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Factorization of an inheritance knowledge base (I)

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Abstract. In [4] we introduced a model of an extended inheritance for knowledge representation. This model allows the multiple inheritance and includes a parameter for each attribute value. This parameter can describe some features of the attribute values, for example the uncertainty. This paper is a starting point for a possible research line to study the decomposition of these knowledge bases into disjoint components. This is named the *factorization problem*. The name comes from the fact that the set of all components of a knowledge base K is the factor set $Obj(K)/\widetilde{\rho}_K$, where $Obj(K)$ is the set of all objects of K and $\widetilde{\rho}_K$ is an equivalence relation defined by means of the inheritance from K. A necessary and sufficient condition for factorization is given. All the results proved in this paper and in [5] constitute the algebraic background of a forthcoming paper as we mention in the last section.

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1. Introduction

Various theoretical and practical aspects of the inheritance mechanism were treated in the last decade: an extension of the disjunctive logic programming (with strong negation) by inheritance ([2]); the inheritance of business rules in the medical insurance domain was studied in [3]; a natural model-theoretic semantics for inheritance in frame-based knowledge bases, which supports inference by inheritance as well as inference via rules was treated in [10]; the use of the lattice theory to characterize the features of the answer mapping in knowledge systems based on inheritance were described in [6], [7] and [8]; the use of the voice interfaces to interrogate an inheritance based knowledge system is given in [9].

Intuitively, a knowledge base K which uses the inheritance mechanism is a finite set of objects and an interrogation of K is defined by a pair (f, a_1), where f is the name of an object of K and a_1 is an attribute. The value of a attribute is of the form (v_1, q_1), where v_1 is a direct value of a_1 or the name of a procedure which returns the value of a_1 and q_1 is a parameter specifying some feature of a_1.

We can say that m is the greatest number of components such that each component is an independent knowledge base that uses the inheritance mechanism. The problem specified above can be named the *factorization problem* of an inheritance knowledge base. This name comes from universal algebra domain, where the factor set X/ρ of the set X with respect to the equivalence relation ρ is the set of all equivalence classes.

The main aspects connected by our research presented in this paper can be shortly described as follows:

(1) Based on the inheritance mechanism from the knowledge base K we define an equivalence relation $\widetilde{\rho}_K$ on the set of objects.

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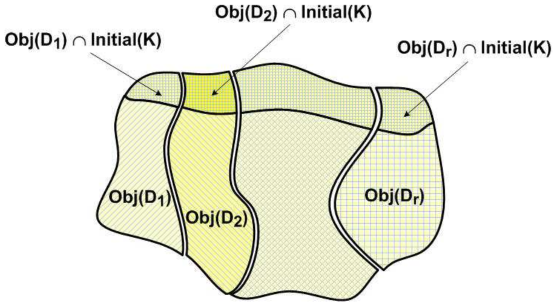

FIGURE 1. The components of a knowledge base

(2) We prove that the set of all components of K is the factor set $K/\widetilde{\rho}_K$ and thus a component is an equivalence class.
(3) We give an method to find the components of a knowledge base. This method is based on the fact that a component is the equivalence class generated by some free of parents objects.
(4) We give a necessary and sufficient condition for the factorization of a knowledge base.

Intuitively we represented in Figure 1 the decomposition of a knowledge base into components. We denoted by $Initial(K)$ the set of all free of parent objects from K and $\{Obj(D_i)\}_{i=1}^{r}$ gives its components. Each $Obj(D_i)$ is "generated" by its initial objects and becomes the support set of the knowledge base $D_i = (Obj(D_i), \widetilde{\rho}_{D_i})$, where $\widetilde{\rho}_{D_i}$ is the restriction of $\widetilde{\rho}_K$.

The paper is organized as follows. Section 2 contains several basic results concerning the relational algebra. Based on [4] in Section 3 we define the concepts of inheritance knowledge base and the answer mapping for such a structure. The concept of inheritance knowledge base described in this section extends the classical model: a parameter is assigned to every attribute value such that this entity can represent the uncertainty or the risk factor if we choose such a representation in knowledge processing. In Section 4 we define the concept of component of a knowledge base. Section 5 contains the algebraic properties of the components of a knowledge base. In Section 6 we present a mathematical study of the factorization problem. Based on this factorization in [5] we show that the answer mapping of a knowledge base extends the answer mapping of each component and the answer mapping can be locally computed in some component. The last section includes the conclusions of our study and we specify several open problems.

2. Elements of relational algebra

In this section we present several algebraic properties of the binary relations, which are used in the next sections. We recall the following properties:

- The usual product operation $\circ$ between binary relations is defined as follows:

$$\rho_1 \circ \rho_2 = \{(x, y) \in X \times X \mid \exists z \in X : (x, z) \in \rho_1, (z, y) \in \rho_2\}$$

- The product operation is an associative one:

$$(\rho_1 \circ \rho_2) \circ \rho_3 = \rho_1 \circ (\rho_2 \circ \rho_3)$$

- The powers of the relation ρ are defined recursively as follows:

$$\begin{cases} \rho^1 = \rho \\ \rho^{n+1} = \rho^n \circ \rho, \ n \geq 0 \end{cases}$$

Definition 2.1. *If $\rho \subseteq X \times X$ is a binary relation then we define the following binary relations on X:*

$\rho^{-1} = \{ \ (y, x) \mid (x, y) \in \rho \ \}$
$\overline{\rho} = \rho \cup \rho^{-1}$
$\widetilde{\rho} = \bigcup_{i \geq 1} \overline{\rho}^{\ i}$

Proposition 2.1. *The relation $\overline{\rho}$ is the least binary relation $r \subseteq X \times X$ such that r is symmetric and $\rho \subseteq r$.*

Proof. We observe that if $(x, y) \in \overline{\rho}$ then $(x, y) \in \rho$ or $(x, y) \in \rho^{-1}$. In the first case we have $(y, x) \in \rho^{-1}$, but $\rho^{-1} \subseteq \overline{\rho}$, therefore $(y, x) \in \overline{\rho}$. Similarly if $(x, y) \in \rho^{-1}$ then $(y, x) \in \rho$, therefore $(y, x) \in \overline{\rho}$ because $\rho \subseteq \overline{\rho}$. Thus $\overline{\rho}$ is a symmetric binary relation. Let us take a binary relation $r \subseteq X \times X$ such that r is symmetric and $\rho \subseteq r$. Suppose that $(x, y) \in \overline{\rho}$. Two cases are possible:
(1) If $(x, y) \in \rho$ then $(x, y) \in r$.
(2) If $(x, y) \in \rho^{-1}$ then $(y, x) \in \rho$. But $\rho \subseteq r$, therefore $(y, x) \in r$. The relation r is a symmetric one, therefore $(x, y) \in r$.

It follows that $\overline{\rho} \subseteq r$. □

Proposition 2.2. *If $\rho \subseteq X \times X$ is a symmetric binary relation then for every $i \geq 1$ the relation $\rho^{\ i}$ is also a symmetric relation. Particularly, $\overline{\rho}^{\ i}$ is a symmetric relation.*

Proof. If $(x, y) \in \rho^{\ i}$ then there are $z_1, \ldots, z_i \in X$ such that $z_0 = x$, $z_i = y$ and $(z_j, z_{j+1}) \in \rho$ for every $j \in \{0, \ldots, i-1\}$. But ρ is a symmetric relation, therefore $(z_{j+1}, z_j) \in \rho$ for every $j \in \{i-1, \ldots, 0\}$. It follows that $(z_i, z_0) \in \rho$. □

Definition 2.2. *An element $x \in X$ is an **isolated element** with respect to ρ if there is no $y \in X$ such that $(x, y) \in \overline{\rho}$.*

Proposition 2.3. *If there is no isolated element with respect to ρ then $\widetilde{\rho}$ is an equivalence relation.*

Proof. We prove first that $\widetilde{\rho}$ is symmetric. Consider a pair $(x, y) \in \widetilde{\rho}$. There is a natural number $i \geq 1$ such that $(x, y) \in \overline{\rho}^{\ i}$. By Proposition 2.2 we deduce that $(y, x) \in \overline{\rho}^{\ i}$, therefore $(y, x) \in \widetilde{\rho}$.
Let us verify the transitivity of the relation $\widetilde{\rho}$. Consider the pairs $(x, z) \in \widetilde{\rho}$ and $(z, y) \in \widetilde{\rho}$. There is $m \geq 1$ and $s \geq 1$ such that $(x, z) \in \overline{\rho}^{\ m}$ and $(z, y) \in \overline{\rho}^{\ s}$. Obviously $(x, y) \in \overline{\rho}^{\ m} \circ \overline{\rho}^{\ s}$. But $\overline{\rho}^{\ m} \circ \overline{\rho}^{\ s} = \overline{\rho}^{\ m+s}$ and $\overline{\rho}^{\ m+s} \subseteq \widetilde{\rho}$. It follows that $(x, y) \in \widetilde{\rho}$.
Let us consider $x \in X$. This is no isolated element with respect to ρ, particularly x is not an isolated element. It follows that there is $y \in X$ such that $(x, y) \in \overline{\rho}$. By the symmetry of $\overline{\rho}$ we deduce that $(y, x) \in \overline{\rho}$. On the other hand $\overline{\rho} \subseteq \widetilde{\rho}$ and the relation $\widetilde{\rho}$ is transitive. It follows that $(x, x) \in \widetilde{\rho}$ because $(x, y) \in \widetilde{\rho}$ and $(y, x) \in \widetilde{\rho}$. This proves the reflexivity of the relation $\widetilde{\rho}$. □

3. Inheritance knowledge bases

In this section we recall the main concepts and results obtained in [4], which are used in the next sections of this paper. The following notations are used:

- L_{obj} is the set of the *object names*. Each element of L_{obj} can designate some object.
- L_{attr} is the language of all *attribute* names.
- V_{dir} is the set of all *direct* values of an attribute.
- L_{proc} is the language of all *procedure* names.
- *Param* is a set of values named *parameters*.

A **slot** is an element of the set $L_{attr} \times (V_{dir} \cup L_{proc}) \times Param$. An **object** is an element of the set

$$L_{obj} \times 2^{L_{obj}} \times 2^{L_{attr} \times (V_{dir} \cup L_{proc}) \times Param}$$

An object is described by the following three components:

- The first component gives the *object name*.
- Every element of the second component is a "direct" parent of the object.
- The last component gives the slots of the object.

We consider a subset $K_0 \subseteq L_{obj} \times 2^{L_{obj}} \times 2^{L_{attr} \times (V_{dir} \cup L_{proc}) \times Param}$. If $x = (m, P, Q) \in K_0$ is an object then m determines uniquely the object x and we denote by $N(x) = m$ the name of x. For this reason an object is denoted by $x = (N(x), P_x, Q_x)$.

Definition 3.1. *([4]) An* ***inheritance knowledge base*** *is a pair $K = (Obj(K), \rho_K)$, where*

(1) *$Obj(K) \subseteq L_{obj} \times 2^{L_{obj}} \times 2^{L_{attr} \times (V_{dir} \cup L_{proc}) \times Param}$ is a finite set of elements named the* ***objects*** *of K, such that if $x = (N(x), P_1, Q_1) \in Obj(K)$, $y = (N(y), P_2, Q_2) \in Obj(K)$ and $N(x) = N(y)$ then $P_1 = P_2$ and $Q_1 = Q_2$.*

(2) *$\rho_K \subseteq Obj(K) \times Obj(K)$ is the relation generated by K, which is defined as follows:*

$$(x, y) \in \rho_K \Longleftrightarrow N(x) \in P_y$$

(3) *$\rho_K^i \cap \rho_K^j = \emptyset$ for $i \neq j$*

Remark 3.1. *We denote by $Proc(K)$ the set of all procedure names appearing in $Obj(K)$. We suppose that the set of procedure names and the set of object names of K are two disjoint sets. We denote by $Attr(K)$ the set of all attributes $a \in L_{attr}$ such that if $a \in Attr(K)$ then there is an object in $Obj(K)$ which contains a slot of the form (a, q), where $q \in Param$.*

Definition 3.2. *([4]) The relation $inh_K = \bigcup_{n \geq 1} \rho_K^n$ is named the* ***inheritance relation*** *generated by K.*

The relation inh_K is the transitive closure of the relation ρ_K.

Definition 3.3. *([4]) An inheritance knowledge base K is an* ***accepted knowledge base*** *if the following conditions are fulfilled:*

- *There is no isolated object in K with respect to ρ_K.*
- *K contains minimal-ρ_K elements.*
- *inh_K is a strict partial order.*

Definition 3.4. *An* ***interrogation*** *of a knowledge base K is an element of the set $Obj(K) \times Attr(K)$. The* ***answer*** *of an interrogation (x, a_1) is the value of the attribute a_1 for $x \in Obj(K)$.*

As we have seen the value of an attribute can be the value given by a procedure. We stipulate here the following assumptions concerning an arbitrary procedure name $p \in Proc(K)$:

- The procedure p has the formal arguments specified in a vector $Arg(p) = (b_1, \ldots, b_r)$, where $b_1, \ldots, b_r \in Attr(K)$.

- In order to call the procedure p we use some vector $(v_1, \ldots, v_r)$ of actual arguments, where each v_i is the value of the attribute b_i. This means that $v_i \in V_{dir} \times Param$.
- We denote by $p(v_1, \ldots, v_r)$ the value returned by p for the actual arguments $v_1, \ldots, v_r$.
- We shall suppose that p is given by a correct algorithm. Particularly This means that no running error can appear if $v_1, \ldots, v_r$ belong to the domain of p (for example division by zero, overflow or underflow operation etc).
- As we specified above the value of an actual argument is the value of an attribute. Two cases are possible:
  - At the time of procedure call the value of the attribute b_i is unknown. This case is encountered when there is no sufficient information to compute the value of the corresponding attribute, but if we update the knowledge base then this value can be computed. In this case we consider the value of the actual parameter $v_i = unknown$, without any parameter.
  - The value of the actual argument can not be computed. This case was treated in [4], where we developed the computability aspects of this case. We suppose that in this case we have $v_i = error$, if b_i is the attribute whose value can not be computed. No parameter is associated to the value $error$.
- We suppose that:
  - If $v_i \in V_{dir} \times Param$ for $i = 1, \ldots, r$ then $p(v_1, \ldots, v_r) \in V_{dir} \times Param$.
  - If there is $i \in \{1, \ldots, r\}$ such that $v_i = unknown$ and $v_j \neq error$ for every $j \neq i$ then $p(v_1, \ldots, v_r) = unknown$.
  - If there is $i \in \{1, \ldots, r\}$ such that $v_i = error$ then $p(v_1, \ldots, v_r) = error$.

We consider a mapping $adj : Param \times Param \longrightarrow Param$. This is used to adjust two parameters.

The following mapping computes the value of an attribute for some object:

$$Val_{attr} : Obj(K) \times Attr(K) \longrightarrow (V_{dir} \times Param) \cup \{unknown, error\}$$

This is defined as follows ([4]):
- If $(x, a_1) \in Obj(K) \times Attr(K)$ and this attribute is inherited with a value $(v_1, q_1) \in V_{dir} \times Param$ then $Val_{attr}(x, a_1) = (v_1, q_1)$.
- If $(x, a_1) \in Obj(K) \times Attr(K)$ and this attribute is inherited with a value $(p_1, q_1) \in Proc(K) \times Param$ then:
  (1) If $Arg(p_1) = (b_1, \ldots, b_r)$ and $Val_{attr}(x, b_1) = (v_1, q_1)$, …, $Val_{attr}(x, b_r) = (v_r, q_r)$ are elements of $V_{dir} \times Param$ then

$$Val_{attr}(x, a_1) = (u, q) \tag{1}$$

  where $p_1(Val_{attr}(x, b_1), \ldots, Val_{attr}(x, b_r)) = (u, s)$ and $q = adj(q_1, s)$.
  (2) Otherwise $Val_{attr}(x, a_1) = error$.
- If a_1 is not an inherited attribute for x then $Val_{attr}(x, a_1) = unknown$.

A necessary and sufficient condition of the case $Val_{attr}(x, a_1) = error$ is given in [4].

Remark 3.2. *In order to underline the recursiveness of the mapping Val_{attr} we observe that the equation (1) can be written without any confusion as follows:*

$$Val_{attr}(x, a_1) = (p_1(Val_{attr}(x, b_1), \ldots, Val_{attr}(x, b_r)), q)$$

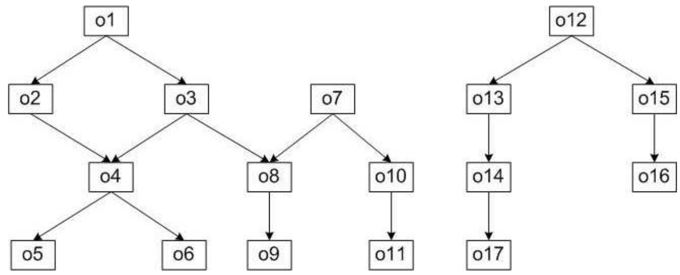

FIGURE 2. An inheritance knowledge base

4. The components of a knowledge base

We consider an inheritance knowledge base K. By an *initial object* of K we understand an object without parents. In other words, an initial object is of the form $(x, \{\}, Q_x)$. The set of all initial objects of K is denoted by $Initial(K)$.

Proposition 4.1. *The binary relation* $\widetilde{\rho}_K$ *is an equivalence relation.*

Proof. Immediate by Proposition 2.3 because there is no isolated element with respect to ρ_K. □

Notation 4.1. *The equivalence class of the object* $x \in Obj(K)$ *with respect to* $\widetilde{\rho}_K$ *is denoted by* $[x]_{\widetilde{\rho}_K}$.

Definition 4.1. *A* ***weak connected subset*** X ***of*** K *is a subset* $X \subseteq Obj(K)$ *such that*

- *There is a minimal-*ρ_K *element in* X.
- *If* $x, y \in X$ *then* $(x, y) \in \widetilde{\rho}_K$.

Remark 4.1. *The first condition of Definition 4.1 can be written as* $X \cap Initial(K) \neq \emptyset$.

Proposition 4.2. *If* X_1 *and* X_2 *are weak connected subsets of* K *such that* $X_1 \cap X_2 \neq \emptyset$ *then* $X_1 \cup X_2$ *is also a weak connected subset of* K.

Proof. Clearly $(X_1 \cup X_2) \cap Initial(K) \neq \emptyset$. Take $x, y \in X_1 \cup X_2$. We have the following cases:

(1) $x, y \in X_i$ for some $i \in \{1, 2\}$; in this case $(x, y) \in \widetilde{\rho}_K$ because X_i is a weak connected subset of K.
(2) $x \in X_i$ and $y \in X_j$, where $i \neq j$; take $z \in X_1 \cap X_2$. It follows that $(x, z) \in \widetilde{\rho}_K$ and $(z, y) \in \widetilde{\rho}_K$, therefore $(x, y) \in \widetilde{\rho}_K$.

The proposition is proved. □

Definition 4.2. *A* ***component*** *of* K *is a maximal weak connected subset of* K.

In Figure 2 we represented an accepted inheritance knowledge base that contains the objects $o_1, \ldots, o_{17}$. The sets $X_1 = \{o_1, \ldots, o_9\}$ and $X_2 = \{o_7, \ldots, o_{11}\}$ are weak connected sets such that $X_1 \cap X_2 \neq \emptyset$. The set $X_1 \cup X_2$ is a weak connected set. Moreover, we observe that there are only two components of this knowledge base: a component given by $\{o_1, \ldots, o_{11}\}$ and other component given by $\{o_{12}, \ldots, o_{17}\}$.

5. Algebraic properties of the components

In this section we give several properties of the components for an inheritance knowledge base and these properties are used to decompose a knowledge base into several components. A necessary and sufficient condition for the factorization problem is given.

Proposition 5.1. *The following sentences are equivalent:*
(1) *X is a component of K.*
(2) *There is $x \in Initial(K)$ such that $X = [x]_{\widetilde{\rho}_K}$.*

Proof. Suppose that X is a component of K. Let us prove that X is an equivalence class with respect to $\widetilde{\rho}_K$. Obviously two arbitrary elements of X are equivalent. Let us prove that if $x \in X$, $y \in Obj(K)$ and $(x, y) \in \widetilde{\rho}_K$ then $y \in X$. Suppose the contrary, that there is $x \in X$ and there is $y \in Obj(K)$ such that $(x, y) \in \widetilde{\rho}_K$ and $y \notin X$. Take $X_y = X \cup \{y\}$. The set X_y is a weak connected set because if $t_1, t_2 \in X_y$ then $(t_1, t_2) \in \widetilde{\rho}_K$. Really, we have the following cases:

- If $t_1, t_2 \in X$ then $(t_1, t_2) \in \widetilde{\rho}_K$ because X is a weak connected set.
- If $t_1 = t_2 = y$ then $(t_1, t_2) \in \widetilde{\rho}_K$ because by Proposition 4.1 the relation $\widetilde{\rho}_K$ is a reflexive one.
- If $t_1 \in X$ and $t_2 = y$ then $(t_1, x) \in \widetilde{\rho}_K$ because X is a weak connected set and $(x, y) \in \widetilde{\rho}_K$ by the choice of y. By Proposition 4.1 we obtain $(t_1, y) \in \widetilde{\rho}_K$.

This shows that X is not a maximal weak connected set because $X \subseteq X_y$ and X_y is a weak connected set. In conclusion X is an equivalence class with respect to $\widetilde{\rho}_K$. Now we observe that $X \cap Initial(K) \neq \emptyset$ because X is a weak connected subset of K. Take an arbitrary element $x \in X \cap Initial(K)$ and we have $X = [x]_{\widetilde{\rho}_K}$.
Suppose that $X = [x]_{\widetilde{\rho}_K}$ for some $x \in Initial(K)$. Obviously X is a weak connected subset of K. Let us prove that X is a maximal subset. Suppose that $X \subseteq Y$, where Y is a weak connected subset of K. If $y \in Y$ is an arbitrary element then $(x, y) \in \widetilde{\rho}_K$ because Y is a weak connected subset and $x \in Y$. But X is the equivalence class defined by x, therefore $y \in [x]_{\widetilde{\rho}_K} = X$. Thus $Y = X$ and the property is proved. □

Corollary 5.1. *If C and D are two arbitrary components of an accepted inheritance knowledge base then either $C = D$ or $C \cap D = \emptyset$.*

Proof. Really, C and D are two equivalence classes. □

We are interested now to decompose a knowledge base into its components. In order to treat this problem several preliminary results are useful and these results are presented in the next propositions.

Definition 5.1. *Consider a set X of initial objects. The $\widetilde{\rho}_K$-closure of X in K is the least subset $X_{\widetilde{\rho}_K}$ of $Obj(K)$ such that*

- *$X \subseteq X_{\widetilde{\rho}_K}$*
- *If $x \in X_{\widetilde{\rho}_K}$ and $(x, y) \in \widetilde{\rho}_K$ then $y \in X_{\widetilde{\rho}_K}$.*

Using a classical model of reasoning in universal algebras we obtain the following proposition.

Proposition 5.2. *Let be $X \subseteq Initial(K)$. The $\widetilde{\rho}_K$-closure of X in K is the set $X_{\widetilde{\rho}_K} = \bigcup_{i \geq 0} M_i$, where*

$$\begin{cases} M_0 = X \\ M_{n+1} = M_n \cup \{y \mid \exists x \in M_n : (x, y) \in \overline{\rho}_K\},\ n \geq 0 \end{cases}$$

Proof. We denote

$$Z = \bigcup_{i \geq 0} M_i \tag{2}$$

We show that Z satisfies the conditions of Definition 5.1. The first condition is satisfied because $Z \supseteq M_0 = X$. Let us verify the second condition. Suppose that $x \in Z$ and $(x, y) \in \widetilde{\rho}_K$. From (2) we deduce that there is $p \geq 0$ such that $x \in M_p$. But $\widetilde{\rho}_K = \bigcup_{i \geq 1} \overline{\rho}_K^{\,i}$ therefore from $(x, y) \in \widetilde{\rho}_K$ we deduce that there is $i \geq 1$ such that $(x, y) \in \overline{\rho}_K^{\,i}$. There are $z_1, \ldots, z_{i+1}$ such that

$x = z_1$
$(z_j, z_{j+1}) \in \overline{\rho}_K$ for $j \in \{1, \ldots, i\}$
$z_{i+1} = y$

By the definition of the sequence $\{M_n\}_{n \geq 0}$ we deduce that $z_{j+1} \in M_{p+j}$ for every $j \in \{1, \ldots, i\}$. It follows that $y = z_{i+1} \in M_{p+i}$. But $M_{p+i} \subseteq Z$ therefore $y \in Z$ and the second condition is also verified.

Let us verify that Z is the least set which satisfies these conditions. Let Y be a set such that

- $X \subseteq Y$
- If $x \in Y$ and $(x, y) \in \widetilde{\rho}_K$ then $y \in Y$.

We verify by induction on $n \geq 0$ the property

$$M_n \subseteq Y \tag{3}$$

The inclusion (3) is true for $n = 0$ because $M_0 = X \subseteq Y$. Suppose that $M_n \subseteq Y$. Take an arbitrary element $t \in M_{n+1}$. If $t \in M_n$ then by the inductive assumption we have $t \in Y$. If $t \in M_{n+1} \setminus M_n$ then there is $x \in M_n$ such that $(x, t) \in \overline{\rho}_K$. By the inductive assumption we have $x \in Y$. Moreover, we have $(x, t) \in \widetilde{\rho}_K$ because $\overline{\rho}_K \subseteq \widetilde{\rho}_K$. From the definition of Y we obtain $t \in Y$. Thus the equation (3) is true for $n+1$. Finally, from (3) we have $\bigcup_{n \geq 0} M_n \subseteq Y$ and the proposition is proved. □

Proposition 5.3. *Let be $X \subseteq Initial(K)$. The following sentences are equivalent:*

- $x \in X_{\widetilde{\rho}_K}$
- $x \in Obj(K)$ *and there is an initial object $r \in X$ such that $(r, x) \in \widetilde{\rho}_K$.*

Proof. Suppose that $x \in X_{\widetilde{\rho}_K}$. We use Proposition 5.2. We verify by induction on n that if $x \in M_n$ then there is an initial object $r \in X$ such that $(r, x) \in \widetilde{\rho}_K$. For $n = 0$ this property is true because $M_0 = X$ and $\widetilde{\rho}_K$ is a reflexive binary relation. Suppose the property is true for M_n and take $x \in M_{n+1}$. If $x \in M_n$ then by the inductive assumption there is $r \in X$ such that $(r, x) \in \widetilde{\rho}_K$. If $x \in M_{n+1} \setminus M_n$ then there is $y \in M_n$ such that $(y, x) \in \overline{\rho}_K$. By the inductive assumption there is $r \in X$ such that $(r, y) \in \widetilde{\rho}_K$. By the definition of $\widetilde{\rho}_K$ we deduce that there is a natural number s such that $(r, y) \in \overline{\rho}_K^{\,s}$. It follows that $(r, x) \in \overline{\rho}_K^{\,s+1}$. But $\overline{\rho}_K^{\,s+1} \subseteq \widetilde{\rho}_K$ and therefore $(r, x) \in \widetilde{\rho}_K$.

Suppose now that there is an initial object $r \in X$ such that $(r, x) \in \widetilde{\rho}_K$. But $X \subseteq X_{\widetilde{\rho}_K}$ and by the second condition of Definition 5.1 we obtain $x \in X_{\widetilde{\rho}_K}$. □

The next proposition shows the monotony of the closure operator with respect to inclusion operation from the set theory.

Proposition 5.4. *If $X \subseteq Y$ then $X_{\widetilde{\rho}_K} \subseteq Y_{\widetilde{\rho}_K}$.*

Proof. We apply Proposition 5.2. We have $X_{\widetilde{\rho}_K} = \bigcup_{i \geq 0} M_i$, where

$$\begin{cases} M_0 = X \\ M_{n+1} = M_n \cup \{y \mid \exists x \in M_n : (x, y) \in \overline{\rho}_K\}, \ n \geq 0 \end{cases}$$

and $Y_{\widetilde{\rho}_K} = \bigcup_{i\geq 0} Q_i$, where

$$\begin{cases} Q_0 = Y \\ Q_{n+1} = Q_n \cup \{y \mid \exists x \in Q_n : (x,y) \in \overline{\rho}_K\},\ n \geq 0 \end{cases}$$

By induction on $n \geq 0$ it is easy to verify that

$$M_n \subseteq Q_n \tag{4}$$

For $n = 0$ this equation is true because $M_0 = X \subseteq Y = Q_0$. Let us suppose that (4) is true. We have $M_{n+1} = M_n \cup \{y \mid \exists x \in M_n : (x,y) \in \overline{\rho}_K\} \subseteq Q_n \cup \{y \mid \exists x \in Q_n : (x,y) \in \overline{\rho}_K\} = Q_{n+1}$. Thus (4) is true for every $n \geq 0$. It follows that $X_{\widetilde{\rho}_K} = \bigcup_{i\geq 0} M_i \subseteq \bigcup_{i\geq 0} Q_i = Y_{\widetilde{\rho}_K}$ and the proposition is proved. □

Definition 5.2. *A subset $X \subseteq Initial(K)$ is a set of **cooperating initial objects** if for every $x, y \in X$ we have $(x,y) \in \widetilde{\rho}_K$.*

Remark 5.1. *The intuitive meaning of this definition can be explained as follows: some object of a component can inherit values of attributes from two or more initial objects. In other words, the corresponding initial objects cooperate to send these values to some object.*

For the case when X is a set of cooperating initial objects we have the property proved in the next proposition.

Proposition 5.5. *Suppose that X is a set of cooperating initial objects. If $x \in X_{\widetilde{\rho}_K}$ then for every initial object $r \in X$ we have $(r,x) \in \widetilde{\rho}_K$.*

Proof. Let us suppose that $x \in X_{\widetilde{\rho}_K}$. We apply Proposition 5.3 and we deduce that there is an initial object $r \in X$ such that $(r,x) \in \widetilde{\rho}_K$. Consider an arbitrary element $t \in X$, $t \neq r$. The set X is a set of cooperating initial objects. It follows that $(t,r) \in \widetilde{\rho}_K$. By transitivity we obtain $(t,x) \in \widetilde{\rho}_K$. □

Proposition 5.6. *If X is a set of cooperating initial objects then $X_{\widetilde{\rho}_K}$ is a weak connected set.*

Proof. The set $X_{\widetilde{\rho}_K}$ includes initial objects because $X \subseteq X_{\widetilde{\rho}_K}$ and X contains only such objects. Suppose that $x, y \in X_{\widetilde{\rho}_K}$. By Proposition 5.3 there are $r_1, r_2 \in X$ such that $(r_1,x) \in \widetilde{\rho}_K$ and $(r_2,y) \in \widetilde{\rho}_K$. By the symmetry of the relation $\widetilde{\rho}_K$ we deduce that $(x,r_1) \in \widetilde{\rho}_K$. But $(r_1,r_2) \in \widetilde{\rho}_K$ because X is a set of cooperating initial objects. By transitivity of the relation $\widetilde{\rho}_K$ we obtain $(x,y) \in \widetilde{\rho}_K$. □

The next two propositions prove a basic property used in the next section: the components of an inheritance knowledge base are "generated" by the maximal sets of cooperating initial objects.

Proposition 5.7. *If T is a component of a knowledge base K and $X = T \cap Initial(K)$ then*

- $T = X_{\widetilde{\rho}_K}$
- *X is a maximal set of cooperating initial objects.*

Proof. Every component contains initial objects, therefore $X \neq \emptyset$. We show that T satisfies the following two conditions:

- $X \subseteq T$
- If $x \in T$ and $(x,y) \in \widetilde{\rho}_K$ then $y \in T$.

The first condition is obviously true. To verify the second condition we suppose that $x \in T$ and $(x,y) \in \widetilde{\rho}_K$. By Proposition 5.1 the set T is an equivalence class with respect to $\widetilde{\rho}_K$. Thus we have $y \in T$.
But $X_{\widetilde{\rho}_K}$ is the least set which satisfies the conditions from Definition 5.1. It follows that

$$X_{\widetilde{\rho}_K} \subseteq T \tag{5}$$

To prove the converse inclusion we take an arbitrary element $x \in T$. If $r \in X$ is an arbitrary element then $(r,x) \in \widetilde{\rho}_K$ because T is a weak connected set. By Proposition 5.3 we obtain $x \in X_{\widetilde{\rho}_K}$. Thus

$$T \subseteq X_{\widetilde{\rho}_K} \tag{6}$$

From (5) and (6) we obtain $T = X_{\widetilde{\rho}_K}$.
Let us prove the second property of this proposition. First we observe that X is a set of cooperating initial objects. Really, if $x, y \in X$ are two arbitrary elements then $x, y \in T$. But T is a weak connected set therefore $(x,y) \in \widetilde{\rho}_K$. We prove now that X is a maximal set of cooperating initial objects. Suppose by contrary that X is not a maximal set of cooperating initial objects. It follows that there is a set U of cooperating initial objects such that

$$X \subset U \tag{7}$$

By Proposition 5.4 it follows that $X_{\widetilde{\rho}_K} \subseteq U_{\widetilde{\rho}_K}$. But $T = X_{\widetilde{\rho}_K}$, therefore $T \subseteq U_{\widetilde{\rho}_K}$. We prove that this is a strict inclusion:

$$T \subset U_{\widetilde{\rho}_K} \tag{8}$$

Based on the strict inclusion (7) we take an element $y \in U \setminus X$. The set X is the set of all initial object of T therefore $y \notin T$. But $y \in U_{\widetilde{\rho}_K}$ and therefore $y \in U_{\widetilde{\rho}_K} \setminus T$. Thus (8) is proved. By Proposition 5.6 the set $U_{\widetilde{\rho}_K}$ is a weak connected set and this fact shows that T is not a maximal connected set. In conclusion X is a maximal set of cooperating initial objects. □

Proposition 5.8. *If X is a maximal set of cooperating initial objects of an accepted inheritance knowledge base K then $X_{\widetilde{\rho}_K}$ is a component of K.*

Proof. By Proposition 5.6 the set $X_{\widetilde{\rho}_K}$ is a weak connected subset of K. It remains to prove that $X_{\widetilde{\rho}_K}$ is a maximal weak connected set. Suppose that $X_{\widetilde{\rho}_K}$ is not a maximal weak connected set. There is component T such that $X_{\widetilde{\rho}_K} \subset T$. Denote by F the set of all initial objects of T. By Proposition 5.7 we have $T = F_{\widetilde{\rho}_K}$. Obviously we have $X \subseteq F$ because $X \subseteq X_{\widetilde{\rho}_K} \subset T$, X is a set of initial objects and F is the set of all initial objects of T. If we suppose that $X = F$ then $X_{\widetilde{\rho}_K} = F_{\widetilde{\rho}_K}$ and therefore $X_{\widetilde{\rho}_K} = T$, which is not true. It follows that $X \subset F$. Take $y \in F \setminus X$. For every $z \in X$ we have $(z,y) \in \widetilde{\rho}_K$ because $z \in T$, $y \in T$ and T is a weak connected set. Thus $X \cup \{y\}$ is a set of cooperating initial objects, which is not true because X is a maximal set of cooperating initial objects. □

Remark 5.2. *We can say that $X_{\widetilde{\rho}_K}$ is ρ_K-generated by X.*

Based on the results presented in this section we can say the components of a knowledge base are the sets generated by the maximal sets of cooperating initial objects.
In conclusion, for a knowledge base the problem of decomposition into its components reduces to the problem of finding the maximal sets of cooperating initial objects.

6. The factorization problem

In this section we treat the problem of decomposition of a knowledge base into its components. This subject is named the *factorization problem*. We show that the set of all components of K is the *factor set* $Obj(K)/\widetilde{\rho}_K$. This can explain why this problem is named the factorization problem of a knowledge base.

We consider the space $(Initial(K), \sigma_K)$, where σ_K is the restriction of $\widetilde{\rho}_K$: for every $x, y \in Initial(K)$ we have $(x, y) \in \sigma_K$ if and only if $(x, y) \in \widetilde{\rho}_K$. The binary relation σ_K is an equivalence relation on $Initial(K)$ because $\widetilde{\rho}_K$ is an equivalence relation on $Obj(K)$. We can consider the factor set $Initial(K)/\sigma_K$.

Proposition 6.1. *If T is a component of a knowledge base K then*

$$T \cap Initial(K) \in Initial(K)/\sigma_K$$

Proof. Let us prove first that if $x, y \in T \cap Initial(K)$ then $(x, y) \in \sigma_K$. We have $(x, y) \in \widetilde{\rho}_K$ because by Proposition 5.1 the set T is a $\widetilde{\rho}_K$-equivalence class. But $x, y \in Initial(K)$, therefore $(x, y) \in \sigma_K$. Thus every two elements of the set $T \cap Initial(K)$ are σ_K-equivalent. It remains to prove that if $x \in T \cap Initial(K)$, $y \in Initial(K)$ and $(x, y) \in \sigma_K$ then $y \in T \cap Initial(K)$. From $(x, y) \in \sigma_K$ it follows that $(x, y) \in \widetilde{\rho}_K$. But $x \in T$ and T is a $\widetilde{\rho}_K$-equivalence class, therefore $y \in T$. □

Proposition 6.2. *If $X \in Initial(K)/\sigma_K$ then X is a maximal set of $\widetilde{\rho}_K$-cooperating initial objects.*

Proof. The proof is based on the following facts:

(1) If $x, y \in X$ then $(x, y) \in \sigma_K$, therefore $(x, y) \in \widetilde{\rho}_K$. Thus X is a set of $\widetilde{\rho}_K$-cooperating initial objects.

(2) Suppose that $Y \subseteq Initial(K)$ satisfies the condition

$$X \subseteq Y \tag{9}$$

and Y is a set of $\widetilde{\rho}_K$-cooperating initial objects. We verify the property

$$Y \subseteq X \tag{10}$$

Consider an element $y \in Y$. Take an element $x \in X$. We have $x \in Y$ because $X \subseteq Y$. The set Y is a set of $\widetilde{\rho}_K$-cooperating initial objects, therefore $(x, y) \in \widetilde{\rho}_K$. On the other hand $x, y \in Initial(K)$ and σ_K is the restriction of $\widetilde{\rho}_K$ on $Initial(K)$. It follows that $(x, y) \in \sigma_K$. But $X \in Initial(K)/\sigma_K$, $x \in X$ and $(x, y) \in \sigma_K$. This shows that $y \in X$, therefore (10) is proved. From (9) and (10) we deduce that $X = Y$. In conclusion, X is a maximal set of $\widetilde{\rho}_K$-cooperating initial elements.

□

Proposition 6.3. *If $X \in Initial(K)/\sigma_K$ then $X_{\widetilde{\rho}_K}$ is a component of K.*

Proof. We apply Proposition 6.2. If $X \in Initial(K)/\sigma_K$ then X is a maximal set of $\widetilde{\rho}_K$-cooperating initial objects. Applying Proposition 5.8 we deduce that $X_{\widetilde{\rho}_K}$ is a component of K. □

Proposition 6.4. *For every $x \in Obj(K) \setminus Initial(K)$ there is $y \in Initial(K)$ such that $(y, x) \in inh_K$.*

Proof. We define the sequence $\{R_n\}_{n \geq 0}$ as follows:

$$\begin{cases} R_0 = \{x\} \\ R_{n+1} = \{y \in Obj(K) \mid \exists t \in R_n : (y, t) \in \rho_K \},\ n \geq 0 \end{cases}$$

We observe first that $R_0 \neq \emptyset$ and $R_1 \neq \emptyset$ because $x \notin Initial(K)$. Let us prove by induction on $i \geq 1$ the following property: if $z \in R_i$ then $(z,x) \in \rho_K^i$. For $i = 1$ this property is true by the definition of R_1. Suppose that this property is true for some $i \geq 1$. Take an element $z \in R_{i+1}$. By definition there is $t \in R_i$ such that $(z,t) \in \rho_K$. By the inductive assumption we have $(t,x) \in \rho_K^i$. It follows that $(z,x) \in \rho_K \circ \rho_K^i = \rho_K^{i+1}$ and the property is proved by induction.
There is a natural number $i \geq 2$ such that $R_i = \emptyset$. Really, suppose by contrary that $R_i \neq \emptyset$ for every $i \geq 2$. But $R_i \subseteq Obj(K)$ for every i and $Obj(K)$ is a finite set. It follows that there are $n \geq 1$ and $m \geq 2$ such that $R_{n+1} \cap R_{n+m} \neq \emptyset$. Take an element $y \in R_{n+1} \cap R_{n+m}$. As we just proved above we know that $(y,x) \in \rho_K^{n+1} \cap \rho_K^{n+m}$. Applying Definition 3.1 we see that this case is not possible because $n+1 < n+m$, therefore $n+1 \neq n+m$.
Thus there is a natural number $i \geq 2$ such that $R_i = \emptyset$. Denote by n the least natural number such that $R_n = \emptyset$. We have $n \geq 2$ and $R_{n-1} \neq \emptyset$. In this case we prove that

$$R_{n-1} \subseteq Initial(K) \tag{11}$$

Suppose by contrary that (11) is not true. In other words there is $z \in R_{n-1} \setminus Initial(K)$. If $z \in Obj(K) \setminus Initial(K)$ then $z = (N(z), P_z, Q_z)$ and $P_z \neq \emptyset$. But if $t \in P_z$ then $t \in R_n$, which is not possible because $R_n = \emptyset$. It follows that (11) is true. Take $t \in R_{n-1}$. As we proved above $(t,x) \in \rho_K^{n-1}$ and from (11) we have $t \in Initial(K)$. But $\rho_K^{n-1} \subseteq inh_K$, therefore we proved that $(t,x) \in inh_K$ and $t \in Initial(K)$. □

Corollary 6.1. *If $T \in Obj(K)/\widetilde{\rho}_K$ then $T \cap Initial(K) \neq \emptyset$.*

Proof. Take an arbitrary element $x \in T$. If $x \in Initial(K)$ then the property is proved. Suppose that $x \notin Initial(K)$. By Proposition 6.4 we know that there is $y \in Initial(K)$ such that $(y,x) \in inh_K$. But $inh_K \subseteq \widetilde{\rho}_K$ therefore $(y,x) \in \widetilde{\rho}_K$. On the other hand T is an equivalence class with respect to $\widetilde{\rho}_K$ and $x \in T$. It follows that $y \in T$ and the corollary is proved. □

Proposition 6.5. *T is a component of K if and only if $T \in Obj(K)/\widetilde{\rho}_K$.*

Proof. If T is a component of K then by Proposition 5.1 the set T is a $\widetilde{\rho}_K$-equivalence class.
Conversely, take an equivalence class $T \in Obj(K)/\widetilde{\rho}_K$ and denote $X = T \cap Initial(K)$. By Corollary 6.1 we have $X \neq \emptyset$. X is a set of cooperating initial objects because $X \subseteq T$ and T is a $\widetilde{\rho}_K$-equivalence class. Moreover, X is a maximal set of cooperating initial objects. Really, suppose that $X \subseteq Y$ and Y is a set of cooperating initial objects. We prove that $X = Y$. If $y \in Y$ and $r \in X$ then $(r,y) \in \widetilde{\rho}_K$ because $r \in Y$ and Y is a set of cooperating initial objects. But $r \in T$ because $X \subseteq T$. On the other hand T is a $\widetilde{\rho}_K$-equivalence class and therefore $y \in T$. But $y \in Initial(K)$ and thus $y \in X$.
We apply Proposition 5.8 and we deduce that $X_{\widetilde{\rho}_K}$ is a component of K. It remains to prove that $T = X_{\widetilde{\rho}_K}$. Consider an arbitrary element $x \in T$. If $r \in X$ then $(r,x) \in \widetilde{\rho}_K$ because T is a $\widetilde{\rho}_K$-equivalence class. But $X_{\widetilde{\rho}_K}$ is the $\widetilde{\rho}_K$-closure of X, therefore $x \in X_{\widetilde{\rho}_K}$. Thus $T \subseteq X_{\widetilde{\rho}_K}$. To prove the converse inclusion we consider an arbitrary element $t \in X_{\widetilde{\rho}_K}$. From Proposition 5.3 we deduce that there is an initial object $r \in X$ such that $(r,t) \in \widetilde{\rho}_K$. But $r \in T$ and T is a $\widetilde{\rho}_K$-equivalence class, therefore $t \in T$. Thus $X_{\widetilde{\rho}_K} \subseteq T$. □

7. Conclusions and future work

In this paper we considered the concept of extended inheritance for knowledge representation introduced in [4] and we treated the computational aspects connected by this concept. As a particular case of this model we can represent uncertain knowledge by inheritance. The main problem of this paper is connected by the factorization of an inheritance knowledge base. Equivalently this means the decomposition of a knowledge base into several components, each component being also an inheritance knowledge base. We give a necessary and sufficient condition for a successful factorization.

The subject presented in this paper will be continued in [5]. Based on the equivalence relation $\widetilde{\rho}_K$ we shall demonstrate that every component is itself an inheritance knowledge base. We prove that an interrogation for the object $x \in Obj(K)$ can be equivalently accomplished in the component which contains the object x. The factorization of a knowledge base is a useful operation in the vision of an implementation on several work stations in a network architecture.

In the future we are interested to imply the mobile agents in a master-slave architecture of such agents. We intend to develop a research line concerning the modeling of the distributed knowledge by inheritance.

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