Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 37(2), 2010, Pages 100–105
ISSN: 1223-6934

Point convexity

Flavia-Corina Minuţă

Abstract. This article is extending the classic definition of convexity.

2010 Mathematics Subject Classification. Primary 52A41; Secondary 34K09.
Key words and phrases. convex function, subdifferential.

1. Introduction

During the last two decades convexity has been the subject of an intensive research. In particular, many improvements, generalizations, and applications of the usual convexity appeared in the literature. The purpose of this paper is to describe a new concept of convex like function. Our investigation parallels the classical approach of convexity as presented in [1] and [2].

2. Point-convexity on an interval

We shall need to define two notions in order to state and prove the main results. We consider I subinterval of $\mathbb{R}$ and $x_0, y_0 \in I$.

Definition 2.1. *A function $f : I \to \mathbb{R}$ is called (x_0, y_0)-convex if*

$$f\left((1-\lambda)\, x_0 + \lambda y_0\right) \leq (1-\lambda)\, f(x_0) + \lambda f(y_0)$$

for all $\lambda \in [0,1]$.

Definition 2.2. *A function $f : I \to \mathbb{R}$ is called x_0-convex if for all $\lambda \in [0,1]$ and all points $y \in I$ we have*

$$f\left((1-\lambda)\, x_0 + \lambda y\right) \leq (1-\lambda)\, f(x_0) + \lambda f(y) .$$

Example 2.1. *Two x_0-convex functions on $\mathbb{R}$, wich are not convex in the classical sense:*

1. The continuous function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} x^2, & x < 1 \\ x, & x \in [1,2] \\ \frac{1}{4}x^2 + 1, & x > 2 \end{cases}$$

is x_0-convex, for all $x_0 \in (-\infty, 0]$.

2. The discontinuous function

Received April 11, 2010. Revision received May 27, 2010.

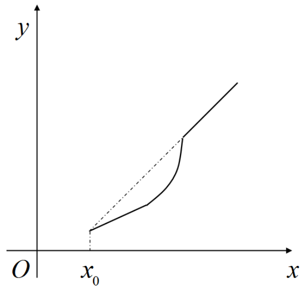

FIGURE 1. The graph of a x_0-convex function

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\begin{cases}-x, & x<0\\ x, & x\in[0,2]\\ 2x, & x>2\end{cases}$$

is x_0-convex, for all $x_0\in(-\infty,0]$.

Reversing the inequalities in the above definitions we will obtain respectively the concept of (x_0,y_0)-*concave function* and of x_0-*concave function.*

A function which is x_0-convex and x_0-concave will be called x_0-*affine.*

The absolute value function is 0-affine though not affine in the ordinary sense. However, every x_0-affine and monotone function is affine (in the usual meaning).

Proposition 2.1. *If $f:[x_0,y_0]\to\mathbb{R}$ is an x_0-convex function, then*

$$\frac{f(y)-f(x_0)}{y-x_0}\le\frac{f(y_0)-f(x_0)}{y_0-x_0}$$

for all $y\in(x_0,y_0]$.

Proof. Indeed, every $y\in(x_0,y_0]$ is of the form $y=(1-\lambda)x_0+\lambda y_0$ for some $\lambda\in(0,1]$, which yields

$$\frac{f((1-\lambda)x_0+\lambda y_0)-f(x_0)}{\lambda(y_0-x_0)}\le\frac{f(y_0)-f(x_0)}{y_0-x_0},$$

equivalently,

$$f((1-\lambda)x_0+\lambda y_0)\le(1-\lambda)f(x_0)+\lambda f(y_0).$$

□

In what follows we will need the function *slope* $s_{x_0}:(x_0,y_0]\to\mathbb{R}$ defined by

$$s_{x_0}(y)=\frac{f(y)-f(x_0)}{y-x_0}.$$

Proposition 2.2. *$f:[x_0,y_0]\to\mathbb{R}$ is x_0-convex if and only if $s_{x_0}:(x_0,y_0]\to\mathbb{R}$ is nondecreasing.*

Proof. If $f:[x_0,y_0]\to\mathbb{R}$ is x_0-convex then

$$f((1-\lambda)x_0+\lambda y)\le(1-\lambda)f(x_0)+\lambda f(y)$$

for all $\lambda\in[0,1]$ and every $y\in[x_0,y_0]$.

That happens if and only if

$$\frac{f((1-\lambda)x_0+\lambda y)-f(x_0)}{\lambda(y-x_0)}\le\frac{f(y)-f(x_0)}{y-x_0}$$

.

We put $z = (1-\lambda) x_0 + \lambda y$, $z \le y$.

$s_{x_0}(z) = \frac{f(z)-f(x_0)}{z-x_0} \le \frac{f(y)-f(x_0)}{y-x_0} = s_{x_0}(y)$ for all z, $y \in [x_0, y_0]$, $z \le y$. □

Remark 2.1. *We notice that* $f(y) \le f(x_0) + \frac{f(y_0)-f(x_0)}{y_0-x_0} \cdot (y - x_0)$, *for all* $y \in [x_0, y_0]$.

If f is x_0-convex for all x_0 in its domain, then it is convex. All results of this article have corresponding tesults in the theory of convex functions. For a detailed discussion see [1].

We can also easily observe that if f is convex in the usual sense, then the above proposition aplies and we get an well known result of Galvani: s_x is nondecreasing for all $x \in [x_0, y_0]$.

Corollary 2.1. *The function* $f : [x_0, y_0] \to \mathbb{R}$ *is a* x_0*-convex function if and only if for all real* y, z *with* $x_0 < y < z < y_0$, *we have* $\begin{vmatrix} 1 & 1 & 1 \\ x_0 & y & z \\ f(x_0) & f(y) & f(z) \end{vmatrix} \ge 0$.

Proposition 2.3. *Let* $f : [x_0, y_0] \to \mathbb{R}$ *be a* x_0*-convex function and* $x_1 < x_2$ *two points of its domain. If* $s_{x_0}(x_1) = s_{x_0}(x_2)$, *then* $f_{|[x_1,x_2]|}$ *is affine.*

Proof. Because of the monotonicity property of s_{x_0}, we have $s_{x_0}(x_1) = s_{x_0}(z)$ for all $z \in (x_1, x_2)$. The point $(z, f(z))$ is then collinear to $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

You can see Figure 1 (an example of such a graph). □

Corollary 2.2. *If the function* $f : I \to \mathbb{R}$ *is the limit function of a pointwise convergent sequence of* x_0*-convex functions, then* f *is also* x_0*-convex.*

Proof. We consider $f_n : I \to R$, $n \in N^*$ a sequence of x_0-convex functions, $f_n \xrightarrow{p} f$.

Then $f_n((1-\lambda) x_0 + \lambda y_0) \le (1-\lambda) f_n(x_0) + \lambda f_n(y_0)$ for all $\lambda \in [0,1]$, all $y \in [x_0, y_0]$ and all $n \in \mathbb{N}$.

Passing to the limit, we get the conclusion. □

Definition 2.3. *The function* $f : I \to \mathbb{R}$ *is called* x_0*-midpoint convex if for all* $y \in I$ *we have*

$$f\left(\frac{x_0 + y}{2}\right) \le \frac{f(x_0) + f(y)}{2}.$$

For a function $f : [x_0, y_0] \to \mathbb{R}$ that is continuous but is not x_0-convex, we define the function

$$\varphi_y(z) = f(z) - f(x_0) - \frac{f(y) - f(x_0)}{y - x_0} \cdot (z - x_0),$$

for all $y \in (x_0, y_0]$. Also we can define $\xi(y) = \inf\{z \in (x_0, y_0) ; \varphi_y(z) > 0\} \in [x_0, y_0]$.

Proposition 2.4. *If* $\xi(y) \ge \frac{x_0+y}{2}$, *for all* $y \in [x_0, y_0]$, *then* f *is* x_0*-midpoint convex .*

Proof. Indeed, $\varphi_y\left(\frac{x_0+y}{2}\right) = f\left(\frac{x_0+y}{2}\right) - f(x_0) - \frac{f(y)-f(x_0)}{y-x_0} \cdot \left(\frac{x_0+y}{2} - x_0\right) \le 0$, for all $y \in (x_0, y_0]$, what brings us to the conclusion that $f\left(\frac{x_0+y}{2}\right) \le \frac{f(x_0)+f(y)}{2}$. □

As you can see in Figure 2, a continuous x_0-midpoint convex function is not necesarily x_0- convex.

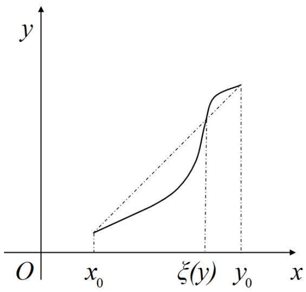

FIGURE 2. x_0- midpoint convex, but not x_0- convex

Theorem 2.1. *The following two statements are equivalent:*

i) The function $f : I \to \mathbb{R}$ is x_0-convex.

ii) For all $J \subseteq I$ compact, with an endpoint x_0, and for all functions L x_0-affine, the function $f + L$ attains its supremum at an endpoint of J.

Proof. We put $J = [x_0, y_0]$.

$i) \Longrightarrow ii)$. In fact,

$$\begin{aligned}\sup_{z\in J}(f+L)(z) &= \sup_{\lambda\in[0,1]}(f+L)((1-\lambda)x_0+\lambda y_0)\\ &\leq \sup_{\lambda\in[0,1]}(1-\lambda)(f+L)(x_0)+\lambda(f+L)(y_0)\\ &= \max\{(f+L)(x_0),(f+L)(y_0)\}\end{aligned}$$

$ii) \Longrightarrow i)$. Let L be an x_0-affine function such that $L(x_0) = f(x_0)$ and $L(y_0) = f(y_0)$. Then

$$\sup_{z\in J}(f-L)(z) = 0.$$

Since every $z \in J$ is of the form $z = (1-\lambda)x_0 + \lambda y_0$, for some $\lambda \in [0,1]$, we get

$$(f-L)((1-\lambda)x_0+\lambda y_0) = f((1-\lambda)x_0+\lambda y_0) - [(1-\lambda)f(x_0)+\lambda f(y_0)] \leq 0.$$

□

Proposition 2.5. *(Properties of x_0-convex functions)*

1) If f and g are two x_0-convex functions defined on the same interval I, then $f+g$ is x_0-convex.

2) If f is x_0-convex on I, $\alpha \geq 0$, then αf is x_0-convex.

3) If f is x_0-convex on I, all restrictions of it to a subinterval of its domain wich contains x_0 are also x_0-convex functions.

4) If f is x_0-convex on I and if g is nondecreasing and $f(x_0)$-convex on $f(I)$, then $g \circ f$ is x_0-convex.

5) If f is x_0-convex on I, bijective and increasing, then its inverse is $f(x_0)$-concave on $f(I)$.

6) If f is x_0-convex on I, bijective and decreasing ,then its inverse is $f(x_0)$-convex on $f(I)$.

Theorem 2.2. *If $f : I \to \mathbb{R}$ is continuous and x_0-convex, then for all $a \in I$ we get*

$$\frac{1}{a-x_0}\int_{x_0}^{a} f(x)\,dx \leq \frac{f(x_0)+f(a)}{2}.$$

Proof.

$$\frac{1}{a-x_0}\int_{x_0}^{a} f(x)\,dx = \int_0^1 f((1-\lambda)x_0+\lambda a)\,d\lambda$$
$$\leq \int_0^1 [(1-\lambda)f(x_0)+\lambda f(a)]\,d\lambda$$
$$= \frac{f(x_0)+f(a)}{2}.$$

□

If f is convex, the last theorem leads us to the right hand side of the Hermite Hadamard inequality.

Theorem 2.3 (Maximum principle for x_0-convex functions)**.** *Lets consider $f:[x_0,y_0]\to\mathbb{R}$, a x_0-convex function. If the point y is a global maximum point and an interior point of its domain, then the function has constant values on $[y,y_0]$.*

Proof. By reductio ad absurdum, we consider that $y\in(x_0,y_0)$ is a maximum point. We choose another point $z\in(y,y_0)$. Then $s_{x_0}(y)\geq s_{x_0}(z)$ and, because of the monotonicity property of s_{x_0}, we deduce that equals. Aplying the proposition 2.3, we saw that the function has constant values on (y,y_0). The points $(x_0,f(x_0))$, $(y,f(y))$ and $(y_0,f(y_0))$ are collinear, $f(y)=\max\limits_{x\in[x_0,y_0]} f(x)$, that implies that the function has constant values on $[y,y_0]$. (See Figure 3.)

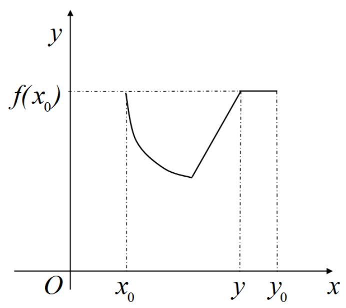

FIGURE 3. Maximum principle

□

From now on we consider that f is a continuous and x_0-convex function on $[x_0,y_0]$.

Remark 2.2. *If f is differentiable on the right side at each point $y\in(x_0,y_0)$, then $s_{x_0}(y)\leq f'_s(y)$.*

3. The left/right subdifferential

Definition 3.1. *We say that f admits a support semiline at the right side of the point y if there exist a λ such that $f(z)\geq f(y)+\lambda(z-y)$ for all $z\in(y,y_0)$. We call the set $\partial f_r(y)$ of all such λ* the right subdifferential *of f at y.*

Definition 3.2. *We say that* f *admits a support semiline at the left side of the point* y *if there exist a* λ *such that* $f(z) \geq f(y) + \lambda(z-y)$ *for all* $z \in (x_0, y)$. *We call the set* $\partial f_l(y)$ *of all such* λ the left subdifferential *of* f *at* y.

If f has a finite left (right) derivative at y, we have $\partial f_r(y) = (-\infty, f'_r(y)]$ ($\partial f_l(y) = [f'_l(y), \infty)$).

If $\partial f_l(y) \cap \partial f_r(y) \neq \Phi$, then the function admits a support line at y. Then $\partial f_l(y) \cap \partial f_r(y) = \partial f(y)$.

Proposition 3.1. *Lets consider* $f : [x_0, y_0] \to R$ *continuous, with finite right derivative at all interior points of its domain. Then is* x_0*-convex if and only if* $s_{x_0}(y) \in \partial f_r(y)$ *for all* $y \in (x_0, y_0)$.

Proof. Direct part of the statement is easy to prove.

The reverse:

$s_{x_0}(y) \in \partial f_r(y) \Longrightarrow f(z) \geq f(y) + s_{x_0}(y)(z-y)$ for all $z \in (y, y_0)$.

We can write all z as a a convex combination of the endpoints, $z = (1-\lambda)x_0 + \lambda y, \lambda \in [0,1]$.

$f((1-\lambda)x_0 + \lambda y) \geq f(y) + s_{x_0}(y)(1-\lambda)(x_0 - y)$

That yields to $f((1-\lambda)x_0 + \lambda y) \geq (1-\lambda) f(x_0) + \lambda f(y)$, for all $\lambda \in [0,1]$. □

We recall the Extreme Value Theorem of Weierstrass: *if a real-valued function* f *is continuous in the closed and bounded interval* $[a,b]$*, then* f *must attain its maximum and minimum value, each at least once.*

Theorem 3.1 (Rolle theorem for x_0-convex functions)**.** *Suppose that* f *is continuous and* x_0*-convex on* $[x_0, y_0]$ *and* $f(x_0) = f(y_0)$. *Then there exists* $y \in (x_0, y_0)$ *such that* $0 \in \partial f_r(y)$.

Proof. If the function is constant, the conclusion is obvious. If is not constant, because of the fact it is continuous (attains ith minimum) and and x_0-convex (the minimum cannot be an endpoint of the interval), we may obviously conclude that there exists at least one interior global minimum point y. The parallel line through $(y, f(y))$ to the Ox axis contains a right support semiline of the function at y and then $0 \in \partial f_r(y)$. □

Theorem 3.2 (Lagrange theorem for x_0-convex functions)**.** *If* f *is continuous and* x_0*-convex on* $[x_0, y_0]$, *with finite right derivative at all interior points of its domain, then there exists* $y \in (x_0, y_0)$ *such that* $s_{x_0}(y_0) \in \partial f_r(y)$.

Proof. We define $g(x) = f(x) - s_{x_0}(y)(x - x_0)$. All conditions of Theorem 3.1 are verified by g. There exists $y \in (x_0, y_0)$ such that $0 \in \partial g_r(y) = \left(-\infty, g'_+(y)\right]$.

It follows that $g'_+(y) = f'_+(y) - s_{x_0}(y) \geq 0$, that is, $s_{x_0}(y_0) \in \left(-\infty, f'_+(y)\right] = \partial f_r(y)$. □

References

[1] C. P. Niculescu and L.-E. Persson, Convex Functions and their Applications. A Contemporary Approach, In *CMS Books in Mathematics* **23**, Springer-Verlag, New York (2006).

[2] J. E. Pecaric, F. Proscan and Y. L. Tong, Convex functions, partial orderings and statistical applications, In *Mathematics in science and engineering* **187**, Academic Press, Boston (1992).

(Flavia-Corina Minuţă) Faculty of Mathematics and Computer Science, University of Craiova, Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania, Tel. & Fax: 40-251412673

E-mail address: `minutacorina@yahoo.com`