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On the weak stability of Picard iteration for some contractive type mappings

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Abstract. We prove a weak stability result for Picard iteration in the class of mappings that satisfy some contraction condition. Illustrative examples are also given.

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1. Introduction

In [8] there are some examples of mapping that satisfy the following conditions:

$$d(Tx,Ty) < \max\{d(x,Tx),d(y,Ty)\} \tag{1}$$

$$d(Tx,Ty) < \max\{d(x,Tx),d(y,Ty),d(x,y)\} \tag{2}$$

$$d(Tx,Ty) < \max\{d(x,Tx),d(y,Ty),d(x,y),d(x,Ty),d(y,Tx)\} \tag{3}$$

$$d(Tx,Ty) < \max\left\{d(x,Tx),d(y,Ty),d(x,y),\frac{d(x,Ty)+d(y,Tx)}{2}\right\} \tag{4}$$

and the associated Picard iteration is not stable.

On the other hand, in [1] is introduced a weaker concept of stability, called weak stability and in [27] there is a study of weak stability of iterative procedures for some coincidence theorems.

The purpose of this paper is to show that for the class of mappings which satisfy the condition (1)-(4), the Picard iteration is weak stable in the meaning of Definition 2.4.

2. Weak stability of fixed point iteration procedures

The concept of stability of fixed point iteration procedure has been systematically studied by Harder in her Ph.D. thesis and published in the papers [7],[8]. The stability of the Picard iteration with respect to α-contractions and Zamfirescu mappings is given in [8] and both these results being established in the framework of a metric space setting. It has also been shown that in a normed linear space setting certain Mann iterations are stable with respect to any Zamfirescu mapping. In the same setting, a similar result was proved for Kirk's iteration procedure, in the class of c-contractions, with $0 \leq c < 1$.

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One of the most general contractive definition for which corresponding stability results have been obtained in the case of Kirk, Mann and Ishikawa iteration procedures in arbitrary Banach spaces appears to be the following class of mappings: for (X, d) a metric space, $T : X \to X$ is supposed to satisfy the condition

$$d\,(Tx, Ty) \leq ad(x, y) + Ld(x, Tx) \tag{5}$$

for some $a \in [0, 1)$, $L \geq 0$ and for all $x, y \in D \subset X$. This condition appears in [15] and other related results may be found in [14], [19], [20].

The concept of stability is not very precise because of the sequence $\{y_n\}_{n=0}^{\infty}$ which is *arbitrary* taken. So, it would be more natural that $\{y_n\}$ to be an *approximate sequence* of $\{x_n\}$. In [1] was introduced a weaker concept of stability, called *weak stability*. Therefore, any stable iteration will be also weakly stable but the reverse is not generally true.

Definition 2.1. [1] *Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty} \subset X$ be a given sequence. We shall say that $\{y_n\}_{n=0}^{\infty} \in X$ is an approximate sequence of $\{x_n\}$ if, for any $k \in \mathbb{N}$, there exists $\eta = \eta(k)$ such that*

$$d(x_n, y_n) \leq \eta, \quad for\ all\ n \geq k.$$

Remark 2.1. *We can have approximate sequences of both convergent and divergent sequences.*

The following result will be useful in the sequel.

Lemma 2.1. [1] *The sequence $\{y_n\}$ is an approximate sequence of $\{x_n\}$ if and only if there exists a decreasing sequence of positive numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ such that*

$$d(x_n, y_n) \leq \eta_n, \quad for\ any\ n \geq k\ (fixed)\ .$$

Definition 2.2. [1] *Let (X, d) be a metric space and $T : X \to X$ be a map. Let $\{x_n\}$ be an iteration procedure defined by $x_0 \in X$ and*

$$x_{n+1} = f(T, x_n), \quad n \geq 0. \tag{6}$$

Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any approximate sequence $\{y_n\} \subset X$ of $\{x_n\}$

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$$

implies

$$\lim_{n\to\infty} y_n = p,$$

then we shall say that (6) is weakly T-stable or weakly stable with respect to T.

In this paper, we consider a new concept of weak stability which is obtained from the Definition 2.2 by replacing of the notion of approximate sequence with the notion of the equivalent sequence.

Definition 2.3. [4] *Two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are equivalent sequences if*

$$d(x_n, y_n) \to 0, \quad as\ \ n \to \infty.$$

Definition 2.4. *Let (X,d) be a metric space and $T : X \to X$ be a map. Let $\{x_n\}$ be an iteration procedure defined by $x_0 \in X$ and*

$$x_{n+1} = f(T, x_n), \quad n \geq 0. \tag{7}$$

Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any equivalent sequence $\{y_n\} \subset X$ of $\{x_n\}$

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$$

implies

$$\lim_{n\to\infty} y_n = p,$$

then we shall say that the iteration procedure is weak w^2-stable with respect to T.

Remark 2.2. *Any equivalent sequence is an approximative sequence but the reverse is not true, as it is shown in the next example.*

Example 2.1.

Let $\{x_n\}_{n=0}^{\infty}$ to be a sequence with $x_n = n$. First, we take an equivalent sequence of $\{x_n\}_{n=0}^{\infty}$ to be $\{y_n\}_{n=0}^{\infty}$, $y_n = n + \frac{1}{n}$. In this case, we have that $d(y_n, x_n) = \frac{1}{n} \to 0, \quad n \to \infty$.

Now, we take an approximate sequence of $\{x_n\}_{n=0}^{\infty}$ to be $\{y_n\}_{n=0}^{\infty}$, $y_n = n + \frac{n}{2n+1}$. Then, $d(y_n, x_n) = \frac{n}{2n+1} \to \frac{1}{2} > 0, \quad n \to \infty$.

The basic results of this paper are the following theorems:

Theorem 2.1. *Let (X,d) a complete metric space and $T : X \to X$ a mapping satisfying (1), i.e.,*

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}.$$

Let $\{x_n\}_{n=0}^{\infty}$ an iterative procedure defined by $x_0 \in X$ and $x_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{x_n\}$ converges to x^*, the unique fixed point of T.

Then, the Picard iteration is w^2-stable.

Proof. Consider $\{y_n\}_{n=0}^{\infty}$ to be an equivalent sequence of $\{x_n\}$. Then, according to Definition 2.4, if $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the Picard iteration is w^2-stable.

In order to prove this, we suppose that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$. Therefore, $\forall \epsilon > 0, \quad \exists n_0 = n(\epsilon)$ such that $d(y_{n+1}, Ty_n) < \epsilon$, $\forall n \geq n_0$.

So, $d(y_{n+1}, x^*) \leq d(y_{n+1}, x_{n+1}) + d(x_{n+1}, x^*) \leq d(y_{n+1}, Ty_n) + d(Ty_n, Tx_n) + d(x_{n+1}, x^*) < d(y_{n+1}, Ty_n) + \max\{d(x_n, Tx_n), d(y_n, Ty_n)\} + d(x_{n+1}, x^*)$.

From the hypothesis, from $x_n \to x^*$, we have that $d(x_n, Tx_n) = d(x_n, x_{n+1}) \leq d(x_n, x^*) + d(x^*, x_{n+1}) \to 0$.

If $\max\{d(x_n, Tx_n), d(y_n, Ty_n)\} = d(x_n, Tx_n)$, by taking to the limit, we obtain that $d(y_{n+1}, x^*) \to 0$.

If $\max\{d(x_n, Tx_n), d(y_n, Ty_n)\} = d(y_n, Ty_n)$, we have that

$$d(y_n, Ty_n) \leq d(y_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y_{n+1}, Ty_n).$$

From Definition 2.3, we have that $d(x_n, y_n) \to 0$ and by taking to the limit, we obtain that $d(y_{n+1}, x^*) \to 0$.

This shows that the Picard iteration is w^2-stable with respect to T. □

Theorem 2.2. *Let (X, d) a complete metric space and $T : X \to X$ a mapping satisfying (2), i.e.,*

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y)\}.$$

Let $\{x_n\}_{n=0}^{\infty}$ an iterative procedure defined by $x_0 \in X$ and $x_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{x_n\}$ converges to x^*, the unique fixed point of T.

Then, the Picard iteration is w^2-stable.

Proof. We follow the same assumptions as in Theorem 2.1, by taking $\{y_n\}_{n=0}^{\infty}$ to be an equivalent sequence of $\{x_n\}$.

According to Definition 2.4, if $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the Picard iteration is w^2-stable.

Theorem 2.1 shows this result if we consider $\max\{d(x, Tx), d(y, Ty)\}$. In this case, there is a new situation, when max could be $d(x, y)$. Therefore, following the same steps, we get that $\max\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, y_n)\} = d(x_n, y_n)$. From Definition 2.3, we have that $d(x_n, y_n) \to 0$ and by taking to the limit as it is shown in the above theorem, we obtain the conclusion. □

Theorem 2.3. *Let (X, d) a complete metric space and $T : X \to X$ a mapping satisfying (3), i.e.,*

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y), d(x, Ty), d(y, Tx)\}.$$

Let $\{x_n\}_{n=0}^{\infty}$ an iterative procedure defined by $x_0 \in X$ and $x_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{x_n\}$ converges to x^*, the unique fixed point of T.

Then, the Picard iteration is w^2-stable.

Proof. We follow the same assumptions as in Theorem 2.2, where is shown this result if we consider $\max\{d(x, Tx), d(y, Ty), d(x, y)\}$. In this case, there are new situations, when max could be $d(x, Ty)$ or $d(y, Tx)$. Again, we follow the same steps.

If $\max\{d(x_n, Tx_n), d(y_n, Ty_n), d(x_n, y_n), d(x_n, Ty_n), d(y_n, Tx_n)\} = d(x_n, Ty_n)$, we have that $d(x_n, Ty_n) \leq d(x_n, y_n) + d(y_n, Ty_n)$. From Definition 2.3, $d(x_n, y_n) \to 0$ and the expression of $d(y_n, Ty_n)$ was treated in Theorem 2.1.

On the other hand, if $\max = d(y_n, Tx_n)$, then $d(y_n, Tx_n) \leq d(y_n, x_n) + d(x_n, Tx_n)$.

By taking to the limit in a same way as in above theorems, we obtain the conclusion. □

Theorem 2.4. *Let (X, d) a complete metric space and $T : X \to X$ a mapping satisfying (4), i.e.,*

$$d(Tx, Ty) < \max\left\{d(x, Tx), d(y, Ty), d(x, y), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Let $\{x_n\}_{n=0}^{\infty}$ an iterative procedure defined by $x_0 \in X$ and $x_{n+1} = Tx_n$, for all $n \geq 0$ and the sequence $\{x_n\}$ converges to x^*, the unique fixed point of T.

Then, the Picard iteration is w^2-stable.

Proof. We follow the same assumptions as in Theorem 2.3, where is shown this result if we consider $\max\{d(x, Tx), d(y, Ty), d(x, y), d(x, Ty), d(y, Tx)\}$. In this case, this is a new situation, when max could be $\frac{d(x,Ty)+d(y,Tx)}{2}$. Then, following the same steps as in Theorem 2.3, we obtain that $d(x_n, Ty_n) \to 0$ and $d(y_n, Tx_n) \to 0$ so, by taking to the limit in the whole expression, we get the result. □

3. Examples of weak stable, w^2-stable but not stable iterations

In [8] there are some examples of mappings which satisfy the conditions (1)-(4) or other related ones and their iteration procedures are not stable.

In this paper, we take these examples and study their weak stability and the w^2-stability.

Example 3.1.

Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \dfrac{1}{2}\right] \\ 0, & x \in \left(\dfrac{1}{2}, 1\right] \end{cases}$$

where $[0,1]$ is endowed with the usual metric. T is continuous at each point of $[0,1]$ except at $\frac{1}{2}$ and T has an unique fixed point at $\frac{1}{2}$.

As shown in [8], T satisfies the condition

$$d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty)\}.$$

Indeed, first let $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left[\frac{1}{2}, 1\right]$. Then, $|Tx - Ty| = |\frac{1}{2} - 0| = \frac{1}{2} < \max\{|x - Tx|, |y - Ty|\} < \max\left\{|\frac{1}{2} - x|, |y - 0|\right\} = \max\left\{\left(\frac{1}{2} - x\right), y\right\} = y$. Now, let $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left[0, \frac{1}{2}\right]$, with $x \neq y$. Then $|Tx - Ty| = 0 < \max\{|x - Tx|, |y - Ty|\} = \max\left\{|\frac{1}{2} - x|, |\frac{1}{2} - y|\right\} = \max\left\{\left(\frac{1}{2} - x\right), \left(\frac{1}{2} - y\right)\right\}$. If $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[\frac{1}{2}, 1\right]$, with $x \neq y$, $|Tx - Ty| = 0 < \max\{|x - Tx|, |y - Ty|\} = \max\{x, y\}$.

In order to study the T-stability, let x_0 be any point in $[0,1]$ and $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \cdots$. Then,

$$x_1 = Tx_0 = \begin{cases} \frac{1}{2}, & x_0 \in \left[0, \dfrac{1}{2}\right] \\ 0, & x_0 \in \left(\dfrac{1}{2}, 1\right] \end{cases}$$

But $x_2 = Tx_1 = \frac{1}{2}$ for either case. Furthermore, $x_n = \frac{1}{2}$, $\forall n \geq 2$ and $\lim_{n\to\infty} x_n = \frac{1}{2} = T\left(\frac{1}{2}\right)$.

Now, let $\{y_n\}_{n=0}^{\infty} = \frac{1}{2}, \frac{1}{4}, \frac{1}{2} + \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{2} + \frac{1}{4^4}, \frac{1}{4^5}, \cdots$. Observe that $\{y_n\}_{n=0}^{\infty}$ is a divergent sequence.

If n is a positive even integer, then

$$\epsilon_n = \left|\frac{1}{4^{n+1}} - T\left(\frac{1}{2} + \frac{1}{4^n}\right)\right| = \left|\frac{1}{4^{n+1}} - 0\right| = \frac{1}{4^{n+1}}.$$

If n is a positive odd integer, then

$$\epsilon_n = \left|\left(\frac{1}{2} + \frac{1}{4^{n+1}}\right) - T\left(\frac{1}{4^n}\right)\right| = \frac{1}{2} + \frac{1}{4^{n+1}} - \frac{1}{2} = \frac{1}{4^{n+1}}.$$

Thus,

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \frac{1}{4^{n+1}} = 0,$$

but $\{y_n\}_{n=0}^{\infty}$ does not converge to $\frac{1}{2}$. So, the Picard iteration is not T-stable.

In order to study the T-weak stability, we take an approximate sequence $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}$. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that

$$|x_n - y_n| \leq \eta_n, \quad n \geq k, \quad k \ fixed.$$

Then, $-\eta_n \leq x_n - y_n \leq \eta_n$ and results that $0 \leq y_n \leq x_n + \eta_n, \quad n \geq k$.

Since $x_n = \frac{1}{2}$, for $n \geq 2$, we obtain $0 \leq y_n \leq \frac{1}{2} + \eta_n, \quad n \geq k_1 = \max\{2, k\}$.

For every choice of nonnegative η_n we have that $0 \leq y_n \leq 1, \ \forall n \geq k_1$. So, $Ty_n = \frac{1}{2}$ or $Ty_n = 0$. So, if $\{y_n\}$ is an approximate sequence, it is not possible to obtain the weak stability.

Now, we take an equivalent sequence $\{y_n\}_{n=0}^{\infty}$ of $\{x_n\}$. Then,

$$|x_n - y_n| \to 0, \quad n \to \infty.$$

So, $x_n - y_n = \alpha_n, \quad \alpha_n \to 0, \quad n \geq k, \quad k \ fixed.$ Since $x_n = \frac{1}{2}$, for $n \geq 2$, we obtain that $y_n = \frac{1}{2} - \alpha_n, \quad n \geq k_1 = \max\{2, k\}$.

Then, $Ty_n = \frac{1}{2}$ and by $\lim_{n\to\infty} |y_{n+1} - Ty_n| = 0$ results that $\lim_{n\to\infty} y_n = \frac{1}{2} = T\left(\frac{1}{2}\right)$. That shows that the Picard iteration is w^2-stable with respect to T..

Example 3.2.

Let $T : [0, 1] \to [0, 1]$ be given by

$$Tx = \begin{cases} 0, & x \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

where $[0, 1]$ is endowed with the usual metric. T is continuous at every point of $[0, 1]$ except at $\frac{1}{2}$ and 0 is the only fixed point of T.

For each $x, y \in [0, 1]$, $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}.$$

Indeed, first let $x \in \left[0, \frac{1}{2}\right]$, $y \in \left[0, \frac{1}{2}\right]$ and $x \neq y$. Then, $|Tx - Ty| = 0 < \max\{x, y\} = \max\{|x - Ty|, |y - Tx|\}$. If $x \in \left(\frac{1}{2}, 1\right]$, $y \in \left(\frac{1}{2}, 1\right]$ and $x \neq y$, then $|Tx - Ty| = 0 < \max\left\{\left(x - \frac{1}{2}\right), \left(y - \frac{1}{2}\right)\right\} = \max\{|x - Ty|, |y - Tx|\}$. If $x \in \left[0, \frac{1}{2}\right]$ and $y \in \left(\frac{1}{2}, 1\right]$, then $|Tx - Ty| = \left|0 - \frac{1}{2}\right| = \frac{1}{2} < y = \max\left\{\left(\frac{1}{2} - x\right), y\right\} =$
$= \max\{|x - Ty|, |y - Tx|\}$.

We will show that the Picard iteration is not T-stable but it is T-weakly stable.

In order to prove the first claim, let (y_n), with

$$y_n = \frac{n+2}{2n}, \quad n \geq 1.$$

Then

$$\epsilon_n = |y_{n+1} - f(T, x_n)| = |y_{n+1} - Ty_n| = |\frac{n+3}{2(n+1)} - \frac{1}{2}|,$$

because $y_n \geq \frac{1}{2}$, for $n \geq 1$.

Therefore, $\lim_{n\to\infty} \epsilon_n = 0$ but $\lim_{n\to\infty} y_n = \frac{1}{2}$, so the Picard iteration is not T-stable.

In order to show the T-weak stability, we take an approximate sequence $\{y_n\}$ of x_n. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that

$$|x_n - y_n| \leq \eta_n, \quad n \geq k.$$

Then, $-\eta_n \leq x_n - y_n \leq \eta_n$ and results that $0 \leq y_n \leq x_n + \eta_n, \quad n \geq k$.

Since $x_n = 0$, for $n \geq 2$, we obtain $0 \leq y_n \leq \eta_n, \quad n \geq k_1 = \max\{2, k\}$.

We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{2},\ n \geq k_1$ and therefore $0 \leq y_n \leq \frac{1}{2},\ \ \forall n \geq k_1$. So, $Ty_n = 0$ and results that $\epsilon_n = |y_{n+1} - Ty_n| = |y_{n+1}| = y_{n+1}$.

Now, it is obvious that $\lim_{n\to\infty} \epsilon_n = 0 \ \Rightarrow\ \lim_{n\to\infty} y_n = 0$, so the iteration $\{y_n\}$is T-weakly stable. Hence, it is w^2-stable, too.

Example 3.3.

Let $T : \mathbb{R} \to \left\{0, \frac{1}{4}, \frac{1}{2}\right\}$ be defined by

$$Tx = \begin{cases} \frac{1}{2}, & x < 0 \\ \frac{1}{4}, & x \in \left[0, \dfrac{1}{2}\right] \\ 0, & x > \frac{1}{2} \end{cases}$$

where $\mathbb{R}$ is endowed with the usual metric. T is continuous at every point in $\mathbb{R}$ except at 0 and $\frac{1}{2}$. The only fixed point of T is $\frac{1}{4}$.

For each $x, y \in \mathbb{R}$, $x \neq y$, T satisfies the condition

$$d(Tx, Ty) < \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Indeed, first choose $x < 0$ and $y \in \left[0, \frac{1}{2}\right]$. Then, $|Tx - Ty| = |\frac{1}{2} - \frac{1}{4}| = \frac{1}{4}$ and $\frac{|x-Tx|+|y-Ty|}{2} \geq \frac{1}{2}\left|x - \frac{1}{2}\right| = \frac{1}{4} - \frac{x}{2} > \frac{1}{4}$. If $x < 0$ and $y > \frac{1}{2}$, then $|Tx - Ty| = |\frac{1}{2} - 0| = \frac{1}{2}$ and $\frac{|x-Tx|+|y-Ty|}{2} = \frac{|x-\frac{1}{2}|+|y-0|}{2} \geq \frac{1}{4} + \frac{y-x}{2} > \frac{1}{4} + \frac{1}{4} > \frac{1}{2}$. If $x \in \left[0, \frac{1}{2}\right]$ and $y > \frac{1}{2}$, then $|Tx - Ty| = |\frac{1}{4} - 0| = \frac{1}{4}$ and $\frac{|x-Tx|+|y-Ty|}{2} = \frac{|x-\frac{1}{4}|+y}{2} \geq \frac{y}{2} > \frac{1}{4}$. If $x < 0$, $y < 0$ and $x \neq y$, then $|Tx - Ty| = 0 < |x - y|$. If $x > \frac{1}{2}$, $y > \frac{1}{2}$ and $x \neq y$, then $|Tx - Ty| = 0 < |x - y|$. If $x \in \left[0, \frac{1}{2}\right]$, $y \in \left[0, \frac{1}{2}\right]$ and $x \neq y$, then $|Tx - Ty| = 0 < |x - y|$. Thus, $|Tx - Ty| < \max\left\{|x - y|, \frac{|x-Tx|+|y-Ty|}{2}, \frac{|x-Ty|+|y-Tx|}{2}\right\}$, for each $x, y \in \mathbb{R}$ such that $x \neq y$.

In order to study the T-stability, let x_0 be any real number and $x_{n+1} = Tx_n$, for $n = 0, 1, 2, \cdots$. Then,

$$x_1 = Tx_0 = \begin{cases} \frac{1}{2}, & x_0 < 0 \\ \frac{1}{4}, & x_0 \in \left[0, \frac{1}{2}\right] \\ 0, & x_0 > \frac{1}{2} \end{cases}$$

In each case, $x_2 = Tx_1 = \frac{1}{4}$ and $x_n = \frac{1}{4},\ \forall n \geq 2$. So, $\lim_{n\to\infty} x_n = \frac{1}{4} = T\left(\frac{1}{4}\right)$.

To show that the Picard iteration is not T-stable, let $\{y_n\}_{n=0}^{\infty}$ be the sequence of real numbers such that $y_0 = x_0$, $y_n = \frac{1}{2} + \frac{1}{n}$, for each positive odd integer and $y_n = -\frac{1}{n}$, for each positive even integer.

If n is a positive even integer, then

$$\epsilon_n = \left|\frac{1}{2} + \frac{1}{n+1} - T\left(-\frac{1}{n}\right)\right| = \left|\frac{1}{2} + \frac{1}{n+1} - \frac{1}{2}\right| = \frac{1}{n+1}.$$

If n is a positive odd integer, then

$$\epsilon_n = \left|\left(-\frac{1}{n+1}\right) - T\left(\frac{1}{2} + \frac{1}{n}\right)\right| = \left|-\frac{1}{n+1} - 0\right| = \frac{1}{n+1}.$$

Thus,

$$\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

but $\{y_n\}_{n=0}^{\infty}$ does not converge to $\frac{1}{4}$. So, the Picard iteration is not T-stable.

In order to study the T-weak stability, we take an approximate sequence $\{y_n\}_{n=0}^{\infty}$ of x_n. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $n \to \infty$ such that

$$|x_n - y_n| \leq \eta_n, \quad n \geq k.$$

Then, $-\eta_n \leq x_n - y_n \leq \eta_n$ and results that $0 \leq y_n \leq x_n + \eta_n, \quad n \geq k$.

Since $x_n = \frac{1}{4}$, for $n \geq 2$, we obtain $0 \leq y_n \leq \frac{1}{4} + \eta_n, \quad n \geq k_1 = \max\{2, k\}$.

We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{4}$, $n \geq k_1$ and therefore $0 \leq y_n \leq \frac{1}{2}$, $\forall n \geq k_1$. So, $Ty_n = \frac{1}{4}$ and by $\lim_{n\to\infty} |y_{n+1} - Ty_n| = 0$ results that $\lim_{n\to\infty} y_n = \frac{1}{4} = T\left(\frac{1}{4}\right)$. That shows that the Picard iteration is weakly T-stable.

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