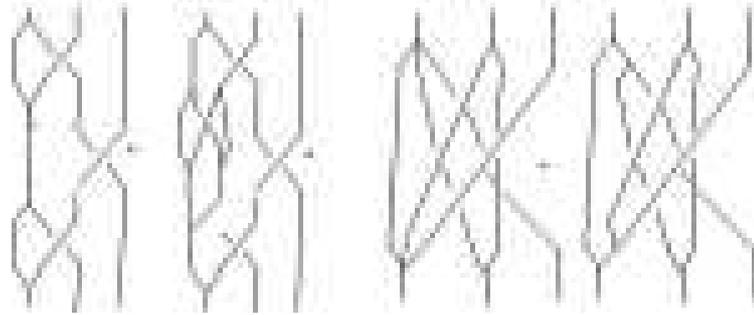
In a strict braided monoidal category, the system introduced above make sense. The first equation of the system (let us call it: the Hopf equation for a braided Hopf algebra in C), has the following diagramatic form:



The braiding morphism is represented diagrammatically as a crossing; the trivalent graphs represent multiplication or comultiplication. The composition of morphisms are read from top to the bottom of the figure.

Theorem 4.2. *The Hopf equation is verified for (T, c) , where c is a braiding of a braided monoidal category, and T is the fusion operator associated with a braided Hopf algebra B in this category.*

Proof. We provide a diagrammatic proof. In the figure below, there are equalities of morphisms in the braided category C ; the morphisms are composed from top to the bottom. The naturality of the braiding allow as to flow trivalent vertices on strings.



We used the axiom $\delta(xy) = (m \otimes m)(id \otimes c \otimes id)(\delta(x) \otimes \delta(y))$, the coassociativity and the associativity of co-multiplication and multiplication of B .

The diagrammatic proof shows that we used the axioms described below:



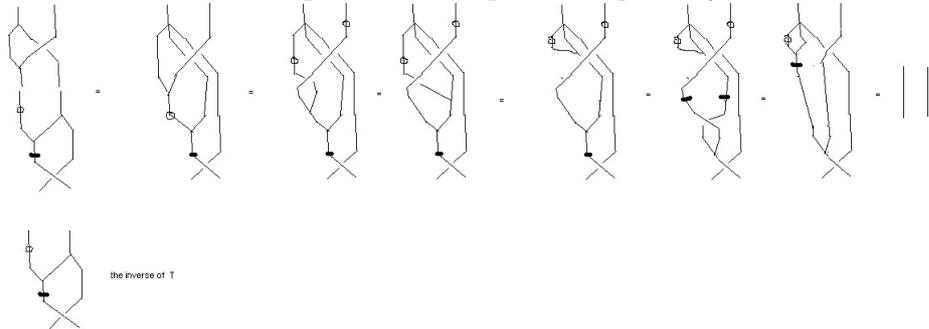
□

In a regular Hopf algebra A with bijective antipode, the inverse of the fusion operator T is $F(a, b) = (b_2, S^{-1}(b_1)a)$.

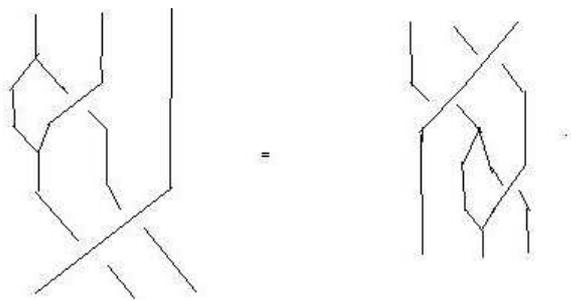
In a braided Hopf algebra with invertible antipode S , the inverse of the fusion operator is

$$c^{-1} \circ (S^{-1} \otimes id) \circ (m \otimes id) \circ (S \otimes \delta).$$

We provide a diagrammatic proof that the above mentioned operator is the inverse of T . The circle and the dot represent the antipode S , respectively its inverse.



The second equation from the system is also satisfied by (T, c) . We used the naturality of the braiding with respect to all the morphisms from the category.



Corollary 4.1. *Any braided Hopf algebra B with invertible antipode provides representations for $R(n)$. There is a group morphism between $R(n)$ and $\text{Aut}(B^{\otimes n})$, which sends $R_{x,x+1}$ into $\text{Id}_{x-1} \otimes T \otimes \text{Id}_{n-x}$, where T is the fusion operator of the braided Hopf algebra B .*

4.1. Future directions. The Mapping class groups and the Cobordism categories of TQFT. Invariants of trivalent knotted graphs which obey the axioms from the last figure above generates invariants for elements of $R(n)$. Examples of braided Hopf algebras, in the braided category of Yetter-Drinfeld modules can be found in [4] [14]. A geometric type of braided Hopf algebra with invertible antipode is provided by the study of certain 3- cobordism categories, suitable to be the source for TQFT (topological quantum field theory) in the sense of Atiyah. According to theorem 4, a braided Hopf algebra will generate a group morphism between $R(n)$ and $\text{Aut}_{\text{Cob}}(B^{\otimes n})$.

The mapping class group $\Gamma_{n,1}$ is the group of isotopy classes of homeomorphisms of an oriented surface of genus n , with one boundary component S^1 , which fix the boundary pointwise. There are two presentations of this group, the first one by generators and relations given by Wajnryb (see also [8] Theorem A and [16] Sec. 6). The second geometric presentation ([16] Thm. 4.1) is given by an isomorphism between $\Gamma_{n,1}$ and a subset of the set of $4n$ -tangles (a q -tangle is a one dimensional manifold in R^3 whose boundary is a fixed set of q points- a generalization of a geometric braid), modulo certain relations called Kirby moves.

The special Lagrangian Mapping class group $L^s\Gamma_{n,1}$ is the subgroup of $\Gamma_{n,1}$, of isotopy classes of homeomorphisms which can be extended to the solid torus of genus n (the 3-solid sphere with n handles attached). It is an infinite index subgroup and a presentation by generators and relations is not known. It was studied in [9] [10](2006) [6] (2008). Using Kontsevich integral, Cheptea , Habiro and Massuyeau found an infinite dimensional linear representation of this group (Lemma 5.5 [6]).Restricting the set of morphisms, they were able to construct an anomaly-free TQFT, able to produce linear representations (not just projective ones) for certain subgroups of the mapping class groups.

In [11] (Thm 1.), Kerler introduced a functor G between the two braided categories Alg and Cob . Alg is the free braided tensor category freely generated by a braided Hopf algebra object A with a ribbon element. The objects of this category are natural numbers. The morphisms are words generated by multiplication , comultiplication, unit , counit, antipode and ribbon, modulo the relations satisfied in any ribbon Hopf algebra. Cob is the category with objects $\Sigma_{g,1} :=$ oriented surfaces with g handles and one boundary component. The morphisms $\text{Mor}(x,y)$ are give by homeomorphism classes of 3-manifolds which bound the $(x+y)$ -handlebody. The functor G produces a braided Hopf algebra object in Cob , given by $\Sigma_{1,1}$ (see also [9] Sec. 6.1).

Theorem 4.3. ([11], Theorem 4) $Aut(\Sigma_{g,1})$ is isomorphic with the mapping class group of $\Sigma_{g,1}$

We can consider just the subcategories of the categories above with the same set of objects, and the set of morphisms given by *invertible* morphisms. Combined with this theorem, we have a the following corollary of Theorem 4 and Cor.2 above: there is a morphism T , from $R(n)$ to the mapping class group of $\Sigma_{g,1}$.

$$T(R_{i,i+1}) = G(Id_{x-1} \otimes \text{the fusion operator of } A \otimes Id_{n-x}).$$

$$G(A^{\otimes n}) = \Sigma_{n,1}.$$

$G(Aut(A^{\otimes n}))$ is included in $L^s\Gamma_{n,1}$, a subset of $Aut(\Sigma_{g,1})$. ([6] Sec. 5.1).

Using the theorems above and the fusion operator of the braided Hopf algebra $\Sigma_{1,1}$, the groups $R(n)$ provide invertible morphisms in the category Alg .

Bar-Nathan and several researchers computed the Kontsevich integral of the unknot, v . The fundamental relations used to compute v are: $n \cdot 0 = 0$, the connected cabling of the unknot is the unknot with a new framing, and the behavior of the Kontsevich integral under connected cabling and frame-changing. The computation of v opened the door of computation of the Kontsevich integral of other knots and links. Knots do not posses in themselves any algebraic structure.

We speculate that the relations between $R(x,y)$ and the cabling from the operad, transferred using Cheptea and Habiro TQFT functor in the algebra of Chinese characters via the representation of the mapping class group as tangles modulo Kirby relations, are able to provide the algebraic relations needed to compute the Kontsevich integral of the unknot.

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