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Generalization of α-open sets in bitopological spaces

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Abstract. In this paper, we introduce and study the notion of (i,j)-$\mathcal{N}$-α-open sets as a generalization of (i,j)-α-open sets in bitopological space.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. The concept of a bitopological space was introduced by Kelly [2]. On the other hand Bose introduced the concept of a α-open sets in bitopological space. In this paper, we introduce and study the notion of (i,j)-$\mathcal{N}$-α-open sets as a generalization of (i,j)-α-open sets in bitopological space. Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, the closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a bitopological space (X,τ_1,τ_2) is said to be (i,j)-α-open [1] if $A \subset \tau_i$-$\mathrm{Int}(\tau_j$-$\mathrm{Cl}(\tau_i$-$\mathrm{Int}(A)))$, where $i,j = 1,2$ and $i \neq j$. A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-α-continuous [1] if the inverse image of every σ_i-open set in (Y,σ_1,σ_2) is (i,j)-α-open in (X,τ_1,τ_2), where $i \neq j$, $i,j = 1$, 2.

2. (i,j)-$\mathcal{N}$-α-open sets

Definition 2.1. *A subset A is said to be (i,j)-$\mathcal{N}$-α-open if for each $x \in A$ there exists an (i,j)-α-open set U_x containing x such that $U_x \backslash A$ is a finite set. The complement of an (i,j)-$\mathcal{N}$-α-open subset is said to be (i,j)-$\mathcal{N}$-α-closed.*
The family of all (i,j)-$\mathcal{N}$-α-open (resp. (i,j)-$\mathcal{N}$-α-closed) subsets of a space (X,τ_1,τ_2) is denoted by (i,j)-$\mathcal{N}\alpha O(X)$ (resp. (i,j)-$\mathcal{N}\alpha C(X)$). The family of all (i,j)-$\mathcal{N}$-α-open subsets of a space (X,τ_1,τ_2) containing a point x is denoted by (i,j)-$\mathcal{N}\alpha O(X,x)$.

It is clear that every (i,j)-α-open set is (i,j)-$\mathcal{N}$-α-open. The following example shows that the converse is not true in general.

Example 2.1. *Let $X = \{a,b,c\}$, $\tau_1 = \{\varnothing,\{a\},X\}$ and $\tau_2 = \{\varnothing,\{b\},X\}$. Then $\{b\}$ is (i,j)-$\mathcal{N}$-α-open but not (i,j)-α-open in (X,τ_1,τ_2).*

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Lemma 2.1. *A subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-$\mathcal{N}$-α-open if and only if for every $x \in A$, there exists an (i, j)-α-open subset U_x containing x and a finite subset C such that $U_x \backslash C \subset A$.*

Proof. Let A be an (i, j)-$\mathcal{N}$-α-open set and $x \in A$, then there exists an (i, j)-α-open subset U_x containing x such that $U_x \backslash A$ is finite. Let $C = U_x \backslash A = U_x \cap (X \backslash A)$. Then $U_x \backslash C \subset A$. Conversely, let $x \in A$. Then there exist an (i, j)-α-open subset U_x containing x and a finite subset C such that $U_x \backslash C \subset A$. Thus $U_x \backslash A \subset C$ and $U_x \backslash A$ is a finite set. □

Theorem 2.1. *Let (X, τ_1, τ_2) be a bitopological space and $F \subset X$. If F is (i, j)-$\mathcal{N}$-α-closed, then $F \subset K \cup C$ for some (i, j)-α-closed subset K and a finite subset C.*

Proof. If F is (i, j)-$\mathcal{N}$-α-closed, then $X \backslash F$ is (i, j)-$\mathcal{N}$-α-open and hence for every $x \in X \backslash F$, there exists an (i, j)-α-open set U containing x and a finite set C such that $U \backslash C \subset X \backslash F$. Thus $F \subset X \backslash (U \backslash C) = X \backslash (U \cap (X \backslash C)) = (X \backslash U) \cup C$. Let $K = X \backslash U$. Then K is an (i, j)-α-closed set such that $F \subset K \cup C$. □

Lemma 2.2. *Any union of family of (i, j)-$\mathcal{N}$-α-open sets is (i, j)-$\mathcal{N}$-α-open.*

Proof. Let $\{U_i : i \in I\}$ be a family of (i, j)-$\mathcal{N}$-α-open subsets of X and $x \in \underset{i \in I}{\cup} U_i$. Then $x \in U_j$ for some $j \in I$. This implies that there exists an (i, j)-α-open subset $V$of X containing x such that $V \backslash U_j$ is finite. Since $V \backslash \underset{i \in I}{\cup} U_i \subseteq V \backslash U_j$, then $V \backslash \underset{i \in I}{\cup} U_i$ is finite. Thus, $\underset{i \in I}{\cup} U_i \in (i, j)$-$\mathcal{N}\alpha O(X)$. □

Definition 2.2. (i) *The intersection of all (i, j)-$\mathcal{N}$-α-closed sets of X containing A is called the (i, j)-$\mathcal{N}$-α-closure of A and is denoted by (i, j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$.*
(ii) *The union of all (i, j)-$\mathcal{N}$-α-open sets of X contained in A is called the (i, j)-$\mathcal{N}$-α-interior and is denoted by (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A)$.*

Theorem 2.2. *Let A and B be subsets of (X, τ_1, τ_2). Then the following properties hold:*
(i) *(i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A)$ is the largest (i, j)-$\mathcal{N}$-α-open subset of X contained in A.*
(ii) *A is (i, j)-$\mathcal{N}$-α-open if and only if $A = (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A)$.*
(iii) *(i, j)-$\mathcal{N}\alpha\,\mathrm{Int}((i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A)) = (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A)$.*
(iv) *If $A \subset B$, then (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(B)$.*
(v) *(i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A) \cup (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A \cup B)$.*
(vi) *$\mathcal{N}\alpha\,\mathrm{Int}(A \cap B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A) \cap \mathcal{N}\alpha\,\mathrm{Int}(B)$.*

Proof. The proof of (i)-(iv) are obvious.
(v). By (iv), (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A \cup B)$ and (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A \cup B)$. Then (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A) \cup (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A \cup B)$.
(vi). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (iv), we have (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A \cap B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A)$ and (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A \cap B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(B)$. Then (i, j)-$\mathcal{N}\alpha\,\mathrm{Int}(A \cap B) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A) \cap (i, j)$-$\mathcal{N}\alpha\,\mathrm{Int}(B)$. □

Theorem 2.3. *Let A and B be subsets of (X, τ_1, τ_2). Then the following properties hold:*
(i) *(i, j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$ is the smallest (i, j)-$\mathcal{N}$-α-closed subset of X containing A.*
(ii) *A is (i, j)-$\mathcal{N}$-α-closed if and only if $A = (i, j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$.*
(iii) *(i, j)-$\mathcal{N}\alpha\,\mathrm{Cl}((i, j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A) = (i, j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$.*
(iv) *If $A \subset B$, then (i, j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A) \subset (i, j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(B)$.*

(v) (i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A\cup B) = (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A) \cup (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(B)$.
(vi) (i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A\cap B) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A) \cap (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(B)$.

Proof. The proofs are obvious. □

Theorem 2.4. *Let (X,τ_1,τ_2) be a bitopological space and $A\subset X$. A point $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$ if and only if $U\cap A\neq\emptyset$ for every $U\in (i,j)$-$\mathcal{N}\alpha O(X,x)$.*

Proof. Suppose that $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. We shall show that $U\cap A\neq\emptyset$ for every $U\in (i,j)$-$\mathcal{N}\alpha O(X,x)$. Suppose that there exists $U\in (i,j)$-$\mathcal{N}\alpha O(X,x)$ such that $U\cap A=\emptyset$. Then $A\subset X\backslash U$ and $X\backslash U$ is (i,j)-$\mathcal{N}$-α-closed. Since $A\subset X\backslash U$, (i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A)\subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(X\backslash U)$. Since $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$, we have $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(X\backslash U)$. Since $X\backslash U$ is (i,j)-$\mathcal{N}$-α-closed, we have $x\in X\backslash U$; hence $x\notin U$, which is a contradiction that $x\in U$. Therefore, $U\cap A\neq\emptyset$. Conversely, suppose that $U\cap A\neq\emptyset$ for every $U\in (i,j)$-$\mathcal{N}\alpha O(X,x)$. We shall show that $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. Suppose that $x\notin (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. Then there exists $U\in (i,j)$-$\mathcal{N}\alpha O(X,x)$ such that $U\cap A=\emptyset$. This is a contradiction to $U\cap A\neq\emptyset$; hence $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. □

Theorem 2.5. *Let (X,τ_1,τ_2) be a bitopological space and $A\subset X$. Then the following properties hold:*
(i) (i,j)-$\mathcal{N}\alpha\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$;
(i) (i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(X\backslash A) = X\backslash(i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A)$.

Proof. (i). Let $x\in X\backslash(i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. Since $x\notin (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$, there exists $V\in (i,j)$-$\mathcal{N}\alpha O(X,x)$ such that $V\cap A=\emptyset$; hence we obtain $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash(i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)\subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(X\backslash A)$. Let $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(X\backslash A)$. Since (i,j)-$\mathcal{N}\alpha\,\mathrm{Int}(X\backslash A)\cap A=\emptyset$, we obtain $x\notin (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$; hence $x\in X\backslash(i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. Therefore, we obtain (i,j)-$\mathcal{N}\alpha\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$.
(ii). Follows from (i). □

Theorem 2.6. *If each nonempty (i,j)-α-open set of a bitopological space (X,τ_1,τ_2) is infinite, then (i,j)-$\alpha\,\mathrm{Cl}(A) = (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$ for each subset $A\in\tau_1\cap\tau_2$.*

Proof. Clearly (i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(A)\subset (i,j)$-$\alpha\,\mathrm{Cl}(A)$. On the other hand, let $x\in (i,j)$-$\alpha\,\mathrm{Cl}(A)$ and B be an (i,j)-$\mathcal{N}$-α-open subset containing x. Then by Lemma 2.1, there exists an (i,j)-α-open set V containing x and a finite set C such that $V\backslash C\subset B$. Thus $(V\backslash C)\cap A\subset B\cap A$ and so $(V\backslash A)\cap C\subset B\cap A$. Since $x\in V$ and $x\in (i,j)$-$\alpha\,\mathrm{Cl}(A)$, $V\cap A\neq\emptyset$ and $V\cap A$ is (i,j)-α-open since V is (i,j)-α-open and $A\in\tau_1\cap\tau_2$. By the hypothesis each nonempty (i,j)-α-open set of X is finite and so is $(V\cap A)\backslash C$. Thus $B\cap A$ is finite. Therefore, $B\cap A\neq\emptyset$ which means that $x\in (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(A)$. □

Corollary 2.1. *If each nonempty (i,j)-α-closed set of a bitopological space (X,τ_1,τ_2) is infinite, then (i,j)-$\alpha\,\mathrm{Int}(A) = (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(A)$ for each subset $A\in\tau_1\cap\tau_2$.*

Theorem 2.7. *If every (i,j)-α-open subset of X is τ_i-open in (X,τ_1,τ_2), then $(X,(i,j)$-$\mathcal{N}\alpha O(X))$ is a topological space.*

Proof. (i). We have $\emptyset$, $X\in (i,j)$-$\mathcal{N}\alpha O(X)$. (ii). Let $U,V\in (i,j)$-$\mathcal{N}\alpha O(X)$ and $x\in U\cap V$. Then there exist (i,j)-α-open sets $G,H\in X$ containing x such that $G\backslash U$ and $H\backslash V$ are finite. And $(G\cap H)\backslash(U\cap V) = (G\cap H)\cap((X\backslash U)\cup(X\backslash V))\subset (G\cap(X\backslash U))\cup(H\cap(X\backslash V))$. Hence $(G\cap H)\backslash(U\cap V)$ is finite and by hypothesis, the intersection of two (i,j)-α-open sets is (i,j)-α-open. Hence $U\cap V\in (i,j)$-$\mathcal{N}\alpha O(X)$. (iii). Let $\{U_i : i\in I\}$ be any family of (i,j)-$\mathcal{N}$-α-open sets of X. Then, by Lemma 2.2 $\underset{i\in I}{\cup} U_i$ is (i,j)-$\mathcal{N}$-α-open. □

3. (i,j)-$\mathcal{N}$-α-continuous functions

Definition 3.1. *A function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$ is said to be (i,j)-$\mathcal{N}$-α-continuous if the inverse image of every σ_i-open set of Y is (i,j)-$\mathcal{N}$-α-open in X, where $i\neq j$, i,j=1, 2.*

It is clear that every (i,j)-α-continuous function is (i,j)-$\mathcal{N}$-α-continuous but not conversely.

Example 3.1. *Let $X=\{a,b,c\}$, $\tau=\{\varnothing,\{a\},X\}$ and $\sigma=\{\varnothing,\{b\},X\}$. Clearly the identity function $f:(X,\tau)\rightarrow(X,\sigma)$ is (i,j)-$\mathcal{N}$-α-continuous but not (i,j)-α-continuous.*

Theorem 3.1. *For a function $f:(X,\tau_1,\tau_2)\rightarrow(Y,\sigma_1,\sigma_2)$, the following statements are equivalent:*
(i) *f is (i,j)-$\mathcal{N}$-α-continuous;*
(ii) *For each point x in X and each σ_i-open set F in Y such that $f(x)\in F$, there is a (i,j)-$\mathcal{N}$-α-open set A in X such that $x\in A$, $f(A)\subset F$;*
(iii) *The inverse image of each σ_i-closed set in Y is (i,j)-$\mathcal{N}$-α-closed in X;*
(iv) *For each subset A of X, $f((i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(A))\subset\sigma_i\text{-}\mathrm{Cl}(f(A))$;*
(v) *For each subset B of Y, $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))$;*
(vi) *For each subset C of Y, $f^{-1}(\sigma_i\text{-}\mathrm{Int}(C))\subset(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}(C))$.*

Proof. (i)⇒(ii): Let $x\in X$ and F be a σ_i-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is (i,j)-$\mathcal{N}$-α-open in X. Let $A=f^{-1}(F)$. Then $x\in A$ and $f(A)\subset F$.
(ii)⇒(i): Let F be σ_i-open in Y and let $x\in f^{-1}(F)$. Then $f(x)\in F$. By (ii), there is an (i,j)-$\mathcal{N}$-α-open set U_x in X such that $x\in U_x$ and $f(U_x)\subset F$. Then $x\in U_x\subset f^{-1}(F)$. Hence $f^{-1}(F)$ is (i,j)-$\mathcal{N}$-α-open in X.
(i)⇔(iii): This follows due to the fact that for any subset B of Y, $f^{-1}(Y\backslash B)=X\backslash f^{-1}(B)$.
(iii)⇒(iv): Let A be a subset of X. Since $A\subset f^{-1}(f(A))$ we have $A\subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(f(A)))$. Now, $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f(A))$ is σ_i-closed in Y and hence $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(A)\subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(f(A)))$, for $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(A)$ is the smallest (i,j)-$\mathcal{N}$-α-closed set containing A. Then $f((i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(A))\subset\sigma_i\text{-}\mathrm{Cl}(f(A))$.
(iv)⇒(iii): Let F be any σ_i-closed subset of Y. Then $f((i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(F)))\subset\sigma_i\text{-}\mathrm{Cl}(f(f^{-1}(F)))\subset\sigma_i\text{-}\mathrm{Cl}(F)=F$. Therefore, $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(F))\subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is (i,j)-$\mathcal{N}$-α-closed in X.
(iv)⇒(v): Let B be any subset of Y. Now, $f((i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(B)))\subset\sigma_i\text{-}\mathrm{Cl}(f(f^{-1}(B)))\subset\sigma_i\text{-}\mathrm{Cl}(B)$. Consequently, $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))$.
(v)⇒(iv): Let $B=f(A)$ where A is a subset of X. Then, $(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(A)\subset(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(B))\subset f^{-1}(\sigma_i\text{-}\mathrm{Cl}(B))=f^{-1}(\sigma_i\text{-}\mathrm{Cl}(f(A)))$. This shows that $f((i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(A))\subset\sigma_i\text{-}\mathrm{Cl}(f(A))$.
(i)⇒(vi): Let B be any subset of Y. Clearly, $f^{-1}(\sigma_i\text{-}\mathrm{Int}(B)$ is (i,j)-$\mathcal{N}$-α-open and we have $f^{-1}(\sigma_i\text{-}\mathrm{Int}(B))\subset(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}\sigma_i\text{-}\mathrm{Int}(B))\subset(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}B)$.
(vi)⇒(i): Let B be a σ_i-open set in Y. Then $\sigma_i\text{-}\mathrm{Int}(B)=B$ and $f^{-1}(B)\subset f^{-1}(\sigma_i\text{-}\mathrm{Int}(B))\subset(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}(B))$. Hence we have $f^{-1}(B)=(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is (i,j)-$\mathcal{N}$-α-open in X. □

Definition 3.2. *(i) A bitopological space (X,τ_1,τ_2) is said to be pairwise α-compact if every cover of X by (i,j)-α-open sets admits a finite subcover.*
(ii) A subset A of a bitopological space (X,τ_1,τ_2) is said to be pairwise α-compact relative to X if every cover of A by (i,j)-α-open sets of X admits a finite subcover.

Theorem 3.2. *If (X, τ_1, τ_2) is a bitopological space such that every (i,j)-α-open subset of X is pairwise α-compact relative to X, then every subset of X is pairwise α-compact relative to X.*

Proof. Let B be an arbitrary subset of X and $\{U_i : i \in I\}$ be cover of B by (i,j)-α-open sets of (X, τ_1, τ_2). Then the family $\{U_i : i \in I\}$ is an (i,j)-α-open cover of the (i,j)-α-open set $\cup\{U_i : i \in I\}$. Hence by hypothesis there is a finite subfamily $\{U_{i_j} : j \in N_0\}$ which covers $\cup\{U_i : i \in I\}$ where N_0 is a finite subset of the naturals N. This subfamily is also a cover of the set B. □

Theorem 3.3. *A subset A of a bitopological space (X, τ_1, τ_2) is pairwise α-compact relative to X if and only if for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by (i,j)-$\mathcal{N}$-α-open sets of X, there exists a finite subset Λ_0 of Λ such that $A \subset \cup\{V_\alpha : \alpha \in \Lambda_0\}$.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of A and $V_\alpha \in (i,j)$-$\mathcal{N}\alpha O(X)$. For each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is (i,j)-$\mathcal{N}$-α-open, there exists an (i,j)-α-open set $U_{\alpha(x)}$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} \backslash V_{\alpha(x)}$ is finite. The family $\{U_{\alpha(x)} : x \in A\}$ is an (i,j)-α-open cover of A. Since A is pairwise α-compact relative to X, there exists a finite subset, $x_1, x_2, ..., x_n$ such that $A \subset \cup\{U_{\alpha(x_i)} : i \in F\}$, where $F = \{1, 2, ..., n\}$. Now, we have $A \subseteq \underset{i\in F}{\cup}((U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cup V_{\alpha(x_i)})$ $=(\underset{i\in F}{\cup}(U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cup (\underset{i\in F}{\cup} V_{\alpha(x_i)})$. For each x_i, $U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda(x_i)$ of Λ such that $(U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cap A \subseteq \cup\{V_\alpha : \alpha \in \Lambda(x_i)\}$. Therefore, we have $A \subseteq (\underset{i\in F}{\cup}(\cup\{V_\alpha : \alpha \in \Lambda(x_i)\})) \cup (\underset{i\in F}{\cup} V_{\alpha(x_i)})$. Hence A is pairwise α-compact relative to X. □

Corollary 3.1. *For any bitopological space (X, τ_1, τ_2), the following properties are equivalent:*
(i) *X is pairwise α-compact.*
(ii) *Every (i,j)-$\mathcal{N}$-α-open cover of X admits a finite subcover.*

Theorem 3.4. *A bitopological space (X, τ_1, τ_2) is pairwise α-compact if and only if every proper (i,j)-$\mathcal{N}$-α-closed set is pairwise α-compact with respect to X.*

Proof. Let A be a proper (i,j)-$\mathcal{N}$-α-closed subset of X. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of A by (i,j)-α-open sets of X. Now for each $x \in X \backslash A$, there is an (i,j)-α-open set V_x such that $V_x \backslash A$ is finite. Then $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X \backslash A\}$ is a (i,j)-α-open cover of X. Since X is pairwise α-compact, there exists a finite subset Λ_1 of Λ and a finite number of points, say, $x_1, x_2, ..., x_n$ in $X \backslash A$ such that $X = (\cup\{U_\alpha : \alpha \in \Lambda_1\}) \cup (\cup\{V_{x_i} : 1 \le i \le n\})$; hence $A \subset (\cup\{U_\alpha : \alpha \in \Lambda_1\}) \cup (\cup\{A \cap V_{x_i} : 1 \le i \le n\})$. Since $A \cap V_{x_i}$ is finite for each i, there exists a finite subset Λ_2 of Λ such that $\cup\{A \cap V_{x_i} : 1 \le i \le n\} \subset \cup\{U_\alpha : \alpha \in \Lambda\}$. Therefore, we obtain $A \subset \cup\{U_\alpha : \alpha \in \Lambda_1 \cup \Lambda_2\}$. This shows that A is pairwise α-compact relative to X. Conversely, Let $\{V_\alpha : \alpha \in \Lambda\}$ be any (i,j)-α-open cover of X. We choose and fix one $\alpha_0 \in \Lambda$. Then $\cup\{V_\alpha : \alpha \in \Lambda \backslash \{\alpha_0\}\}$ is an (i,j)-α-open cover of the (i,j)-$\mathcal{N}$-α-closed set $X \backslash V_{\alpha_0}$. Then there exists a finite set $\Lambda_0 \subset \Lambda$ such that $X \backslash V_{\alpha_0} \subset \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore, $X = \cup\{V_\alpha : \alpha \in \Lambda_0 \cup \{\alpha_0\}\}$. This shows that X is pairwise α-compact. □

Theorem 3.5. *Let (X, τ_1, τ_2) be a bitopological space such that (i,j)-$\alpha O(X) = \tau_i$. Then (X, τ_1, τ_2) is pairwise α-compact if and only if $(X, (i,j)$-$\mathcal{N}\alpha O(X))$ is compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $(X, (i,j)$-$\mathcal{N}\alpha O(X))$. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is (i,j)-$\mathcal{N}$-α-open, there exists an (i,j)-α-open set $U_{\alpha(x)}$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} \backslash V_{\alpha(x)}$ is finite. The family

$\{U_{\alpha(x)} : x \in X\}$ is a (i,j)-α-open cover of (X,τ_1,τ_2). Since (X,τ_1,τ_2) is pairwise α-compact, there exists a finite subset, $x_1, x_2, ..., x_n$ such that $X = \cup\{U_{\alpha(x_i)} : i \in F\}$, where $F = \{1, 2, ..., n\}$. Now, we have $X = \underset{i\in F}{\cup}((U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cup V_{\alpha(x_i)}) = (\underset{i\in F}{\cup}(U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cup (\cup_{i\in F} V_{\alpha(x_i)})$. For each x_i, $U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda(x_i)$ of Λ such that $(U_{\alpha(x_i)} \backslash V_{\alpha(x_i)}) \cap A \subseteq \cup\{V_\alpha : \alpha \in \Lambda(x_i)\}$. Therefore, we have $X = (\underset{i\in F}{\cup}(\cup\{V_\alpha : \alpha \in \Lambda(x_i)\}) \cup (\underset{i\in F}{\cup} V_{\alpha(x_i)})$. Hence $(X, (i,j)$-$\mathcal{N}\alpha O(X))$ is compact. Conversely, let $\mathcal{U}$ be an (i,j)-α-open cover of (X,τ_1,τ_2). Then $\mathcal{U} \subseteq (i,j)$-$\mathcal{N}\alpha O(X)$. Since $(X, (i,j)$-$\mathcal{N}\alpha O(X))$ is compact, there exists a finite subcover of $\mathcal{U} \subset (i,j)$-$\mathcal{N}\alpha O(X)$. Hence (X,τ_1,τ_2) is pairwise α-compact. □

Definition 3.3. *(i) A bitopological space (X,τ_1,τ_2) is said to be pairwise compact if every cover of X by τ_i-open sets admits a finite subcover.*
(ii) A subset A of a bitopological space (X,τ_1,τ_2) is said to be pairwise compact relative to X if every cover of A by τ_i-open sets of X admits a finite subcover.

Theorem 3.6. *Let f be an (i,j)-$\mathcal{N}$-α-continuous function from a space X onto a space Y. If X is pairwise α-compact, then Y is pairwise compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of Y by σ_i-open sets of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an (i,j)-$\mathcal{N}$-α-open cover of X. Since X is pairwise α-compact, there exists a finite subset Λ_0 of Λ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Lambda_0\}$; hence $Y = \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore Y is pairwise compact. □

4. (i,j)-$\mathcal{N}$-α-irresolute functions

Definition 4.1. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be (i,j)-$\mathcal{N}$-α-irresolute if the inverse image of every (i,j)-α-$\mathcal{N}$-open set of Y is (i,j)-$\mathcal{N}$-α-open in X, where $i \neq j$, i,j=1, 2.*

Proposition 4.1. *Every (i,j)-$\mathcal{N}$-α-irresolute function is (i,j)-$\mathcal{N}$-α-continuous but not conversely.*

Proof. Straightforward. □

The function f defined as in example 3.1 is (i,j)-$\mathcal{N}$-α-continuous but not (i,j)-$\mathcal{N}$-α-irresolute.

Theorem 4.1. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a function, then*
(1) *f is (i,j)-$\mathcal{N}$-α-irresolute;*
(2) *the inverse image of each (i,j)-α-$\mathcal{N}$-closed subset of Y is (i,j)-$\mathcal{N}$-α-closed in X;*
(3) *for each $x \in X$ and each $V \in (i,j)$-$\mathcal{N}\alpha O(Y)$ containing $f(x)$, there exists $U \in (i,j)$-$\mathcal{N}\alpha O(X)$ containing x such that $f(U) \subset V$.*

Proof. The proof is obvious from the fact that the arbitrary union of (i,j)-$\mathcal{N}$-α-open subsets is (i,j)-$\mathcal{N}$-α-open. □

Theorem 4.2. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a function, then*
(i) *f is (i,j)-$\mathcal{N}$-α-irresolute;*
(ii) *(i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V))$ for each subset V of Y;*
(iii) *$f((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(U) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(U))$ for each subset U of X.*

Proof. $(i) \Rightarrow (ii)$: Let V be any subset of Y. Then $V \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V)$ and $f^{-1}(V) \subset f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V))$. Since f is (i,j)-$\mathcal{N}$-α-irresolute, $f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V))$ is a (i,j)-$\mathcal{N}$-α-closed subset of X. Hence

(i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(V)) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V))) = f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V))$.
$(ii) \Rightarrow (iii)$: Let U be any subset of X. Then $f(U) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(U))$ and (i,j)-$\mathcal{N}\alpha\,\mathrm{Cl}(U) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(f(U))) \subset f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(U)))$. This implies that $f((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(U)) \subset f(f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(U)))) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(U))$.
$(iii) \Rightarrow (i)$: Let V be an (i,j)-$\mathcal{N}$-α-closed subset of Y. Then $f((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(V)) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(f^{-1}(V))) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V) = V$. This implies that

$$(i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(f((i,j)\text{-}\mathcal{N}\alpha\,\mathrm{Cl}(f^{-1}(V)))) \subset f^{-1}(V).$$

Therefore, $f^{-1}(V)$ is an (i,j)-$\mathcal{N}$-α-closed subset of X and consequently f is an (i,j)-$\mathcal{N}$-α-irresolute function. □

Theorem 4.3. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is an (i,j)-$\mathcal{N}$-α-irresolute if and only if $f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(V)) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}(V))$ for each subset V of Y.*

Proof. Let V be any subset of Y. Then (i,j)-$\mathcal{N}\alpha\,\mathrm{Int}(V) \subset V$. Since f is (i,j)-$\mathcal{N}$-α-irresolute, $f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(V))$ is an (i,j)-$\mathcal{N}$-α-open subset of X. Hence $f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(V)) = (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(V))) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}(V))$.

Conversely, let V be an (i,j)-α-$\mathcal{N}$-open subset of Y. Then $f^{-1}(V) = f^{-1}((i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(V)) \subset (i,j)$-$\mathcal{N}\alpha\,\mathrm{Int}(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is an (i,j)-$\mathcal{N}$-α-open subset of X and consequently f is an (i,j)-$\mathcal{N}$-α-irresolute function. □

Definition 4.2. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be:*
(i) *(i,j)-$\mathcal{N}$-α-open if $f(U)$ is a (i,j)-$\mathcal{N}$-α-open set of Y for every τ_i-open set U of X.*
(ii) *(i,j)-$\mathcal{N}$-α-closed if $f(U)$ is a (i,j)-$\mathcal{N}$-α-closed set of Y for every τ_i-closed set U of X.*

Corollary 4.1. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a function. Then f is (i,j)-$\mathcal{N}$-α-closed and (i,j)-$\mathcal{N}$-α-irresolute if and only if $f((i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(V)) = (i,j)$-$\mathcal{N}\alpha\,\mathrm{Cl}(f(V)))$ for every subset V of X.*

Theorem 4.4. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (i,j)-$\mathcal{N}$-α-irresolute function. If Y is pairwise α-compact, then X is pairwise α-compact.*

Proof. The proof is clear. □

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