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Weighted Lipschitz Estimates for Multilinear Commutator of Pseudo-differential Operator

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Abstract. In this paper, we prove the boundedness for some multilinear commutators generated by the pseudo-differential operator and lipschitz functions.

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1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied(see [4-7]). In [4-7],[15-16], the authors prove that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [4] [14], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1][11], the weighted boundedness for the commutators generated by the singular integral operators and BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. The purpose of this paper is to prove the weighted boundedness on Lebesgue spaces for some multilinear operators associated to the pseudo-differential operators and the weighted Lipschitz functions. To do this, we first prove a sharp function estimate for the multilinear operators. Our results are new, even in the unweighted cases.

2. Notations and Theorems

In order to state our results, we begin by introducing the relevant notions and definitions.

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. The A_1 weight is defined by

$$A_1 = \{0 < \omega \in L^1_{loc} : \sup_{Q \ni x} |Q|^{-1} \int_Q \omega(y)dy \le c\omega(x), a.e.\}.$$

For $\omega \in A_1$ and $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(\omega)$ is the space of functions b such that

$$||b||_{Lip_\beta(\omega)} = \sup_Q \omega(Q)^{-1-\frac{\beta}{n}} \int_Q |b(y) - b_Q|dy < \infty,$$

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where $b_Q = |Q|^{-1}\int_Q b(y)dy$.

For $\omega \in A_1$ and $1 < p < \infty$, the weighted Lebesgue space $L^p(\omega)$ is the space of functions f such that

$$||f||_{L^p(\omega)} = \left(\int_{R^n} |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} < \infty.$$

For $\omega \in A_1$ and $\beta > 0$ and $p > 1$, let $\tilde{F}_p^{\beta,\infty}(\omega)$ be the weighted homogeneous Triebel-Lizorkin space.

Given some function $b_j \in Lip_\beta(\omega), 1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\gamma = \{\gamma(1), \cdots, \gamma(j)\}$ of j different elements in $\{1, \cdots, m\}$, where $\gamma(i) < \gamma(j)$ when $i < j$.

For $\gamma \in C_j^m$, we denote $\gamma^c = \{1, \cdots, m\}\backslash\gamma$.

For $\vec{b} = \{b_1, \cdots, b_m\}$ and $\gamma = \{\gamma(1), \cdots, \gamma(j)\} \in C_j^m$, we denote $\vec{b_\gamma} = \{b_{\gamma(1)}, \cdots, b_{\gamma(j)}\}$, and $b_\gamma = b_{\gamma(1)} \cdots b_{\gamma(j)}$, and $||\vec{b_\gamma}||_{Lip_\beta(\omega)} = ||b_{\gamma(1)}||_{Lip_\beta(\omega)} \cdots ||b_{\gamma(j)}||_{Lip_\beta(\omega)}$.

We say $\delta(x,\xi) \in S_{\epsilon,\sigma}^m$, if for $x, \xi \in R^n$, $\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial \xi^\beta}\delta(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\epsilon|\beta|+\sigma|\alpha|}$.

The pseudo-differential operators $\psi \cdot d \cdot o \cdot$ with symbols $\delta(x,\xi) \in S_{\epsilon,\sigma}^m$ is given by

$$T(f)(x) = \int_{R^n} e^{2\pi i(x,\xi)}\delta(x,\xi)\hat{f}(\xi)d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f.

The pseudo-differential operators $\psi \cdot d \cdot o \cdot$ also have another expression

$$T(f)(x) = \int_{R^n} K(x, x-y)f(y)dy,$$

where $K(x, x-y) = \int_{R^n} \delta(x,\xi)e^{2\pi i(x-y,\xi)}d\xi$.

Let $b_j, 1 \le j \le m$ be the fixed locally integrable functions on R^n. The multilinear commutator associated to the pseudo-differential operator is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} K(x, x-y)\prod_{j=1}^m (b_j(x) - b_j(y))f(y)dy.$$

Now, we state the main results as follows.

Theorem 2.1. *Let T be a $\psi \cdot d \cdot o \cdot$ with symbol $\delta(x,\xi) \in S_{1-a,\sigma}^{-\frac{na}{2}}$ for $0 \le \sigma < 1-a$ and $0 < a < 1$. Suppose $0 < \beta < \frac{1}{m(1-a)}$ and $\omega \in A_1$ and $b_j \in Lip_\beta(\omega)$ for $1 \le j \le m$ and $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}$. Then $T_{\vec{b}}$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-m+\frac{(q-1)m\beta}{n}})$.*

Theorem 2.2. *Let T be a $\psi \cdot d \cdot o \cdot$ with symbol $\delta(x,\xi) \in S_{1-a,\sigma}^{-\frac{na}{2}}$ for $0 \le \sigma < 1-a$ and $0 < a < 1$. Suppose $0 < \beta < \frac{1}{m(1-a)}$ and $\omega \in A_1$ and $\omega^{-1} \in A_1$ and $b_j \in Lip_\beta(\omega)$ for $1 \le j \le m$ and $1 < p < \infty$. Then $T_{\vec{b}}$ is bounded from $L^p(\omega)$ to $\tilde{F}_p^{m\beta,\infty}(\omega^{1-m-\frac{m\beta}{n}})$.*

3. Preliminary Lemmas

Lemma 3.1. *(see [10]) $\chi_Q \in A_1$ for any cube Q.*

Lemma 3.2. *(see [9][11]) For $0 < \beta < 1$ and $\omega \in A_1$ and $b \in Lip_\beta(\omega)$ and $1 \le p \le \infty$, we have*

$$||b||_{Lip_\beta(\omega)} \approx \sup_Q \omega(Q)^{-\frac{\beta}{n}}\left(\omega(Q)^{-1}\int_Q |b(x) - b_Q|^p \omega(y)^{1-p}dx\right)^{1/p}.$$

Lemma 3.3. *(see [10][11]) For $0<\beta<1$ and $\omega\in A_1$ and $b\in Lip_\beta(\omega)$ and any cube Q, we have*

$$\sup_{x\in Q}|b(x)-b_Q|\le C||b||_{Lip_\beta(\omega)}\omega(Q)^{1+\frac{\beta}{n}}|Q|^{-1}.$$

Lemma 3.4. *For $0<\beta<1$ and $\omega\in A_1$ and $b\in Lip_\beta(\omega)$ and any cube Q, there exists $\tilde{x}\in Q$ such that*

$$|b_{2^kQ}-b_Q|\le Ck\omega(\tilde{x})\omega(2^kQ)^{\frac{\beta}{n}}||b||_{Lip_\beta(\omega)}.$$

Lemma 3.5. *(see [3]) Let $\delta(x,\xi)\in S^{-\frac{na}{2}}_{1-a,\sigma}$ for $0\le\sigma<1-a$ and $0<a<1$. $K(x,w)$ denote the inverse Fourier transformations in the ξ-variable and in the distribution sense of $\delta(x,\xi)$, that is informally $K(x,w)=\int_{R^n}\delta(x,\xi)e^{2\pi i(w,\xi)}d\xi$.*

Then for $|x-x_0|\le d\le\frac{1}{2}$ and $N\ge 0$,

$$\left(\int_{(2^Nd)^{1-a}\le|y-x_0|\le(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|^2dy\right)^{\frac{1}{2}}\le$$

$$\le C|x-x_0|^{(1-a)(h-\frac{n}{2})}(2^{N+1}d)^{-h(1-a)},$$

where h is an integer such that $\frac{n}{2}<h<\frac{n}{2}+\frac{1}{1-a}$.

Lemma 3.6. *(see [3]) Let $\delta(x,\xi)\in S^0_{\epsilon,\sigma}$ for $0<\epsilon<1$ and as usual $K(x,w)=\int_{R^n}\delta(x,\xi)e^{2\pi i(w,\xi)}d\xi$.*

Then for $|w|\ge 1/4$ and arbitrarily large M, $|K(x,w)|\le C_M|w|^{-2M}$.

Lemma 3.7. *(see [14]) For $0<\beta<1$ and $\omega\in A_1$ and $1<p<\infty$ and $m>0$, we have*

$$||f||_{\tilde{F}^{m\beta,\infty}_p(\omega)}\approx\left|\left|\sup_{Q\ni\tilde{x}}|Q|^{-1-\frac{m\beta}{n}}\int_Q|f(x)-f_Q|dx\right|\right|_{L^p(\omega)} \quad (1)$$

$$\approx\left|\left|\sup_{Q\ni\tilde{x}}\inf_{c\in C_Q}|Q|^{-1-\frac{m\beta}{n}}\int_Q|f(x)-c|dx\right|\right|_{L^p(\omega)}. \quad (2)$$

Lemma 3.8. *(see [3]) Let $\delta(x,\xi)\in S^{-\frac{na}{2}}_{1-a,\sigma}$ for $0\le\sigma<1-a$ and $0<a<1$ and $\omega\in A_{\frac{p}{2}}$ with $2<p<\infty$.*

Then $||T(f)||_{L^p(\omega)}\le C_p||f||_{L^p(\omega)}$ for $f\in C_0^\infty(R^n)$.

Lemma 3.9. *(see [2][14]) Let $M_{r,m\beta}(f)(x)=\sup_{x\in Q}\left(|Q|^{-1+\frac{rm\beta}{n}}\int_Q|f(y)|^rdy\right)^{\frac{1}{r}}$ for $1<r<\infty$ and $\beta>0$ and $m>0$ and $\omega\in A_1$ and $r<p<\frac{m\beta}{n}$ and $\frac{1}{q}=\frac{1}{p}-\frac{m\beta}{n}$. Then*

$$||M_{r,m\beta}(f)(x)||_{L^q(\omega^q)}\le C||f||_{L^p(\omega^p)}.$$

Lemma 3.10. *(see [18]) Suppose $\omega\in A_1$ and $1<p<\infty$ and $1<r<\infty$. Then*

$$\left(\int_{R^n}M_r(f)(x)^p\omega(x)dx\right)^{\frac{1}{p}}\le C\left(\int_{R^n}|f(x)|^p\omega(x)dx\right)^{\frac{1}{p}}.$$

4. Proof of Theorem

Proof. Proof of Theorem 1

We first prove that for any $Q = Q(x_0, d)$, there exists some constant c and $1 < r < \infty$ such that for $f \in L^p(\omega)$ and $\tilde{x} \in Q$,

$$|Q|^{-1}\int_Q |T_{\vec{b}}(f)(x) - c|dx \le C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}(M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})).$$

We consider the case $m = 1$ and $d \le 1$.

Let $f(x) = f_1(x) + f_2(x)$ with $f_1(x) = f(x)\chi_J(x)$ and $f_2(x) = f(x)\chi_{J^c}(x)$, and $(b_1)_J = |J|^{-1}\int_J b_1(y)dy$, where J is a cube concentric with Q of side-length d^{1-a}.

$$\begin{aligned}
T_{\vec{b_1}}(f)(x) &= \int_{R^n} K(x, x-y)((b_1(x) - (b_1)_J) - (b_1(y) - (b_1)_J))f(y)dy = \\
&= (b_1(x) - (b_1)_J)\int_{R^n} K(x, x-y)f(y)dy - \\
&\quad - \int_{R^n} K(x, x-y)(b_1(y) - (b_1)_J)f_1(y)dy - \\
&\quad - \int_{R^n} K(x, x-y)(b_1(y) - (b_1)_J)f_2(y)dy = \\
&= (b_1(x) - (b_1)_J)T(f)(x) - T((b_1 - (b_1)_J)f_1)(x) - T((b_1 - (b_1)_J)f_2)(x),
\end{aligned}$$

thus

$$\begin{aligned}
& |Q|^{-1}\int_Q |T_{\vec{b_1}}(f)(x) - T(((b_1)_J - b_1)f_2)(x_0)|dx \\
\le\ & |Q|^{-1}\int_Q |b_1(x) - (b_1)_J||T(f)(x)|dx + |Q|^{-1}\int_Q |T((b_1 - (b_1)_J)f_1)(x)|dx \\
& + |Q|^{-1}\int_Q |T((b_1 - (b_1)_J)f_2)(x) - T((b_1 - (b_1)_J)f_2)(x_0)|dx \\
=\ & A_{1,1} + A_{1,2} + A_{1,3}.
\end{aligned}$$

For $A_{1,1}$, by Hölder's inequality with exponent $\frac{1}{r} + \frac{1}{r'} = 1$ and lemma 3, we have

$$\begin{aligned}
A_{1,1} &\le C|Q|^{-1}\left(\int_Q |b_1(x) - (b_1)_J|^{r'}dx\right)^{\frac{1}{r'}}\left(\int_Q |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\le C|Q|^{-1}\sup_{x\in J}|b_1(x) - (b_1)_J||J|^{\frac{1}{r'}}\left(\int_J |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\le C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(J)^{1+\frac{\beta}{n}}|J|^{-1}|J|^{-\frac{\beta}{n}}\left(|J|^{-1+\frac{r\beta}{n}}\int_Q |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\le C||\vec{b_1}||_{Lip_\beta(\omega)}\left(\frac{\omega(J)}{|J|}\right)^{1+\frac{\beta}{n}} M_{r,\beta}(T(f))(\tilde{x}) \\
&\le C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(T(f))(\tilde{x}).
\end{aligned}$$

For $A_{1,2}$, by Hölder's inequality with exponent $\frac{1}{r} + \frac{1}{r'} = 1$ and lemma 3,8, we have

$$\begin{aligned}
A_{1,2} &\leq C|Q|^{-1}\left(\int_{R^n}|T((b_1-(b_1)_J)f_1)(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
&\leq C|Q|^{-1}\left(\int_{R^n}|b_1(x)-(b_1)_J|^r|f_1(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
&\leq C|Q|^{-1}\left(\int_J|b_1(x)-(b_1)_J|^r|f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
&\leq C|Q|^{-1}\sup_{x\in J}|b_1(x)-(b_1)_J|\left(\int_J|f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
&\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(J)^{1+\frac{\beta}{n}}|J|^{-1}|J|^{-\frac{\beta}{n}}\left(|J|^{-1+\frac{r\beta}{n}}\int_Q|f(x)|^r dx\right)^{\frac{1}{r}}\\
&\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\left(\frac{\omega(J)}{|J|}\right)^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
&\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x}).
\end{aligned}$$

For $A_{1,3}$, choose h such that $\frac{n}{2}<h<\frac{n}{2}+\frac{1}{1-a}$. By Hölder's inequality with exponent $\frac{1}{r}+\frac{1}{s}+\frac{1}{2}=1$ and lemma 3,4,5, we have

$$\begin{aligned}
A_{1,3}(x) &= |T((b_1-(b_1)_J)f_2)(x)-T((b_1-(b_1)_J)f_2)(x_0)|\\
&= \left|\int_{R^n}(K(x,x-y)-K(x_0,x_0-y))(b_1(y)-(b_1)_J)f_2(y)dy\right|\\
&\leq \int_{|y-x_0|>d^{1-a}}|K(x,x-y)-K(x_0,x_0-y)||b_1(y)-(b_1)_J||f(y)|dy\\
&= \sum_{N=0}^{\infty}\int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)||b_1(y)-(b_1)_J||f(y)|dy\\
&= \sum_{N=0}^{\infty}\int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|\\
&\quad |b_1(y)-(b_1)_{2^{(N+1)(1-a)}J}||f(y)|dy\\
&+ \sum_{N=0}^{\infty}\int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|\\
&\quad |(b_1)_{2^{(N+1)(1-a)}J}-(b_1)_J||f(y)|dy\\
&\leq C\sum_{N=0}^{\infty}\left(\int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|^2dy\right)^{\frac{1}{2}}\\
&\times \sup_{y\in 2^{(N+1)(1-a)}J}|b_1(y)-(b_1)_{2^{(N+1)(1-a)}J}|(2^{N+1}d)^{\frac{n(1-a)}{s}}\\
&\quad\left(\int_{|y-x_0|\leq(2^{N+1}d)^{1-a}}|f(y)|^r dy\right)^{\frac{1}{r}}\\
&\quad +C\sum_{N=0}^{\infty}\left(\int_{(2^Nd)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|^2dy\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\times|(b_1)_{2^{(N+1)(1-a)}J}-(b_1)_J|(2^{N+1}d)^{\frac{n(1-a)}{s}}\left(\int_{|y-x_0|\le(2^{N+1}d)^{1-a}}|f(y)|^r dy\right)^{\frac{1}{r}}\\
\le\ & C\sum_{N=0}^{\infty}|x-x_0|^{(1-a)(h-\frac{n}{2})}(2^{N+1}d)^{-h(1-a)+\frac{n(1-a)}{2}}\\
&\times||\vec{b_1}||_{Lip_\beta(\omega)}\omega(2^{(N+1)(1-a)}J)^{1+\frac{\beta}{n}}|2^{(N+1)(1-a)}J|^{-1}|2^{(N+1)(1-a)}J|^{-\frac{\beta}{n}}\\
&\times\left((2^{N+1}d)^{n(1-a)(-1+\frac{r\beta}{n})}\int_{|y-x_0|\le(2^{N+1}d)^{1-a}}|f(y)|^r dy\right)^{\frac{1}{r}}\\
&+C\sum_{N=0}^{\infty}|x-x_0|^{(1-a)(h-\frac{n}{2})}(2^{N+1}d)^{-h(1-a)+\frac{n(1-a)}{2}}\\
&\times(N+1)\omega(\tilde{x})\omega(2^{(N+1)(1-a)}J)^{\frac{\beta}{n}}||\vec{b_1}||_{Lip_\beta(\omega)}|2^{(N+1)(1-a)}J|^{-\frac{\beta}{n}}\\
&\times\left((2^{N+1}d)^{n(1-a)(-1+\frac{r\beta}{n})}\int_{|y-x_0|\le(2^{N+1}d)^{1-a}}|f(y)|^r dy\right)^{\frac{1}{r}}\\
\le\ & C\textstyle\sum_{N=0}^{\infty}d^{(1-a)(h-\frac{n}{2})}(2^{N+1}d)^{(1-a)(\frac{n}{2}-h)}||\vec{b_1}||_{Lip_\beta(\omega)}\left(\frac{\omega(2^{(N+1)(1-a)}J)}{|2^{(N+1)(1-a)}J|}\right)^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
+\ & C\sum_{N=0}^{\infty}(N+1)d^{(1-a)(h-\frac{n}{2})}(2^{N+1}d)^{(1-a)(\frac{n}{2}-h)}||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})\\
&\left(\frac{\omega(2^{(N+1)(1-a)}J)}{|2^{(N+1)(1-a)}J|}\right)^{\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
\le\ & C\sum_{N=0}^{\infty}(N+2)2^{(N+1)(1-a)(\frac{n}{2}-h)}||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
\le\ & C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x}),
\end{aligned}
$$

thus

$$A_{1,3}\le C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m=1$ and $d\le 1$.

In case $m=1$ and $d>1$, we proceeds the case as follows.

Let $f(x)=f_1(x)+f_2(x)$ with $f_1(x)=f(x)\chi_{2Q}(x)$ and $f_2(x)=f(x)\chi_{(2Q)^c}(x)$ and $(b_1)_{2Q}=|2Q|^{-1}\int_{2Q}b_1(y)dy$. We have

$$
\begin{aligned}
T_{\vec{b_1}}(f)(x) &= \int_{R^n}K(x,x-y)((b_1(x)-(b_1)_{2Q})-(b_1(y)-(b_1)_{2Q}))f(y)dy\\
&= (b_1(x)-(b_1)_{2Q})\int_{R^n}K(x,x-y)f(y)dy-\\
&\quad -\int_{R^n}K(x,x-y)(b_1(y)-(b_1)_{2Q})f_1(y)dy-\\
&\quad -\int_{R^n}K(x,x-y)(b_1(y)-(b_1)_{2Q})f_2(y)dy\\
&= (b_1(x)-(b_1)_{2Q})T(f)(x)-T((b_1-(b_1)_{2Q})f_1)(x)-T((b_1-(b_1)_{2Q})f_2)(x),
\end{aligned}
$$

thus

$$\begin{aligned}
& |Q|^{-1}\int_Q |T_{\vec{b_1}}(f)(x)|dx \\
\leq\ & |Q|^{-1}\int_Q |b_1(x)-(b_1)_{2Q}||T(f)(x)|dx \\
+\ & |Q|^{-1}\int_Q |T((b_1-(b_1)_{2Q})f_1)(x)|dx \\
+\ & |Q|^{-1}\int_Q |T((b_1-(b_1)_{2Q})f_2)(x)|dx \\
=\ & A_{2,1}+A_{2,2}+A_{2,3}.
\end{aligned}$$

Similar to $A_{1,1}$, $A_{2,1}\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(T(f))(\tilde{x})$.

Similar to $A_{1,2}$, $A_{2,1}\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})$.

For $A_{2,3}$, by Hölder's inequality with exponent $\frac{1}{r}+\frac{1}{r'}=1$ and lemma 3,4,6, we have

$$\begin{aligned}
A_{2,3}(x) &= |T((b_1-(b_1)_{2Q})f_2)(x)| \\
&= \left|\int_{R^n} K(x,x-y)(b_1(y)-(b_1)_{2Q})f_2(y)dy\right| \\
&\leq \int_{|y-x_0|>2d} |K(x,x-y)||b_1(y)-(b_1)_{2Q}||f(y)|dy \\
&\leq C\int_{|y-x_0|>2d} |x-y|^{-2n}|b_1(y)-(b_1)_{2Q}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\int_{2^N d\leq|y-x_0|\leq 2^{N+1}d} |x-y|^{-2n}|b_1(y)-(b_1)_{2^{N+1}Q}||f(y)|dy \\
&\quad +C\sum_{N=1}^{\infty}\int_{2^N d\leq|y-x_0|\leq 2^{N+1}d} |x-y|^{-2n}|(b_1)_{2^{N+1}Q}-(b_1)_{2Q}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n}\sup_{y\in 2^{N+1}Q}|b_1(y)-(b_1)_{2^{N+1}Q}|\left(2^{N+1}d\right)^{\frac{n}{r'}} \\
&\quad \left(\int_{|y-x_0|\leq 2^{N+1}d}|f(y)|^r dy\right)^{\frac{1}{r}} \\
&+ C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n}\left|(b_1)_{2^{N+1}Q}-(b_1)_{2Q}\right|(2^{N+1}d)^{\frac{n}{r'}}\left(\int_{|y-x_0|\leq 2^{N+1}d}|f(y)|^r dy\right)^{\frac{1}{r}} \\
&\leq C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n+n}||\vec{b_1}||_{Lip_\beta(\omega)}\omega(2^{N+1}Q)^{1+\frac{\beta}{n}}|2^{N+1}Q|^{-1}|2^{N+1}Q|^{-\frac{\beta}{n}} \\
&\quad \times\left((2^{N+1}d)^{n(-1+\frac{r\beta}{n})}\int_{|y-x_0|\leq 2^{N+1}d}|f(y)|^r dy\right)^{\frac{1}{r}} \\
&\quad +C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-2n+n}(N+1)\omega(\tilde{x})\omega(2^{N+1}Q)^{\frac{\beta}{n}}||\vec{b_1}||_{Lip_\beta(\omega)}|2^{N+1}Q|^{-\frac{\beta}{n}} \\
&\quad \times\left((2^{N+1}d)^{n(-1+\frac{r\beta}{n})}\int_{|y-x_0|\leq 2^{N+1}d}|f(y)|^r dy\right)^{\frac{1}{r}}
\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{N=1}^{\infty}\left(2^{N+1}d\right)^{-n}||\vec{b_1}||_{Lip_\beta(\omega)}\left(\frac{\omega(2^{N+1}Q)}{|2^{N+1}Q|}\right)^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
&\quad +C\sum_{N=1}^{\infty}(N+1)\left(2^{N+1}d\right)^{-n}\omega(\tilde{x})\left(\frac{\omega(2^{N+1}Q)}{|2^{N+1}Q|}\right)^{\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
&\leq C\sum_{N=1}^{\infty}(N+2)\left(2^{N+1}d\right)^{-n}||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
&\leq C\sum_{N=1}^{\infty}(N+2)2^{-n(N+1)}d^{-n}||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x})\\
&\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x}),
\end{aligned}$$

thus

$$A_{2,3}\leq C||\vec{b_1}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{1+\frac{\beta}{n}}M_{r,\beta}(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m=1$ and $d>1$.

Now, we consider the case $m\geq 2$ and $d\leq 1$.

Let $f(x)=f_1(x)+f_2(x)$ with $f_1(x)=f(x)\chi_J(x)$ and $f_2(x)=f(x)\chi_{J^c}(x)$, and $(b_j)_J=|J|^{-1}\int_J b_j(y)dy$ for $1\leq j\leq m$, where J is a cube concentric with Q of side-length d^{1-a}.

$$\begin{aligned}
T_{\vec{b}}(f)(x) &= \int_{R^n}K(x,x-y)\prod_{j=1}^{m}((b_j(x)-(b_j)_J)-(b_j(y)-(b_j)_J))f(y)dy\\
&= \sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(-1)^{m-j}(b(x)-b_J)_\gamma\int_{R^n}K(x,x-y)(b(y)-b_J)_{\gamma^c}f(y)dy\\
&= \prod_{j=1}^{m}(b_j(x)-(b_j)_J)\int_{R^n}K(x,x-y)f(y)dy\\
&\quad +\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(-1)^{m-j}(b(x)-b_J)_\gamma\int_{R^n}K(x,x-y)(b(y)-b_J)_{\gamma^c}f(y)dy\\
&\quad +(-1)^m\int_{R^n}K(x,x-y)\prod_{j=1}^{m}(b_j(y)-(b_j)_J)f_1(y)dy\\
&\quad +(-1)^m\int_{R^n}K(x,x-y)\prod_{j=1}^{m}(b_j(y)-(b_j)_J)f_2(y)dy\\
&= \prod_{j=1}^{m}(b_j(x)-(b_j)_J)T(f)(x)+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(b(x)-b_J)_\gamma T((b-b_J)_{\gamma^c}f)(x)\\
&\quad +(-1)^mT(\prod_{j=1}^{m}(b_j-(b_j)_J)f_1)(x)+(-1)^mT(\prod_{j=1}^{m}(b_j-(b_j)_J)f_2)(x),
\end{aligned}$$

thus

$$|Q|^{-1}\int_Q|T_{\vec{b}}(f)(x)-T(\prod_{j=1}^{m}((b_j)_J-b_j)f_2)(x_0)|dx$$

$$
\begin{aligned}
&\leq |Q|^{-1}\int_Q \prod_{j=1}^m |b_j(x)-(b_j)_J||T(f)(x)|dx \\
&+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1}\int_Q |(b(x)-b_J)_\gamma||T((b-b_J)_{\gamma^c}f)(x)|dx \\
&+|Q|^{-1}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_J)f_1)(x)|dx \\
&+|Q|^{-1}\int_Q |T(\prod_{j=1}^m (b_j-(b_j)_J)f_2)(x)-T(\prod_{j=1}^m (b_j-(b_j)_J)f_2)(x_0)|dx \\
&= A_{3,1}+A_{3,2}+A_{3,3}+A_{3,4}.
\end{aligned}
$$

For $A_{3,1}$, by Hölder's inequality with exponent $\frac{1}{r_1}+\cdots+\frac{1}{r_m}+\frac{1}{r}=1$ and lemma 3, we have

$$
\begin{aligned}
A_{3,1} &\leq C|Q|^{-1}\prod_{j=1}^m\left(\int_J |b_j(x)-(b_j)_J|^{r_j}dx\right)^{\frac{1}{r_j}}\left(\int_J |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C|Q|^{-1}\prod_{j=1}^m\left(\sup_{x\in J}|b_j(x)-(b_j)_J||J|^{\frac{1}{r_j}}\right)\left(\int_J |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C|Q|^{-1}\left(\prod_{j=1}^m \sup_{x\in J}|b_j(x)-(b_j)_J|\right)|J|^{1-\frac{1}{r}}\left(\int_J |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C\left(\prod_{j=1}^m ||b_j||_{Lip_\beta(\omega)}\right)\omega(J)^{m+\frac{m\beta}{n}}|J|^{-m}|J|^{-\frac{m\beta}{n}}\left(|J|^{-1+\frac{rm\beta}{n}}\int_J |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}\left(\frac{\omega(J)}{|J|}\right)^{m+\frac{m\beta}{n}}\left(|J|^{-1+\frac{rm\beta}{n}}\int_J |T(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(T(f))(\tilde{x}).
\end{aligned}
$$

For $A_{3,2}$, by Hölder's inequality with exponent $\frac{1}{r}+\frac{1}{r'}=1$, and lemma 1,3,8, we have

$$
\begin{aligned}
A_{3,2} &\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1}\left(\int_J |(b(x)-b_J)_\gamma|^{r'}dx\right)^{\frac{1}{r'}}\left(\int_{R^n}|T((b-b_J)_{\gamma^c}f)(x)|^r\chi_J(x)dx\right)^{\frac{1}{r}} \\
&\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1}\left(\int_J |(b(x)-b_J)_\gamma|^{r'}dx\right)^{\frac{1}{r'}}\left(\int_{R^n}|(b(x)-b_J)_{\gamma^c}|^r|f(x)|^r\chi_J(x)dx\right)^{\frac{1}{r}} \\
&\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1}\sup_{x\in J}|(b(x)-b_J)_\gamma||J|^{\frac{1}{r'}}\left(\int_J |(b(x)-b_J)_{\gamma^c}|^r|f(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1}\sup_{x\in J}|(b(x)-b_J)_\gamma||J|^{\frac{1}{r'}}\sup_{x\in J}|(b(x)-b_J)_{\gamma^c}|\left(\int_J |f(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||\vec{b_\gamma}||_{Lip_\beta(\omega)}\omega(J)^{j+\frac{j\beta}{n}}|J|^{-j}||\vec{b_{\gamma^c}}||_{Lip_\beta(\omega)}\omega(J)^{m-j+\frac{(m-j)\beta}{n}}|J|^{-(m-j)}
\end{aligned}
$$

$$
\begin{aligned}
&\times|J|^{-\frac{m\beta}{n}}\left(|J|^{-1+\frac{rm\beta}{n}}\int_J|f(x)|^r dx\right)^{\frac{1}{r}}\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||\vec{b}||_{Lip_\beta(\omega)}\left(\frac{\omega(J)}{|J|}\right)^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x})\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x})\\
\leq\ & C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x}).
\end{aligned}
$$

For $A_{3,3}$, by Hölder's inequality with exponent $\frac{1}{r}+\frac{1}{r'}=1$ and lemma 3,8, we have

$$
\begin{aligned}
A_{3,3}\ \leq\ & C|Q|^{-1}\left(\int_{R^n}|T(\prod_{j=1}^m(b_j-(b_j)_J)f_1)(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
\leq\ & C|Q|^{-1}\left(\int_{R^n}\prod_{j=1}^m|b_j(x)-(b_j)_J|^r|f_1(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
\leq\ & C|Q|^{-1}\left(\int_J\prod_{j=1}^m|b_j(x)-(b_j)_J|^r|f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
\leq\ & C|Q|^{-1}\left(\prod_{j=1}^m\sup_{x\in J}|b_j(x)-(b_j)_J|\right)\left(\int_J|f(x)|^r dx\right)^{\frac{1}{r}}|Q|^{\frac{1}{r'}}\\
\leq\ & C\left(\prod_{j=1}^m||b_j||_{Lip_\beta(\omega)}\right)\omega(J)^{m+\frac{m\beta}{n}}|J|^{-m}|J|^{-\frac{m\beta}{n}}\left(|J|^{-1+\frac{rm\beta}{n}}\int_J|f(x)|^r dx\right)^{\frac{1}{r}}\\
\leq\ & C||\vec{b}||_{Lip_\beta(\omega)}\left(\frac{\omega(J)}{|J|}\right)^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x})\\
\leq\ & C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x}).
\end{aligned}
$$

For $A_{3,4}$, choose h such that $\frac{n}{2}<h<\frac{n}{2}+\frac{1}{1-a}$. By Hölder's inequality with exponent $\frac{1}{s}+\frac{1}{r}+\frac{1}{2}=1$ and lemma 3,4,5, we have

$$
\begin{aligned}
A_{3,4}(x)\ =\ & \left|\int_{R^n}(K(x,x-y)-K(x_0,x_0-y))\prod_{j=1}^m(b_j(y)-(b_j)_J)f_2(y)dy\right|\\
\leq\ & \int_{|y-x_0|>d^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|\prod_{j=1}^m|b_j(y)-(b_j)_J||f(y)|dy\\
\leq\ & \sum_{N=0}^\infty\int_{(2^N d)^{1-a}\leq|y-x_0|\leq(2^{N+1}d)^{1-a}}|K(x,x-y)-K(x_0,x_0-y)|\\
& \prod_{j=1}^m|b_j(y)-(b_j)_J||f(y)|dy
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{N=0}^{\infty} \int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x, x-y) - K(x_0, x_0 - y)| \\
&\quad \prod_{j=1}^{m} \left(|b_j(y) - (b_j)_{2^{(N+1)(1-a)}J}| + |(b_j)_{2^{(N+1)(1-a)}J} - (b_j)_J| \right) |f(y)| dy \\
&\leq \sum_{N=0}^{\infty} \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} |(b_{2^{(N+1)(1-a)}J} - b_J)_\gamma| \\
&\quad \times \int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x, x-y) - K(x_0, x_0 - y)| \\
&\quad |(b(y) - b_{2^{(N+1)(1-a)}J})_{\gamma^c}| |f(y)| dy \\
&\leq C \sum_{N=0}^{\infty} \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} |(b_{2^{(N+1)(1-a)}J} - b_J)_\gamma| \\
&\quad \times \left(\int_{(2^N d)^{1-a} \leq |y-x_0| \leq (2^{N+1}d)^{1-a}} |K(x, x-y) - K(x_0, x_0 - y)|^2 dy \right)^{\frac{1}{2}} \\
&\quad \times \sup_{y \in 2^{(N+1)(1-a)}J} |(b(y) - b_{2^{(N+1)(1-a)}J})_{\gamma^c}| \left(2^{N+1}d\right)^{\frac{n(1-a)}{s}} \\
&\quad \left(\int_{|y-x_0| \leq (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{\frac{1}{r}} \\
&\leq C \sum_{N=0}^{\infty} \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} (N+1)^j \omega(\tilde{x})^j \omega(2^{(N+1)(1-a)}J)^{\frac{j\beta}{n}} ||\vec{b_\gamma}||_{Lip_\beta(\omega)} \\
&\quad \times |x - x_0|^{(1-a)(h - \frac{n}{2})} (2^{N+1}d)^{-h(1-a) + \frac{n(1-a)}{2}} \\
&\quad \times \omega(2^{(N+1)(1-a)}J)^{m-j+\frac{(m-j)\beta}{n}} |2^{(N+1)(1-a)}J|^{-(m-j)} ||\vec{b_{\gamma^c}}||_{Lip_\beta(\omega)} \\
&\quad \times |2^{(N+1)(1-a)}J|^{-\frac{m\beta}{n}} \\
&\quad \left((2^{N+1}d)^{n(1-a)(-1+\frac{rm\beta}{n})} \int_{|y-x_0| \leq (2^{N+1}d)^{1-a}} |f(y)|^r dy \right)^{\frac{1}{r}} \\
&\leq C \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} \sum_{N=0}^{\infty} (N+1)^j \omega(\tilde{x})^j \left(\frac{\omega(2^{(N+1)(1-a)}J)}{|2^{(N+1)(1-a)}J|} \right)^{m-j+\frac{m\beta}{n}} \\
&\quad d^{(1-a)(h-\frac{n}{2})} (2^{N+1}d)^{(1-a)(\frac{n}{2}-h)} \times ||\vec{b}||_{Lip_\beta(\omega)} M_{r,m\beta}(f)(\tilde{x}) \\
&\leq C \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} \sum_{N=0}^{\infty} (N+1)^j 2^{(N+1)(1-a)(\frac{n}{2}-h)} \\
&\quad ||\vec{b}||_{Lip_\beta(\omega)} \omega(\tilde{x})^{m+\frac{m\beta}{n}} M_{r,m\beta}(f)(\tilde{x}) \\
&\leq C \sum_{j=0}^{m} \sum_{\gamma \in C_j^m} ||\vec{b}||_{Lip_\beta(\omega)} \omega(\tilde{x})^{m+\frac{m\beta}{n}} M_{r,m\beta}(f)(\tilde{x}) \\
&\leq C ||\vec{b}||_{Lip_\beta(\omega)} \omega(\tilde{x})^{m+\frac{m\beta}{n}} M_{r,m\beta}(f)(\tilde{x}),
\end{aligned}
$$

thus

$$A_{3,4}\le C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m\ge 2$ and $d\le 1$.

In case $m\ge 2$ and $d>1$, we proceeds the case as follows.

Let $f(x)=f_1(x)+f_2(x)$, with $f_1(x)=f(x)\chi_{2Q}(x)$ and $f_2(x)=f(x)\chi_{(2Q)^c}(x)$, and $(b_j)_{2Q}=|2Q|^{-1}\int_{2Q}b_j(y)dy$ for $1\le j\le m$.

$$\begin{aligned}T_{\vec{b}}(f)(x) &= \prod_{j=1}^m(b_j(x)-(b_j)_{2Q})T(f)(x)\\ &\quad+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}(b(x)-b_{2Q})_\gamma T((b-b_{2Q})_{\gamma^c}f)(x)\\ &\quad+(-1)^mT(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_1)(x)\\ &\quad+(-1)^mT(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(x),\end{aligned}$$

thus

$$\begin{aligned}&|Q|^{-1}\int_Q|T_{\vec{b}}(f)(x)|dx\\ \le\ &|Q|^{-1}\int_Q\prod_{j=1}^m|b_j(x)-(b_j)_{2Q}||T(f)(x)|dx\\ &+\sum_{j=1}^{m-1}\sum_{\gamma\in C_j^m}|Q|^{-1}\int_Q|(b(x)-b_{2Q})_\gamma||T((b-b_{2Q})_{\gamma^c}f)(x)|dx\\ &+|Q|^{-1}\int_Q|T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_1)(x)|dx\\ &+|Q|^{-1}\int_Q|T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(x)|dx\\ =\ &A_{4,1}+A_{4,2}+A_{4,3}+A_{4,4}.\end{aligned}$$

Similar to $A_{3,1}$, $A_{4,1}\le C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(T(f))(\tilde{x})$.
Similar to $A_{3,2}$, $A_{4,2}\le C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x})$.
Similar to $A_{3,3}$, $A_{4,3}\le C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x})$.

For $A_{4,4}$, by Hölder's inequality with exponent $\frac{1}{r}+\frac{1}{r'}=1$ and lemma 3,4,6, we have

$$\begin{aligned}A_{4,4}(x) &= \left|T(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(x)\right|\\ &= \left|\int_{R^n}K(x,x-y)\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f_2(y)dy\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{|y-x_0|>2d} |K(x,x-y)| \prod_{j=1}^{m} |b_j(y)-(b_j)_{2Q}||f(y)|dy \\
&\leq C\int_{|y-x_0|>2d} |x-y|^{-2n} \prod_{j=1}^{m} |b_j(y)-(b_j)_{2Q}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\int_{2^N d\leq|y-x_0|\leq 2^{N+1}d} |x-y|^{-2n} \prod_{j=1}^{m} |b_j(y)-(b_j)_{2Q}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}(2^{N+1}d)^{-2n}\int_{2^N d\leq|y-x_0|\leq 2^{N+1}d} \\
&\quad \prod_{j=1}^{m}(|b_j(y)-(b_j)_{2^{N+1}Q}|+|(b_j)_{2^{N+1}Q}-(b_j)_{2Q}|)|f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(2^{N+1}d)^{-2n}|(b_{2^{N+1}Q}-b_{2Q})_\gamma| \\
&\quad \int_{|y-x_0|\leq 2^{N+1}d}|(b(y)-b_{2^{N+1}Q})_{\gamma^c}||f(y)|dy \\
&\leq C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(2^{N+1}d)^{-2n}|(b_{2^{N+1}Q}-b_{2Q})_\gamma| \\
&\quad \sup_{y\in 2^{N+1}Q}|(b(y)-b_{2^{N+1}Q})_{\gamma^c}|(2^{N+1}d)^{\frac{n}{r'}}\times\left(\int_{|y-x_0|\leq 2^{N+1}d}|f(x)|^r dy\right)^{\frac{1}{r}} \\
&\leq C\sum_{N=1}^{\infty}\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}(2^{N+1}d)^{-2n+n}(N+1)^j\omega(\tilde{x})^j\omega(2^{N+1}Q)^{\frac{j\beta}{n}}||\vec{b_\gamma}||_{Lip_\beta(\omega)} \\
&\quad \times\omega(2^{N+1}Q)^{m-j+\frac{(m-j)\beta}{n}}|2^{N+1}Q|^{-(m-j)}||\vec{b_{\gamma^c}}||_{Lip_\beta(\omega)} \\
&\quad \times|2^{N+1}Q|^{-\frac{m\beta}{n}}\left((2^{N+1}d)^{n(-1+\frac{rm\beta}{n})}\int_{|y-x_0|\leq 2^{N+1}d}|f(x)|^r dy\right)^{\frac{1}{r}} \\
&\leq C\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}\sum_{N=1}^{\infty}(2^{N+1}d)^{-n}(N+1)^j\omega(\tilde{x})^j \\
&\quad \left(\frac{\omega(2^{N+1}Q)}{|2^{N+1}Q|}\right)^{m-j+\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta(\omega)}M_{r,m\beta}(f)(\tilde{x}) \\
&\leq C\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}\sum_{N=1}^{\infty}(N+1)^j 2^{-n(N+1)}d^{-n}||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x}) \\
&\leq C\sum_{j=0}^{m}\sum_{\gamma\in C_j^m}||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x}) \\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x}),
\end{aligned}
$$

thus

$$A_{4,4}\leq C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_r(f)(\tilde{x}).$$

Combining all the estimates, we finish the case $m \geq 2$ and $d > 1$.

So, for $m \geq 1$ and any cube Q, there exists c and $1 < r < \infty$ such that for $f \in L^p(\omega)$,

$$|Q|^{-1}\int_Q |T_{\vec{b}}(f)(x) - c|dx \leq C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}(M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})).$$

Further, we have

$$(T_{\vec{b}}(f))^{\#}(\tilde{x}) \leq C||\vec{b}||_{Lip_\beta}\omega(\tilde{x})^{m+\frac{m\beta}{n}}(M_{r,m\beta}(T(f))(\tilde{x}) + M_{r,m\beta}(f)(\tilde{x})).$$

By Minkowski's inequality and lemma 8,9, we have

$$\begin{aligned}
&||T_{\vec{b}}(f)||_{L^q(\omega^{1-m+\frac{(q-1)m\beta}{n}})} \leq C||M(T_{\vec{b}}(f))(\tilde{x})||_{L^q(\omega^{1-m+\frac{(q-1)m\beta}{n}})}\\
&\leq C||(T_{\vec{b}}(f))^{\#}(\tilde{x})||_{L^q(\omega^{1-m+\frac{(q-1)m\beta}{n}})}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(T(f))(\tilde{x})||_{L^q(\omega^{1-m+\frac{(q-1)m\beta}{n}})}\\
&+C||\vec{b}||_{Lip_\beta(\omega)}||\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_{r,m\beta}(f)(\tilde{x})||_{L^q(\omega^{1-m+\frac{(q-1)m\beta}{n}})}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||M_{r,m\beta}(T(f))||_{L^q(\omega^{\frac{q}{p}})} + ||\vec{b}||_{Lip_\beta(\omega)}||M_{r,m\beta}(f)||_{L^q(\omega^{\frac{q}{p}})}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||T(f)||_{L^p(\omega)} + ||\vec{b}||_{Lip_\beta(\omega)}||f||_{L^p(\omega)}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||f||_{L^p(\omega)}.
\end{aligned}$$

This completes the proof of Theorem 1. □

Proof. Proof of Theorem 2

Similar to Theorem 1, for any$Q = Q(x_0, d)$, there exists some constant c and $1 < r < \infty$ such that for $f \in L^p(\omega)$ and $\tilde{x} \in Q$,

$$|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(f)(x) - c|dx \leq C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}(M(T(f))(\tilde{x}) + M_r(f)(\tilde{x})).$$

Further, we have

$$\sup_{Q\ni\tilde{x}}\inf_{c\in C_Q}|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(x) - c|dx \leq C||\vec{b}||_{Lip_\beta(\omega)}\omega(\tilde{x})^{m+\frac{m\beta}{n}}(M_r(T(f))(\tilde{x}) + M_r(f)(\tilde{x})).$$

By Minkowski's inequality and lemma 7,8,10, we have

$$\begin{aligned}
||T_{\vec{b}}(f)||_{\dot{F}_p^{m\beta,\infty}(\omega^{1-m-\frac{m\beta}{n}})} &\approx \left|\left|\sup_{Q\ni\tilde{x}}\inf_{c\in C_Q}|Q|^{-1-\frac{m\beta}{n}}\int_Q |T_{\vec{b}}(f)(x) - c|dx\right|\right|_{L^p(\omega^{1-m-\frac{m\beta}{n}})}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_r(T(f))(\tilde{x})||_{L^p(\omega^{1-m-\frac{m\beta}{n}})}\\
&\quad +C||\vec{b}||_{Lip_\beta(\omega)}||\omega(\tilde{x})^{m+\frac{m\beta}{n}}M_r(f)(\tilde{x})||_{L^p(\omega^{1-m-\frac{m\beta}{n}})}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||M_r(T(f))||_{L^p(\omega)}\\
&\quad +C||\vec{b}||_{Lip_\beta(\omega)}||M_r(f)||_{L^p(\omega)}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||T(f)||_{L^p(\omega)} + ||\vec{b}||_{Lip_\beta(\omega)}||f||_{L^p(\omega)}\\
&\leq C||\vec{b}||_{Lip_\beta(\omega)}||f||_{L^p(\omega)}.
\end{aligned}$$

This completes the proof of Theorem 2. □

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