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On a weaker form of ω-continuity

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Abstract. In [5], Hdeib introduced and investigated a new type of continuity called ω-continuity. In [1], Al-Omari and Noorani have introduced the notion of almost weak ω-continuity. It is the objective of this paper to study almost weak ω-continuity and present some of its basic properties.

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1. Introduction

In this paper, a space will always mean a topological space on which no separation axioms assumed unless explicitly stated.

A subset A of a space (X, τ) is called ω-closed [4] if it contains all its condensation points. The complement of an ω-closed set is called ω-open, or equivalently, if for each $x \in A$ there exists an open set U containing x such that $\mid U \setminus A \mid \leq \aleph_0$ (see [8]). The family of all ω-open subsets of a space (X, τ), denoted by $\omega O(X)$, forms a topology on X finer than τ.

ω-closure and ω-interior of a subset A of a space X, that were defined in an analogous manner to $cl(A)$ and $int(A)$, respectively, will be denoted by ω-$cl(A)$ and ω-$int(A)$, respectively.

Definition 1.1. *A subset A is said to be*
(1) regular open [9] if $A = int(cl(A))$,
(2) regular closed [9] if $A = cl(int(A))$,
(3) preopen [7] if $A \subset int(cl(A))$.

A point $x \in X$ is said to be in the θ-closure [10] of a subset A of X, denoted by θ-$cl(A)$, if $cl(G) \cap A \neq \emptyset$ for each open set G of X containing x. A subset A of a space X is called θ-closed if $A = \theta$-$cl(A)$. The complement of a θ-closed set is called θ-open.

Lemma 1.1. *([4]) Let A be a subset of a space X. Then*
(1) A is ω-closed in X if and only if $A = \omega$-$cl(A)$.
(2) ω-$cl(X \setminus A) = X \setminus \omega$-$int(A)$.
(3) ω-$cl(A)$ is ω-closed in X.
(4) $x \in \omega$-$cl(A)$ if and only if $A \cap G \neq \emptyset$ for each ω-open set G containing x.

Definition 1.2. *A function $f : X \to Y$ is said to be ω-continuous [5] if $f^{-1}(A) \in \omega O(X)$ for each open set A of Y.*

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2. Almost weakly ω-continuous functions

Definition 2.1. *A function $f : X \to Y$ is said to be*

(1) almost weakly ω-continuous at $x \in X$ [1] if for each open set A of Y containing $f(x)$, there exists an ω-open set B containing x such that $f(B) \subset cl(A)$.

(2) almost weakly ω-continuous [1] if for each $x \in X$, f is almost weakly ω-continuous at $x \in X$.

Remark 2.1. *(1) Every weakly continuous function is almost weakly ω-continuous [1].*

(2) Every ω-continuous function is almost weakly ω-continuous.

(3) None of the above implications is reversible as shown in the following example and in [1].

Example 2.1. *Let $X = \{a, b, c, d\}$ and $\sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Consider the set of real numbers R with the standard topology τ. Then the function $f : (R, \tau) \to (X, \sigma)$ defined by $f(x) = \begin{cases} a & x \in R \setminus Q \\ c & x \in Q \end{cases}$, where Q is the rational numbers is almost weakly ω-continuous but it is not ω-continuous.*

Theorem 2.1. *The following are equivalent for a function $f : X \to Y$:*

(1) f is almost weakly ω-continuous,

(2) ω-$cl(f^{-1}(int(cl(A)))) \subset f^{-1}(cl(A))$ for every subset A of Y,

(3) ω-$cl(f^{-1}(int(K))) \subset f^{-1}(K)$ for every regular closed set K of Y,

(4) ω-$cl(f^{-1}(B)) \subset f^{-1}(cl(B))$ for every open set B of Y,

(5) $f^{-1}(B) \subset \omega$-$int(f^{-1}(cl(B)))$ for every open set B of Y,

(6) ω-$cl(f^{-1}(B)) \subset f^{-1}(cl(B))$ for each preopen set B of Y,

(7) $f^{-1}(B) \subset \omega$-$int(f^{-1}(cl(B)))$ for each preopen set B of Y.

Proof. (1) $\Rightarrow$ (2) : Let $A \subset Y$ and $x \in X\backslash f^{-1}(cl(A))$. We have $f(x) \in Y\backslash cl(A)$. This implies that there exists an open set B containing $f(x)$ such that $B \cap A = \emptyset$. Also, $cl(B) \cap int(cl(A)) = \emptyset$. Since f is almost weakly ω-continuous, then there exists an ω-open set S containing x such that $f(S) \subset cl(B)$. We have $S \cap f^{-1}(int(cl(A))) = \emptyset$ and hence $x \in X\backslash\omega$-$cl(f^{-1}(int(cl(A))))$. Thus, ω-$cl(f^{-1}(int(cl(A)))) \subset f^{-1}(cl(A))$.

(2) $\Rightarrow$ (3) : Let K be any regular closed set in Y. We have

$$\begin{aligned} \omega - cl(f^{-1}(int(K))) &= \omega - cl(f^{-1}(int(cl(int(K))))) \\ &\subset f^{-1}(cl(int(K))) = f^{-1}(K). \end{aligned} \tag{1}$$

(3) $\Rightarrow$ (4) : Let B be an open subset of Y. Since $cl(B)$ is regular closed in Y, ω-$cl(f^{-1}(B)) \subset \omega$-$cl(f^{-1}(int(cl(B)))) \subset f^{-1}(cl(B))$.

(4) $\Rightarrow$ (5) : Let B be any open set of Y. Since $Y\backslash cl(B)$ is open in Y, by Lemma 1.1,

$$\begin{aligned} X \setminus \omega - int(f^{-1}(cl(B))) &= \omega - cl(f^{-1}(Y \setminus cl(B))) \\ &\subset f^{-1}(cl(Y \setminus cl(B))) \subset X \setminus f^{-1}(B). \end{aligned} \tag{2}$$

Thus, $f^{-1}(B) \subset \omega$-$int(f^{-1}(cl(B)))$.

(5) $\Rightarrow$ (1) : Let $x \in X$ and B be any open subset of Y containing $f(x)$. We have $x \in f^{-1}(B) \subset \omega$-$int(f^{-1}(cl(B)))$. Take $S = \omega$-$int(f^{-1}(cl(B)))$. This implies that $f(S) \subset cl(B)$ and hence f is almost weakly ω-continuous at x in X.

(1) $\Rightarrow$ (6) : Let B be any preopen set of Y and $x \in X\backslash f^{-1}(cl(B))$. Then there exists an open set R containing $f(x)$ such that $R \cap B = \emptyset$. We have $cl(R \cap B) = \emptyset$.

Since B is preopen, then

$$B \cap cl(R) \subset int(cl(B)) \cap cl(R) \subset cl(int(cl(B)) \cap R) \subset cl(int(cl(B) \cap R)) \subset cl(int(cl(B \cap R))) \subset cl(B \cap R) = \emptyset. \tag{3}$$

Since f is almost weakly ω-continuous and R is an open set containing $f(x)$, there exists an ω-open set S in X containing x such that $f(S) \subset cl(R)$. We have $f(S) \cap B = \emptyset$ and hence $S \cap f^{-1}(B) = \emptyset$. This implies that $x \in X \backslash \omega\text{-}cl(f^{-1}(B))$ and hence $\omega\text{-}cl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(6) $\Rightarrow$ (7) : Let B be any preopen set of Y. Since $Y \backslash cl(B)$ is open in Y, by Lemma 1.1,

$$X \setminus \omega - int(f^{-1}(cl(B))) = \omega - cl(f^{-1}(Y \setminus cl(B))) \subset f^{-1}(cl(Y \setminus cl(B))) \subset X \setminus f^{-1}(B). \tag{4}$$

Thus, $f^{-1}(B) \subset \omega\text{-}int(f^{-1}(cl(B)))$.

(7) $\Rightarrow$ (1) : Let $x \in X$ and B any open set of Y containing $f(x)$. Then $x \in f^{-1}(B) \subset \omega\text{-}int(f^{-1}(cl(B)))$. Take $S = \omega\text{-}int(f^{-1}(cl(B)))$. Then $f(S) \subset cl(B)$ and hence f is almost weakly ω-continuous at x in X.

Definition 2.2. *A function $f : X \to Y$ is said to be (ω, s)-open if $f(B) \in SO(Y)$ for every ω-open set B of X.*

Definition 2.3. *A function $f : X \to Y$ is said to be neatly weak ω-continuous if for each $x \in X$ and each open set A of X containing $f(x)$, there exists an ω-open set B containing x such that $Int(f(B)) \subset Cl(A)$.*

Theorem 2.2. *If a function $f : X \to Y$ is neatly weak ω-continuous and (ω, s)-open, then f is almost weakly ω-continuous.*

Proof. Let $x \in X$ and $A \in SO(Y, f(x))$. Since f is neatly weak ω-continuous, there exists an ω-open set B of X containing x such that $Int(f(B)) \subset Cl(A)$. Since f is (ω, s)-open, then $f(B) \in SO(Y)$. This implies that $f(B) \subset Cl(Int(f(B))) \subset Cl(A)$. Therefore f is almost weakly ω-continuous.

Definition 2.4. *A function $f : X \to Y$ is relatively weak ω-continuous if for each open set A of Y, the set $f^{-1}(A)$ is ω-open in the subspace $f^{-1}(Cl(A))$.*

Theorem 2.3. *A function $f : X \to Y$ is ω-continuous if and only if f is almost weakly ω-continuous and relatively weak ω-continuous.*

Proof. "Necessity". Obvious.
"Sufficiency". Suppose that A is an open set of Y. Since f is relatively weak ω-continuous, there exists an ω-open set B of X such that $f^{-1}(A) = B \cap f^{-1}(Cl(A))$. We have $f^{-1}(A) \subset \omega\text{-}int(f^{-1}(cl(A)))$. Therefore, $f^{-1}(A) = B \cap \omega\text{-}int(f^{-1}(cl(A)))$. This shows that $f^{-1}(A)$ is ω-open in X and hence f is ω-continuous.

Definition 2.5. *A function $f : X \to Y$ is said to be (ω, p)-continuous if $f^{-1}(A) \in \omega O(X)$ for each preopen set A of Y.*

Definition 2.6. *A function $f : X \to Y$ is relatively weak ωp-continuous if for each preopen set A of Y, the set $f^{-1}(A)$ is ω-open in the subspace $f^{-1}(Cl(A))$.*

Observe that relatively weak ω-continuous and relatively weak ωp-continuous are equivalent with each other.

Theorem 2.4. *A function $f : X \to Y$ is (ω, p)-continuous if and only if f is almost weakly ω-continuous and relatively weak ωp-continuous.*

Proof. Similar to the proof of Theorem 2.3.

Theorem 2.5. *The following are equivalent for a function $f : X \to Y$:*
(1) f is almost weakly ω-continuous,
(2) $f(\omega\text{-}cl(A)) \subset \theta\text{-}cl(f(A))$ for each subset A of X,
(3) $\omega\text{-}cl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}cl(B))$ for each subset B of Y,
(4) $\omega\text{-}cl(f^{-1}(int(\theta\text{-}cl(B)))) \subset f^{-1}(\theta\text{-}cl(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2) : Let $A \subset X$ and $x \in \omega\text{-}cl(A)$. Suppose that U is any open set of Y containing $f(x)$. Then there exists an ω-open set S containing x such that $f(S) \subset cl(U)$. Since $x \in \omega\text{-}cl(A)$, by Lemma 1.1, $S \cap A \neq \emptyset$. Thus, $\emptyset \neq f(S) \cap f(A) \subset cl(U) \cap f(A)$ and hence $f(x) \in \theta\text{-}cl(f(A))$. Thus, $f(\omega\text{-}cl(A)) \subset \theta\text{-}cl(f(A))$.

(2) $\Rightarrow$ (3) : Let $B \subset Y$. We have $f(\omega\text{-}cl(f^{-1}(B))) \subset \theta\text{-}cl(B)$. Thus, $\omega\text{-}cl(f^{-1}(B)) \subset f^{-1}(\theta\text{-}cl(B))$.

(3) $\Rightarrow$ (4) : Let $B \subset Y$. Since $\theta\text{-}cl(B)$ is closed in Y, then

$$\begin{aligned} \omega - cl(f^{-1}(int(\theta - cl(B)))) &\subset f^{-1}(\theta - cl(int(\theta - cl(B)))) \\ &= f^{-1}(cl(int(\theta - cl(B)))) \\ &\subset f^{-1}(\theta - cl(B)). \end{aligned} \tag{5}$$

(4) $\Rightarrow$ (1) : Let U be any open set of Y. Then $U \subset int(cl(U)) = int(\theta\text{-}cl(U))$. Thus,

$$\omega - cl(f^{-1}(U)) \subset \omega - cl(f^{-1}(int(\theta - cl(U)))) \subset f^{-1}(\theta - cl(U)) = f^{-1}(cl(U)). \tag{6}$$

By Theorem 2.1, f is almost weakly ω-continuous.

Theorem 2.6. *The following hold for a function $f : X \to Y$:*
(1) If f is almost weakly ω-continuous, then $f^{-1}(A)$ is ω-closed in X for every θ-closed set A of Y.
(2) If f is almost weakly ω-continuous, then $f^{-1}(A)$ is ω-open in X for every θ-open set A of Y.
(3) If $f^{-1}(\theta\text{-}cl(A))$ is ω-closed in X for every subset A of Y, then f is almost weakly ω-continuous.

Proof. (1) and (2) follows from Theorem 2.5.

(3) Let $A \subset Y$. Since $f^{-1}(\theta\text{-}cl(A))$ is ω-closed in X, then $\omega\text{-}cl(f^{-1}(A)) \subset \omega\text{-}cl(f^{-1}(\theta\text{-}cl(A))) = f^{-1}(\theta\text{-}cl(A))$. By Theorem 2.5, f is almost weakly ω-continuous.

A space X is called p-space if countable intersections of open subsets are open.

Theorem 2.7. *The following are equivalent for a function $f : X \to Y$ where X is a p-space:*
(1) f is almost weakly ω-continuous,
(2) f is weakly continuous.

Proof. It follows from the fact that $\tau = \omega O(X)$ [2].

Lemma 2.1. *If $f : X \to Y$ is almost weakly ω-continuous and $g : Y \to Z$ is continuous, then the composition $gof : X \to Z$ is almost weakly ω-continuous.*

Proof. Let $x \in X$ and A be an open set of Z containing $g(f(x))$. Then $g^{-1}(A)$ is an open set of Y containing $f(x)$. This implies that there exists an ω-open set B containing x such that $f(B) \subset cl(g^{-1}(A))$. Since g is continuous, then $(gof)(B) \subset g(cl(g^{-1}(A))) \subset cl(A)$. Hence, gof is almost weakly ω-continuous.

Theorem 2.8. *Let $\{A_i : i \in I\}$ be an ω-open cover of a space X. The following are equivalent for a function $f : X \to Y$:*
(1) f is almost weakly ω-continuous,
(2) for each $i \in I$, the restriction $f_{A_i} : A_i \to Y$ is almost weakly ω-continuous.

Proof. (1) $\Rightarrow$ (2) : Let $i \in I$ and A_i be an ω-open set of X. Suppose that $x \in A_i$ and U is an open set of Y containing $f_{A_i}(x) = f(x)$. Since f is almost weakly ω-continuous, then there exists an ω-open set B containing x such that $f(B) \subset cl(U)$. On the other hand, $B \cap A_i$ is ω-open in A_i containing x and $f_{A_i}(B \cap A_i) = f(B \cap A_i) \subset f(B) \subset cl(U)$. Thus, f_{A_i} is almost weakly ω-continuous.

(2) $\Rightarrow$ (1) : Let $x \in X$ and U be an open set containing $f(x)$. There exists $i \in I$ such that $x \in A_i$. Since $f_{A_i} : A_i \to Y$ is almost weakly ω-continuous, there exists an ω-open set B in A_i containing x such that $f_{A_i}(B) \subset cl(U)$. Since A_i is ω-open in X, then B is ω-open in X containing x and $f(B) \subset cl(U)$. Thus, f is almost weakly ω-continuous.

Definition 2.7. *A function $f : X \to Y$ is said to be faintly ω-continuous if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists an ω-open set U containing x such that $f(U) \subset V$.*

Theorem 2.9. *Let $f : X \to Y$ be a function. The following are equivalent:*
(1) f is faintly ω-continuous,
(2) $f^{-1}(A)$ is ω-open in X for every θ-open set A of Y,
(3) $f^{-1}(B)$ is ω-closed in X for every θ-closed set B of Y.

Theorem 2.10. *Let $f : X \to Y$ be a function. If Y is regular, then the following are equivalent. Otherwise, the implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) hold:*
(1) f is ω-continuous,
(2) $f^{-1}(\theta\text{-}cl(A))$ is ω-closed in X for every subset A of Y,
(3) f is almost weakly ω-continuous,
(4) f is faintly ω-continuous.

Proof. (1) $\Rightarrow$ (2) : Let $A \subset Y$. Since θ-$cl(A)$ is closed, then $f^{-1}(\theta\text{-}cl(A))$ is ω-closed in X.

(2) $\Rightarrow$ (3) : It follows from Theorem 2.6.

(3) $\Rightarrow$ (4) : Let A be a θ-closed subset of Y. By Theorem 2.5, $\omega\text{-}cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}cl(A)) = f^{-1}(A)$. Hence, $f(A)$ is ω-closed. Thus, f is faintly ω-continuous.

Let Y be regular and let A be any open set of Y. Since Y is regular, A is θ-open in Y. Since f is faintly ω-continuous, then $f^{-1}(A)$ is ω-open in X. Hence, f is ω-continuous. Thus, the implication (4) $\Rightarrow$ (1) holds.

Theorem 2.11. *The following properties equivalent for a function $f : X \to Y$:*
(1) $f : X \to Y$ is almost weakly ω-continuous at $x \in X$.
(2) $x \in \omega\text{-}int(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$.

Proof. (1) $\Rightarrow$ (2) : Let A be any neighborhood of $f(x)$. There exists an ω-open set B containing x such that $f(B) \subset cl(A)$. We have $B \subset f^{-1}(cl(A))$. Since B is ω-open, then $x \in B \subset \omega\text{-}int(B) \subset \omega\text{-}int(f^{-1}(cl(A)))$.

(2) $\Rightarrow$ (1) : Let $x \in \omega\text{-}int(f^{-1}(cl(A)))$ for each neighborhood A of $f(x)$. Take $U = \omega\text{-}int(f^{-1}(cl(A)))$. Then $f(U) \subset cl(A)$. Moreover, U is ω-open. Thus, f is almost weakly ω-continuous at $x \in X$.

Definition 2.8. *A subset A is said to be ω-semi-open if there exists ω-open set U such that $U \subset A \subset cl(U)$.*

Theorem 2.12. *Let $f : X \to Y$ be almost weakly ω-continuous at $x \in X$. The following properties hold:*

(1) For each neighborhood A of $f(x)$ and each ω-neighborhood B of x, there exists a nonempty ω-open set $U \subset B$ such that $U \subset \omega\text{-}cl(f^{-1}(cl(A)))$.

(2) For each neighborhood A of $f(x)$, there exists a ω-semi-open set B containing x such that $B \subset \omega\text{-}cl(f^{-1}(cl(A)))$.

Proof. (1) : Let A be any neighborhood of $f(x)$ and B be an open set of X containing x. Since $x \in \omega\text{-}int(f^{-1}(cl(A)))$, then $B \cap \omega\text{-}int(f^{-1}(cl(A)))) \neq \emptyset$. Take $U = B \cap \omega\text{-}int(f^{-1}(cl(A)))$. Thus, U is a nonempty ω-open set and hence $U \subset B$ and $U \subset \omega\text{-}int(f^{-1}(cl(A))) \subset \omega\text{-}cl(f^{-1}(cl(A)))$.

(2) : Suppose that (1) holds. Let B be ω-open containing x and A be any neighborhood of $f(x)$. There exists a nonempty ω-open set U_B such that $U_B \subset \omega\text{-}cl(f^{-1}(cl(A)))$. Take $U = \cup\{U_B : B \text{ is } \omega\text{-open in } X \text{ containing } x\}$. Then U is ω-open, $x \in \omega\text{-}cl(U)$ and $U \subset \omega\text{-}cl(f^{-1}(cl(A)))$. Take $S = U \cup \{x\}$. Then $U \subset S \subset \omega\text{-}cl(U)$. Thus, S is ω-semi-open set containing x and $S \subset \omega\text{-}cl(f^{-1}(cl(A)))$.

3. Further properties

Recall that a space is rim-compact [6] if it has a basis of open sets with compact boundaries. The graph of a function $f : X \to Y$, denoted by $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Theorem 3.1. *Let $f : X \to Y$ be a function with the closed graph. If Y is a rim-compact space, then the following are equivalent:*

(1) f is almost weakly ω-continuous,

(2) f is ω-continuous.

Proof. (2) $\Rightarrow$ (1) : Obvious.

(1) $\Rightarrow$ (2) : Let $x \in X$ and A be any open set of Y containing $f(x)$. Since Y is rim-compact, there exists an open set B of Y such that $x \in B \subset A$ and ∂B is compact. For each $y \in \partial B$, $(x, y) \in X \times Y \backslash G(f)$. This implies that there exist open sets $U_y \subset X$ and $V_y \subset Y$ such that $x \in U_y$, $y \in V_y$. Since $G(f)$ is closed, then $f(U_y) \cap V_y = \emptyset$. The family $\{V_y\}_{y \in \partial B}$ is an open cover of ∂B. This implies that there exist a finite number of points of ∂B, say, y_1, y_2, ..., y_n such that $\partial B \subset \cup\{V_{y_i}\}_{i=1}^n$. Take

$$S = \cap\{U_{y_i}\}_{i=1}^n \quad and \quad R = \cup\{V_{y_i}\}_{i=1}^n. \tag{7}$$

Then S and R are open sets such that $x \in S$, $\partial B \subset R$ and $f(S) \cap \partial B \subset f(S) \cap R = \emptyset$. Since f is almost weakly ω-continuous, there exists an ω-open set N containing x such that $f(N) \subset cl(B)$. Take $U = S \cap N$. Then, U is ω-open containing x, $f(U) \subset cl(B)$ and $f(U) \cap \partial B = \emptyset$. Thus, $f(U) \subset B \subset A$ and hence f is ω-continuous.

Theorem 3.2. *Let $f : X \to Y$ be a function where Y is a rim-compact Hausdorff space. Then the following are equivalent:*

(1) f is ω-continuous,

(2) f is almost weakly ω-continuous.

Proof. (1) $\Rightarrow$ (2) : Obvious.

(2) $\Rightarrow$ (1) : Since a rim-compact Hausdorff space is regular, by Theorem 2.10, f is ω-continuous.

Definition 3.1. *([1]) If a space X can not be written as the union of two nonempty disjoint ω-open sets, then X is said to be ω-connected.*

Theorem 3.3. *If $f : X \to Y$ is an almost weakly ω-continuous surjection and X is ω-connected, then Y is connected.*

Proof. Suppose that Y is not connected. Then there exist nonempty open sets A and B of Y such that $Y = A \cup B$ and $A \cap B = \emptyset$. This implies that A and B are clopen in Y. By Theorem 2.1, $f^{-1}(A) \subset \omega\text{-}int(f^{-1}(cl(A))) = \omega\text{-}int(f^{-1}(A))$. Hence $f^{-1}(A)$ is ω-open in X. Similarly, $f^{-1}(B)$ is ω-open in X. We have $f^{-1}(A) \cap f^{-1}(B) = \emptyset$, $X = f^{-1}(A) \cup f^{-1}(B)$. Also, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Thus, X is not ω-connected.

Corollary 3.1. *If $f : X \to Y$ is a ω-continuous surjection and X is ω-connected, then Y is connected.*

For a function $f : X \to Y$, the graph function $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$ for each $x \in X$.

Theorem 3.4. *The following are equivalent for a function $f : X \to Y$:*
(1) f is almost weakly ω-continuous,
(2) the graph function g is almost weakly ω-continuous.

Proof. (1) $\Rightarrow$ (2) : Let f be almost weakly ω-continuous and $x \in X$. Suppose that A is an open set containing $g(x)$. There exist open sets $B \subset X$ and $C \subset Y$ such that $g(x) = (x, f(x)) \in B \times C \subset A$. Since f is almost weakly ω-continuous, there exists ω-open set D containing x such that $f(D) \subset cl(C)$. Take $S = B \cap D$. Then S is an ω-open set containing x and $g(S) \subset cl(A)$. Thus, g is almost weakly ω-continuous.

(2) $\Rightarrow$ (1) : Let g be almost weakly ω-continuous and $x \in X$ and A be an open set of X containing $f(x)$. Then $X \times A$ is an open set containing $g(x)$. Then there exists an ω-open set B containing x such that $g(B) \subset cl(X \times A) = X \times cl(A)$. Thus, $f(B) \subset cl(A)$ and hence f is almost weakly ω-continuous.

Theorem 3.5. *If $f, g : X \to Y$ is almost weakly ω-continuous and Y is Urysohn, then the set $A = \{x \in X : f(x) = g(x)\}$ is ω-closed in X.*

Proof. Let $x \in X \backslash A$. Then $f(x) \neq g(x)$. Since Y is Urysohn, then there exist open sets S and R of Y such that $f(x) \in S$, $g(x) \in R$ and $cl(S) \cap cl(R) = \emptyset$. Since f is almost weakly ω-continuous, there exists ω-open set U in X containing x such that $f(U) \subset cl(S)$. Since g is almost weakly ω-continuous, there exists an ω-open set B of X containing x such that $g(B) \subset cl(R)$. Take $H = U \cap B$. Then H is ω-open containing x and $f(H) \cap g(H) \subset cl(S) \cap cl(R) = \emptyset$. Thus, $H \cap A = \emptyset$ and hence A is ω-closed in X.

Theorem 3.6. *Let $f : X \to Y$ be an almost weakly ω-continuous function and A be a θ-closed set of $X \times Y$. Then $p(A \cap G(f))$ is ω-closed in X where p is the projection of $X \times Y$ onto X and $G(f) = \{(x, f(x)) : x \in X\}$.*

Proof. Let $x \in \omega\text{-}cl(p(A \cap G(f)))$. Suppose that U is an open set of X containing x and V is an open set of Y containing $f(x)$. Since f is almost weakly ω-continuous, by Theorem 2.1, $x \in f^{-1}(V) \subset \omega\text{-}int(f^{-1}(cl(V)))$. We have $U \cap \omega\text{-}int(f^{-1}(cl(V)))$ and $x \in U \cap \omega\text{-}int(f^{-1}(cl(V)))$. Since $x \in \omega\text{-}cl(p(A \cap G(f)))$, then $(U \cap \omega\text{-}int(f^{-1}(cl(V)))) \cap p(A \cap G(f))$ contains a $a \in X$. Then $(a, f(a)) \in A$ and $f(a) \in cl(V)$. Therefore

$$\emptyset \neq (U \times cl(V)) \cap A \subset cl(U \times V) \cap A \tag{8}$$

and $(x, f(x)) \in \theta\text{-}cl(A)$. Since A is θ-closed, $(x, f(x)) \in A \cap G(f)$ and $x \in p(A \cap G(f))$. Thus, $p(A \cap G(f))$ is ω-closed in X.

Corollary 3.2. *If $f : X \to Y$ has the θ-closed graph and $g : X \to Y$ is almost weakly ω-continuous, then the set $A = \{x \in X : f(x) = g(x)\}$ is ω-closed in X.*

Proof. Let $G(f)$ be θ-closed. We have $p(G(f) \cap G(g)) = \{x \in X : f(x) = g(x)\}$. By Theorem 3.6, $A = \{x \in X : f(x) = g(x)\}$ is ω-closed in X.

Theorem 3.7. *If $f : X \to Y$ is almost weakly ω-continuous and Y is Hausdorff, then for each $(x, y) \notin G(f)$, there exist an ω-open set $A \subset X$ and an open set $B \subset Y$ containing x and y, respectively, such that $f(A) \cap int(cl(B)) = \varnothing$.*

Proof. Let $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets B and C containing y and $f(x)$, respectively. Thus, $int(cl(B)) \cap cl(C) = \varnothing$. Since f is almost weakly ω-continuous, there exists an ω-open set A containing x such that $f(A) \subset cl(C)$. Hence, $f(A) \cap int(cl(B)) = \varnothing$. □

Definition 3.2. *A subset S of a space X is said to be N-closed relative to X [3] if for each cover $\{A_i : i \in I\}$ of S by open sets of X, there exists a finite subfamily $I_0 \subset I$ such that $S \subset \cup_{i \in I_0} cl(A_i)$.*

Theorem 3.8. *Let $f : X \to Y$ be a function. Suppose that for each $(x, y) \notin G(f)$, there exist an ω-open set $A \subset X$ and an open set $B \subset Y$ containing x and y, respectively, such that $f(A) \cap int(cl(B)) = \varnothing$. Then inverse image of each N-closed set of Y is ω-closed in X.*

Proof. Suppose that there exists a N-closed set $S \subset Y$ such that $f^{-1}(S)$ is not ω-closed in X. Then, there exists a point $x \in \omega\text{-}cl(f^{-1}(S)) \backslash f^{-1}(S)$. Since $f(x) \notin f^{-1}(S)$, then $(x, y) \notin G(f)$ for each $y \in S$. This implies that there exist ω-open sets $A_y(x) \subset X$ and an open set $B(y) \subset Y$ containing x and y, respectively, such that $f(A_y(x)) \cap int(cl(B(y))) = \varnothing$. The family $\{B(y) : y \in S\}$ is a cover of S by open sets of Y. Since S is N-closed, there exist a finite number of points y_1, y_2, ..., y_n in S such that $S \subset \cup_{j=1}^{n} int(cl(B(y_j)))$. Take $A = \cap_{j=1}^{n} A_{y_j}(x)$. Then $f(A) \cap S = \varnothing$. Since $x \in \omega\text{-}cl(f^{-1}(S))$, then $f(A) \cap S \neq \varnothing$. This is a contradiction. □

Corollary 3.3. *Let Y be Hausdorff such that every closed set is N-closed. Then the following are equivalent:*

(1) f is ω-continuous,
(2) f is almost weakly ω-continuous.

Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be any two families of topological spaces. The product space of $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ is denoted by $\prod X_i$ and $\prod Y_i$, respectively. Let $f_i : X_i \to Y_i$ be a function for each $i \in I$. Let $f : \prod X_i \to \prod Y_i$ be the product function defined as follows: $f(\{x_i\}) = (f_i(x_i))$ for each $(x_i) \in \prod X_i$. The projection of $\prod X_i$ and $\prod Y_i$ onto X_i and Y_i, respectively is denoted by p_i and q_i.

Theorem 3.9. *If $f_i : X_i \to Y_i$ is almost weakly ω-continuous for each $i \in I$, then a function $f : \prod X_i \to \prod Y_i$ is almost weakly ω-continuous.*

Proof. Let $x = (x_i) \in \prod X_i$. Suppose that A is an open set containing $f(x)$. Then there exists an open set $\prod B_i$ such that $f(x) \in \prod_{i=1}^n B_i \times \prod_{i \neq j} Y_j \subset A$, where B_i is open in Y_i. Since f_i is almost weakly ω-continuous, there exists ω-open sets S_i in X_i containing x_i such that $f_i(S_i) \subset cl(B_i)$ for each $i = 1, 2, ..., n$. Take $S = \prod_{i=1}^n S_i \times \prod_{i \neq j} X_j$, then S is ω-open in $\prod X_i$ containing x and

$$f(S) \subset \prod_{i=1}^{n} f_i(S_i) \times \prod_{i \neq j} Y_j \subset \prod_{i=1}^{n} cl(B_i) \times \prod_{i \neq j} Y_j \subset cl(A). \tag{9}$$

Thus, f is almost weakly ω-continuous.

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