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On New Class of Generalized Closed Sets

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Abstract. This paper introduces a new class of sets called generalised sg-closed sets in topological spaces to prove that this class lies between the class of closed sets and g-closed sets. Basic properties of generalised sg-closed sets are analysed and the inference is that the class of generalised sg-closed sets form a topology.

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1. Introduction

The concept of generalised closed sets introduced by Levine[10] plays a significant role in General Topology. This notion has been studied extensively in recent years by many topologists. The investigation of generalized closed sets has led to several new and interesting concepts, e.g. new covering properties and new separation axioms weaker than T_1. Some of these separation axioms have been found to be useful in computer science and digital topology. As an example, the well-known digital line is a $T_{3/4}$ space but fails to be a T_1 space (see e.g. [5]). After the introduction of generalized closed sets there are many research papers which deal with different types of generalized closed sets. Devi et al. [3] and Maki et al. [11] introduced semi-generalised closed sets (briefly sg-closed), generalised semi-closed sets (briefly gs-closed), generalised α-closed (briefly gα-closed) sets and α-generalised closed (briefly αg-closed) sets respectively. Jafari et al. [9] and Donchev [6] have introduced sg-compact spaces and studied their properties using sg-open and sg-closed sets. Also Ganster et al. [7] introduced sg-regular and sg-normal spaces using sg-open and sg-closed sets. In this paper a new class of closed sets called generalised sg-closed sets is introduced to prove that the class forms a topology. It has been proved that the class of generalised closed sets lies between the class of closed sets and the class of generalised closed sets.

2. Preliminaries

We list some definitions which are useful in the following sections. The interior and the closure of a subset A of a topological space (X, τ) are denoted by $Int(A)$ and $Cl(A)$, respectively. Throughout the present paper (X, τ) and (Y, σ) (or X and Y) represent non-empty topological spaces on which no separation axiom is defined, unless otherwise mentioned.

Definition 2.1. *A subset A of a topological space (X, τ) is called*
i: *a semi-open set* [1] *if $A \subseteq Cl(Int(A))$*

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ii: *a pre-open set* [2] *if* $A\subseteq Int(Cl(A))$
iii: *an* α *open set* [11] *if* $A \subseteq Int(Cl(Int(A)))$
iv: *an* β *open set* [4] *if (=semi-pre open)if* $A \subseteq Cl(Int(Cl(A)))$
v: *a regular open set* [8] *if* $A = Int(Cl(A))$.

The semi closure [3]*(resp* α*-closure* [11]*) of a subset A of X denoted by* $sCl(A)$ *(*$\alpha Cl(A)$*) is defined to be the intersection of all semi-closed (*α*-closed) sets containing A. The semi interior* [3] *of A denoted by sInt(A) is defined to be the union of all semi-open sets contained in A. If* $A\subseteq B\subseteq X$ *then* $Cl_B(A)$ *and* $Int_B(A)$ *denote the closure of A relative to B and interior of A relative to B.*

Definition 2.2. *A subset A of* (X, τ) *is called*
i: *a generalised closed set* [10]*(briefly g-closed) if Cl(A)* $\subseteq$ *U whenever A* $\subseteq$ *U and U is open in X.*
ii: *a semi-generalised closed set* [3]*(briefly sg-closed)if sCl(A)* $\subseteq$ *U whenever A* $\subseteq$ *U and U is semi-open in X.*
iii: *a generalised semi-closed set* [3]*(briefly gs-closed)if sCl(A)* $\subseteq$ *U whenever A* $\subseteq$ *U and U is open in X.*
iv: *an* α*-generalised closed set* [11]*(briefly* α*g-closed) if* $\alpha Cl(A)\subseteq U$ *whenever A* $\subseteq$ *U and U is open in X.*
v: *a generalised* α*-closed set* [11]*(briefly g*α*-closed) if* $\alpha Cl(A)\subseteq U$ *whenever A* $\subseteq$ *U and U is* α*-open in X.*
vi: *a* α^{**}*g-closed set* [11] *if* $\alpha Cl(A) \subseteq$ *Int(Cl(U)) whenever A* $\subseteq$ *U and U is open in X.*
vii: *a generalised semi-pre-closed set* [8] *(briefly gsp-closed)if spCl(A)* $\subseteq$ *U whenever A* $\subseteq$ *U and U is open in X.*
viii: *a generalised pre-closed set* [12] *(briefly gp-closed) if pCl(A)* $\subseteq$ *U whenever A* $\subseteq$ *U and U is open in X.*
ix: *an* ω*-closed set* [13] *if Cl(A)* $\subseteq$ *U whenever A* $\subseteq$ *U and U is semi-open in X.*

The complements of the above sets are called their respective open sets.

Definition 2.3. *A topological space* (X, τ) *is called a*
i: $T_{1/2}$*-space [10] if every g-closed set in it is closed.*
ii: T_ω*-space [13] if every* ω*-closed set in it is closed.*
iii: T_b*-space [3] if every gs-closed set in it is closed.*

3. Generalised sg-closed and generalised sg-open sets

Definition 3.1. *A subset A of a topological space* (X, τ) *is called a generalised sg-set (briefly gsg-closed) if* $Cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is sg-open in X.*

Proposition 3.1. *Every closed set is gsg-closed.*

Proof. Let A be any closed set and U be any sg-open set containing A then $Cl(A) = A \subseteq U$. Hence A is gsg-closed. □

Example 3.1. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a, b\}\}$*. The set* $\{a, c\}$ *is gsg-closed but not closed.*

Proposition 3.2. *Every gsg-closed set is g-closed.*

Proof. Let A be any gsg-closed set and U be any open set containing A. Since any open set is sg-open, we have $Cl(A) \subseteq U$. Hence A is g-closed. □

Example 3.2. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{c\}\}$. *The set* $\{a\}$ *is g-closed but not gsg-closed.*

Proposition 3.3. *Every gsg-closed set is* ω*-closed.*

Proof. Let A be any gsg -closed set and U be any semi-open set containg A. Since any semi open set is sg-open, we have $Cl(A) \subseteq U$. Hence A is ω-closed. □

Example 3.3. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. *The set* $\{b\}$ *is* ω*-closed but not gsg-closed.*

Proposition 3.4. *Every gsg-closed set is* αg*-closed.*

Proof. Let A be any gsg-closed set and U be any open set containing A. Since any open set is sg-open, we have $\alpha Cl(A) \subseteq Cl(A) \subseteq U$. Hence A is αg-closed. □

Example 3.4. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. *The set* $\{b\}$ *is* αg*-closed but not gsg-closed.*

Proposition 3.5. *Every gsg-closed set is* $\alpha^{**}g$*-closed.*

Proof. Let A be any gsg-closed set and by Proposition 3.4 A is αg-closed and hence by Maki et al. [11] A is α^{**}g-closed. □

Example 3.5. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. *The set* $\{b\}$ *is* $\alpha^{**}g$*-closed but not gsg-closed.*

Proposition 3.6. *Every gsg-closed set is* $g\alpha$*-closed and pre-closed.*

Proof. Let A be any gsg-closed set and U be any α-open set containing A. Since any α-open set is semi-open which is sg-open, we have $Cl(A) \subseteq U$. Hence A is $g\alpha$-closed. It has been proved that every [11] $g\alpha$-closed set is pre-closed. Hence A is pre-closed. Observe that each gsg-closed subset of X is $g\alpha$-closed if and only if X is extremally disconncted. A space (X, τ) is called extremally disconnected if $Cl(U) \in \tau$ for all $U \in \tau$. □

Example 3.6. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{c\}\}$. *The set* $\{a\}$ *is* $g\alpha$*-closed and pre-closed but not gsg-closed.*

Proposition 3.7. *Every gsg-closed set is sg-closed and* β*-closed.*

Proof. Let A be any gsg-closed set and U be any semi-open set containg U. Since any semi-open set is sg-open, we have $sCl(A) \subseteq Cl(A) \subseteq U$. Hence A is sg-closed. In [3] it has been proved that every sg-closed set is β-closed. Hence A is β-closed. □

Example 3.7. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. *The set* $\{a\}$ *is sg-closed and* β *but not gsg-closed.*

Proposition 3.8. *Every gsg-closed set is gs-closed,gsp-closed and gp-closed.*

Proof. Let A be any gsg-closed and U be any semi-open set containing A. Then $sCl(A) \subseteq Cl(A) \subseteq$U. Hence A is gs-closed.[resp $spCl(A) \subseteq U, pCl(A) \subseteq U$] □

Example 3.8. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{c\}\}$. *The set* $\{a\}$ *is gs-closed,gsp-closed and gp-closed but not gsg-closed.*

Remark 3.1. *The following examples show that the gsg-closed sets are independent of* α*-closed sets and semi-closed sets.*

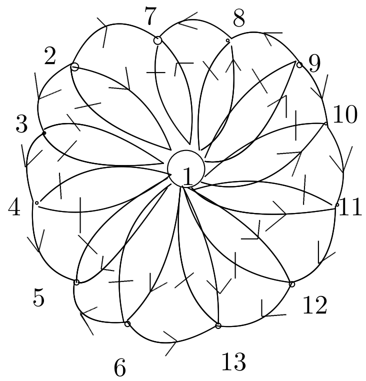
Example 3.9. *Let us consider the topological space $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{c\}\}$. The set $\{a\}$ is semi-closed and α-closed but not gsg-closed.*

Example 3.10. *Let us consider the topological space $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a, b\}\}$. The set $\{a, c\}$ is gsg-closed but not semi-closed and α-closed.*

Remark 3.2. *From the above discussions and known results we have the following implications where an arc with an arrow from A to B represents A implies B but not conversely and an with a small line from A to B implies they are independent.*

1.gsg-closed sets. 2.closed sets. 3.α-closed sets. 4.semi-closed sets. 5.β-closed sets. 6.pre-closed sets. 7.gp-closed sets. 8.αg-closed sets. 9.g-closed sets. 10.ω-closed sets. 11.sg-closed sets. 12.gs-closed sets.13.gsp-closed sets.

Thus the class of generalised sg-closed sets lies between the class of closed sets and the class of generalised closed sets.



Definition 3.2. *A subset A of X is called gsg-open if and only if A^c is gsg-closed in X.*

Theorem 3.1. **i:** *Every open set is gsg-open.*
ii: *Every gsg-open set is g-open and ω-open.*
iii: *Every gsg-open set is gs-open,sg-open,β-open and gsp-open.*
iv: *Every gsg-open set is gα-open,pre-open and αg-open.*

4. Characterization of gsg-closed and gsg-open sets

Theorem 4.1. *If A and B are gsg-closed sets in X then $A \cup B$ is gsg-closed in X.*

Proof. Let A and B are gsg-closed sets in X and U be any sg-open set containing A and B. Therefore we have $Cl(A) \subseteq U, Cl(B) \subseteq U$. Since $A \subseteq U, B \subseteq U$ then $A \cup B \subseteq U$. Since $Cl(A \cup B) = Cl(A) \cup Cl(B) \subseteq U$. Hence $A \cup B$ is gsg-closed. □

Theorem 4.2. *If a set A is gsg-closed then $Cl(A) - A$ contains no non empty closed set.*

Proof. Let F be a closed subset of $Cl(A) - A$. Then $A \subseteq (F)^c$. Since A is gsg-closed $Cl(A) \subseteq (F^c)$. Hence $F \subseteq (Cl(A))^c$. We have $F \subseteq (Cl(A) \cap (Cl(A))^c$ and hence F is empty. □

Remark 4.1. *The converse of Theorem 4.2 need not be true.*

Example 4.1. *Let us consider the topological space $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{c\}\}$. If $A = \{a\}$ then $Cl(A) - A = \{a, b\} - \{a\} = \{b\}$ does not contain non empty closed set and A is not gsg-closed in X.*

Theorem 4.3. *A set A is gsg-closed if and only if $Cl(A) - A$ contains no non empty sg-closed set.*

Proof. Necessity. Suppose that A is gsg-closed. Let D be a sg-closed subset of $Cl(A) - A$. Thus $A \subseteq D^c$. Since A is gsg-closed we have $Cl(A) \subseteq (D)^c$. Consequently $D \subseteq Cl(A) \cap (Cl(A)^c = \phi$. Thus D is empty.

Sufficiency. Suppose that $Cl(A) - A$ contains no non empty sg-closed set. Let $A \subseteq G$ and G be sg-open. If Cl(A) is not a subset of G then $Cl(A) \cap (G)^c$ is non empty. Since Cl(A) is a closed set, G^c is a sg-closed set, $Cl(A) \cap G^c$is a non empty sg-closed subset of $Cl(A) - A$ which is a contradiction. Therefore $Cl(A) \subseteq G$ and hence A is gsg-closed. □

Theorem 4.4. *If A is gsg-closed in X and $A \subseteq B \subseteq Cl(A)$ then B is gsg-closed in X.*

Proof. Since $B \subseteq Cl(A)$, we have $Cl(B) \subseteq Cl(A)$ then $Cl(B) - B \subseteq Cl(A) - A$. By Theorem 4.3 $Cl(A) - A$ has no non empty sg-closed subset of X and hence $Cl(B) - B$ contains no non empty sg-closed subset of X. Hence by Theorem 4.3 B is gsg-closed in X. □

Theorem 4.5. *Let $A \subseteq Y \subseteq X$ and if A is gsg-closed in X then A is gsg-closed relative to Y.*

Proof. Let $A \subseteq Y \cap G$ where G is sg-open in X. Then $A \subseteq G$ and hence $Cl(A) \subseteq G$. This implies that $Y \cap Cl(A) \subseteq Y \cap G$. Thus A is gsg-closed relative to Y. □

Theorem 4.6. *In a topological space X, $SGO(X) = F$(closed sets in X) if and only if every subset of X is a gsg-closed .*

Proof. If $SGO(X) = F$. Let A be a subset of X such that $A \subseteq G$ where $G \in SGO(X)$ then $Cl(G) = G$. Also $Cl(A) \subseteq Cl(G) = G$. Hence A is gsg-closed in X.

Sufficiency. If every subset of X is gsg-closed in X. Let $G \in SGO(X)$ therefore $G \subseteq G$ and G is gsg-closed in X we have $Cl(G) \subseteq G$. Thus $Cl(G) = G$. Therefore $SGO(X) \subseteq F$. If $S \in F$ then S^c is open and hence it is sg-open. Therefore $S^c \in SGO(X) \subseteq F$ and hence $S \in F^c$. Thus $SGO(X) = F$. □

Theorem 4.7. *If A is sg-open and gsg-closed in X, then A is closed in X.*

Proof. Since A is sg-open and gsg-closed in X then, $Cl(A) \subseteq A$ and hence A is closed in X. □

Theorem 4.8. *For each $x \in X$ either $\{x\}$ is sg-closed or $\{x\}^c$ is gsg-closed in X.*

Proof. If $\{x\}$ is not sg-closed in X then $\{x\}^c$ is not sg-open and the only sg-open set containing $\{x\}^c$ is the space X itself. Therefore $Cl(\{x\}^c \subseteq X$ and so $\{x\}^c$ is gsg-closed in X. □

Lemma 4.1. [2] *Let x be a point of* (X, τ). *Then* $\{x\}$ *is either nowhere dense or pre-open (A set is said to be nowhere dense if* $Int(Cl(A)) = \phi$*).*

Remark 4.2. *In the notation of Lemma 4.1 ,we consider the following decomposition of a given topological space* (X, τ), *namely* $X = X_1 \cup X_2$ *where* $X_1 = \{x \in X : \{x\} is\ no\ where\ dense\}$ *and* $X_2 = \{x \in X : \{x\} is\ pre-open\}$.

Definition 4.1. *The intersection of all sg-open subsets of* (X, τ) *containing A is called the sg-kernel of A and is denoted by sg-ker(A).*

Lemma 4.2. *A subset A of* (X, τ) *is gsg-closed if and only if* $Cl(A) \subseteq sg - ker(A)$.

Proof. Suppose that A is gsg-closed in X. Then $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ). Let $x \in Cl(A)$. If $x \notin sg - ker(A)$ then there is a sg-open set U containing such that $x \notin U$. Since U is a sg-open set containing A, we have $x \notin Cl(A)$, a contradiction.
Conversely let $Cl(A) \subseteq sg - ker(A)$. If U is any sg- open set containing A, then $Cl(A) \subseteq sg - ker(A) \subseteq U$. Therefore A is gsg-closed. □

Proposition 4.1. *For any subset A of X,* $X_2 \cap Cl(A) \subseteq sg - ker(A)$.

Proof. Let $x \in X_2 \cap Cl(A)$ and if $x \notin sg - ker(A)$. Then there is a sg-open set U containing A such that $x \notin U$. If $F = X - U$,then F is sg-closed. Since $Cl(\{x\}) \subseteq Cl(A)$, we have $int(Cl(A)) \subseteq A \cup int(Cl(A))$. Since $x \in X_2$, we have $x \notin X_1$ and so $int(Cl(\{x\}) \neq \phi$. Therefore there has to be some point $y \in A \cap int(Cl(\{x\})$ and hence $y \in F \cap A$. A contradiction. □

Theorem 4.9. *A subset A of* (X, τ) *is gsg-closed if and only if* $X_1 \cap Cl(A) \subseteq A$.

Proof. Suppose that A is gsg-closed. Let $x \in X_1 \cap Cl(A)$. Then $x \in X_1$ and $x \in Cl(A)$. Since $x \in X_1$, $int(Cl(\{x\}) = \phi$. Therefore $\{x\}$ is semi-closed, since $int(Cl(\{x\}) \subseteq \{x\}$. Since every semi-closed set is sg-closed, $\{x\}$ is sg-closed. If $x \notin A$ and $U = X - \{x\}$, then U is a sg-open set containing A and so $Cl(A) \subseteq U$, a contradiction.

Conversely, let $X_1 \cap Cl(A) \subseteq A$. Then $X_1 \cap Cl(A) \subseteq sg - ker(A)$. Since $A \subseteq sg - ker(A)$. Now $Cl(A) = X \cap Cl(A) = (X_1 \cup X_2) \cap Cl(A) = (X_1 \cap Cl(A)) \cup (X_2 \cap Cl(A)) \subseteq sg - ker(A)$. Since $X_1 \cap Cl(A) \subseteq sg - ker(A)$ and by Proposition 4.1 and Lemma 4.2 A is gsg-closed. □

Theorem 4.10. *Arbitrary intersection of gsg-closed sets is gsg-closed.*

Proof. Let $F = \{A_i : i \in \Lambda\}$ be a family of gsg-closed sets and let $A = \bigcap_{i \in \Lambda} A_i$. Since $A \subseteq A_i$ for each i,$X_1 \cap Cl(A) \subseteq X_1 \cap Cl(A_i)$ for each i. By Theorem 4.9 for each closed gsg-closed set A_i, we have $X_1 \cap Cl(A_i) \subseteq A_i$ for each i and so $X_1 \cap Cl(A_i) \subseteq A$ for each i. Thus $X_1 \cap Cl(A) \subseteq X_1 \cap Cl(A_i) \subseteq A$ for each $i \in \Lambda$. That is $X_1 \cap Cl(A) \subseteq A$ and so A is gsg-closed by Theorem 4.9. □

Remark 4.3. *Thus Theorem 4.1 and Theorem 4.10 leads us into another class of closed sets namely gsg-closed sets which are closed under finite union and arbitrary intersection. Note that while no other class of closed sets form a topology this class forms a topology.*

Proposition 4.2. *If A and B are gsg-open sets then* $A \cap B$ *is gsg-open.*

Theorem 4.11. *A set A is gsg-open if and only if* $F \subseteq Int(A)$, *where F is sg-closed and* $F \subseteq A$.

Proof. If $F \subseteq Int(A)$ where F is sg-closed and $F \subseteq A$. Let $A^c \subseteq G$ where $G = F^c$ is sg-open. Then $G^c \subseteq A$ and $G^c \subseteq Int(A)$. Thus A^c is gsg -closed. Hence A is gsg-open.
Conversely if A is gsg-open,$F \subseteq A$ and F is sg-closed. Then F^c is sg-open and $A^c \subseteq F^c$. Therefore $Cl(A^c) \subseteq (F^c)$. But $Cl(A^c) = (Int(A))^c$. Hence $F \subseteq Int(A)$ □

Theorem 4.12. *If $A \subseteq B \subseteq X$ where A is gsg-open relative to B and B is gsg-open in X, then A is gsg-open in X.*

Proof. Let F be a sg-closed set in X and suppose that $F \subseteq A$. Then $F = F \cap B$ is sg-closed in B. But A is gsg-open relative to B. Therefore $F \subseteq Int_B(A)$. Since $Int_B(A)$ is an open set relative to B. We have $F \subseteq G \cap B \subseteq A$, for some open set G in X. Since B is gsg-open in X, We have $F \subseteq Int(B) \subseteq B$. Therefore $F \subseteq Int(B) \cap G \subseteq B \cap G \subseteq A$. It follows that $F \subseteq Int(A)$. Therefore A is gsg-open in X. □

Theorem 4.13. *If $Int(B) \subseteq B \subseteq A$ and if A is gsg-open in X, then B is gsg-open in X.*

Proof. Suppose that $Int(A) \subseteq B \subseteq A$ and A is gsg-open in X then $A^c \subseteq B^c \subseteq Cl(A^c)$ and since A^c is gsg-closed in X, by Theorem 4.6 B is gsg-open in X. □

Theorem 4.14. *A set A is gsg-closed in X if and only if $Cl(A) - A$ is gsg-open in X.*

Proof. Necessity. Suppose that A is gsg-closed in X. Let $F \subseteq Cl(A) - A$ where F is sg-closed. By theorem 4.5 F is empty. Therefore $F \subseteq Int(Cl(A) - A)$ and by Theorem 4.5 $Cl(A) - A$ is gsg-open in X.
Sufficiency. Let $A \subseteq G$ where G is sg-open set in X. Then $Cl(A) \cap G^c \subseteq Cl(A) \cap A^c = Cl(A) - A$. Since $Cl(A) \subseteq G^c$ where G^c is sg-closed and $Cl(A) - A$ is gsg-open, by Theorem 4.20 we have $Cl(A) \cap G^c \subseteq Int(Cl(A) - A) = \phi$. Hence A is gsg-closed in X. □

Theorem 4.15. *For a subset A of X the following are equivalent.*
- **i:** *A is gsg-closed.*
- **ii:** *$Cl(A) - A$ contains no non empty sg-closed set.*
- **iii:** *$Cl(A) - A$ is gsg-open .*

Proof. Follows from Theorem 4.5 and Theorem 4.23. □

Theorem 4.16. *Let (X, τ) be a normal space and if Y is a gsg-closed subset of (X, τ), then the subspace Y is normal.*

Proof. If G_1 and G_2 disjoint closed sets in (X, τ) such that $(Y \cap G_1) \cap (Y \cap G_2) = \phi$. Then $Y \subseteq (G_1 \cap G_2)^c$ and $(G_1 \cap G_2)^c$ is sg-open. Y is gsg-closed in (X, τ). Therefore $Cl(Y) \subseteq (G_1 \cap G_2)^c$ and hence $(Cl(Y) \cap G_1) \cap (Cl(Y) \cap G_2) = \phi$. Since (X, τ) is normal, there exists disjoint open sets A and B such that $Cl(Y) \cap G_1 \subseteq A$ and $Cl(Y) \cap G_2 \subseteq B$. Thus $Y \cap A$ and $Y \cap B$ are disjoint open sets of Y such that $Y \cap G_1 \subseteq Y \cap A$ and $Y \cap G_2 \subseteq Y \cap B$. Hence Y is normal. □

5. gsg-Closure and gsg-Interior

Definition 5.1. *Let (X, τ) be a topological space and $B \subseteq X$. We define the gsg-closure of B (briefly gsg-Cl(B)) to be the intersection of all gsg-closed sets containing B which is denoted by $gsg - Cl(B) = \bigcap\{A : B \subseteq A \ andA \in GSGC(X, \tau)\}$ where $GSGC(X, \tau)$ is set of all gsg-closed subsets of X.*

Lemma 5.1. *For any* $B \subseteq X, B \subseteq gsg - Cl(B) \subseteq Cl(B)$

Proof. It follows from Proposition 3.2. □

Remark 5.1. *The relations in Lemma 5.2 may be proper as seen from the following example.*

Example 5.1. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{\phi, X, \{a, b\}\}$. *Let* $B = \{b\}$. *Then* $gsg-Cl(B) = \{b, c\}$ *and so* $B \subseteq gsg-Cl(B) \subseteq Cl(B)$

Theorem 5.1. *The gsg-closure is a Kuratowski closure operator on X.*

Proof. **i:** $gsg - Cl(\phi) = \phi$

ii: $B \subseteq gsg - Cl(B)$ by Lemma 5.2

iii: Let $B_1 \cup B_2 \subseteq A$ and $A \in GSGC(X, \tau)$,then $B_i \subseteq A$ and by definition 5.1 $gsg-Cl(B_i) \subseteq A$ for i=1,2. Therefore $gsg-Cl(B_1) \cup gsg-Cl(B_2) \subseteq \bigcap\{A : B_1 \cup B_2 \subseteq A \; and A \in GSGC(X, \tau)\} = gsg - Cl(B_1 \cup B_2)$.For the reverse inclusion,let $x \in gsg - Cl(B_1 \cup B_2)$ and suppose that $x \notin gsg - Cl(B_1) \cup gsg - Cl(B_2)$.Then there exist gsg-closed sets A_1 and A_2 with $B_1 \subseteq A_1, B_2 \subseteq A_2 \; and \; x \notin A_1 \cup A_2$.We have $B_1 \cup B_2 \subseteq A_1 \cup A_2 \; and \; A_1 \cup A_2$ is a gsg-closed set by theorem 4.1 such that $x \notin A_1 \cup A_2$.Thus $x \notin gsg - Cl(B_1 \cup B_2)$ which is a contradiction to $x \in gsg - Cl(B_1 \cup B_2)$.Hence $gsg - Cl(B_1) \cup gsg - Cl(B_2) = gsg - Cl(B_1 \cup B_2)$.

iv: Let $B \subseteq A$ and $A \in GSGC(X, \tau)$.Then by definition 5.1 $gsg - Cl(B) \subseteq A \; and \; gsg - Cl(gsg - Cl(B)) \subseteq A$.We have $gsg - Cl(gsg - Cl(B)) \subseteq \bigcap\{A : B \subseteq A and A \in GSGC(X, \tau)\} = gsg - Cl(B)$.By Lemma 5.2 $gsg - Cl(B) \subseteq gsg - Cl(gsg - Cl(B))$ and therefore $gsg - Cl(B) = gsg - Cl(gsg - Cl(B))$. Hence gsg-closure is a Kuratowski closure operator on X.

□

Proposition 5.1. *Let* (X, τ) *be a topological space and* $B \subseteq A$. *The following properties hold.*

i: *gsg-Cl(B) is the smallest gsg-closed set containing B.*

ii: *B is gsg-closed if and only if* $gsg - Cl(B) = B$.

Proposition 5.2. *For any two subsets A and B of* (X, τ)

i: *If* $A \subseteq B$, *then* $gsg - Cl(A) \subseteq gsg - Cl(B)$.

ii: $gsg - Cl(A \cap B) \subseteq gsg - Cl(A) \cap gsg - Cl(B)$.

Definition 5.2. *A space* (X, τ) *is called a* T_{gsg}*-space if every gsg-closed set in it is closed.*

Proposition 5.3. *Every* $T_{1/2}$*-space is a* T_{gsg}*-space but not conversely.*

Proof. Let (X, τ) be a $T_{1/2}$-space and let A be a gsg-closed subset of (X, τ). By Proposition 3.4 A is a g-closed set in (X, τ).Since (X, τ) is a $T_{1/2}$-space A is closed in (X, τ). Hence (X, τ) is a T_{gsg}-space. □

Example 5.2. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}\}$. $GC(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *The space* (X, τ) *is a* T_{gsg}*-space but not a* $T_{1/2}$*-space.*

Proposition 5.4. *Every* T_ω*-space is a* T_{gsg}*-space but not conversely.*

Proof. Since every gsg-closed set is ω-closed the proof follows from Proposition 3.7. $\square$

Example 5.3. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. *Since every subset of* (X, τ) *is* ω*-closed* $GSGC(X) = \{\phi, X, \{a\}, \{b, c\}\}$. *The space* (X, τ) *is a* T_{gsg}*-space but not a* T_ω*-space.*

Proposition 5.5. *Every* T_b*-space is a* T_{gsg}*-space but not conversely.*

Proof. Follows from Proposition 3.16. $\square$

Example 5.4. *Let us consider the topological space* $X = \{a, b, c\}$ *with* $\tau = \{X, \phi, \{a\}\}$. $GSC(X) = \{\phi, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. *The space* (X, τ) *is a* T_{gsg}*-space but not a* T_b*-space.*

Theorem 5.2. *For a space* (X, τ) *the following are equivalent.*

i: (X, τ) *is a* T_{gsg}*-space.*

ii: *Every singleton of* (X, τ) *is either sg-closed or open.*

Proof. $(i) \Rightarrow (ii)$ Assume that for some $x \in X$ the set $\{x\}$ is not a sg-closed set in (X, τ). Then the only sg-open set containing $\{x\}^c$ is the space X itself and so $\{x\}^c$ is gsg-closed in (X, τ). By assumption $\{x\}^c$ is closed in (X, τ)or equivalently $\{x\}$ is open.

$(ii) \Rightarrow (i)$ Let A be a gsg-closed subset of (X, τ) and let $x \in Cl(A)$. By assumption $\{x\}$ is either sg-closed or open.

case(1):Suppose $\{x\}$ is sg-closed. If $x \notin A$ then $Cl(A) - A$ contains a non-empty sg-closed set $\{x\}$ which is a contradiction to Theorem 4.5. Therefore $x \in A$.

case(2): Suppose $\{x\}$ is open. Since $x \in Cl(A), \{x\} \cap A \neq \phi$ and therefore $Cl(A) \subseteq A$ or equivalently A is a closed subset of (X, τ). $\square$

6. Conclusion

The class of generalised sg-closed sets defined using sg-open sets forms a topology and lies between the class of closed sets and the class of generalised closed sets. The gsg-closed sets can be used to derive a new decomposition of continuity and new separation axioms. This concept can be extended to bitopological and fuzzy topological spaces.

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