Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(1), 2011, Pages 11–26
ISSN: 1223-6934

A nonlinear equation that unifies the quantum Yang-Baxter equation and the Hopf equation. Solutions and applications

Gefry Barad

Abstract. We define a nonlinear equation , called (UE),which unify QYBE and Hopf equations, and which provide representations for the new crossed-simplicial groups R(n) defined in [2]. We continue the study of a system of mixed Yang-Baxter type equations presented in [2], which provide solutions for UE. Similar to the study of Hopf, Long or QYBE, we find sufficient conditions for a bilinear on a Hopf algebra to provide canonical solutions on any H-comodule, and also sufficient conditions for an entwining structure, such that the canonical application for an entwined module $R(m \otimes n) = m_1 n \otimes m_0$ on $M \otimes M$ verifies the equation.
Any solution of the equation generates a twisted factorisation structure on a tensor algebra. Theory of Hopf algebras with a weak projection satisfying certain properties provides solutions. A new nonlinear equation is proposed.

2010 Mathematics Subject Classification. Primary 16T25; Secondary 46L87, 16S40, 81R50.
Key words and phrases. Yang-Baxter Equation, Hopf equation, entwined modules, Hopf algebras with a weak projection, braided Hopf algebra, algebra factorisations.

1. Introduction

Non-linear equations on multiple tensor products of vector spaces play an important role in the theory of integrable systems and of quantum groups. Hopf and quantum Yang-Baxter equations were studied in [3], [12] using a Tannaka-Krein approach.Also, Militaru mentioned that the Yetter- Drinfel'd modules and Hopf modules, which are modules and comodules satisfying certain compatibility conditions and which provide solutions to QYBE and PE equations are particular cases of a Doi-Koppinen datum [6], a concept generalized by Brzezinski to the concept of entwining structure [9].

Our purpose is to present a non-linear equation such that solutions of QYBE and Hopf equations are automatically solutions of it. Solutions to a system of mixed Yang-Baxter type equations came from a particular Doi-Koppinen datum.

2. Non - linear equations on tensor product of vector spaces

The quantum Yang-Baxter equation (QYBE) is the following equation involving R: $M \otimes M \to M \otimes M$; $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} : M \otimes M \otimes M \to M \otimes M \otimes M$. A solution of it generates representations for the type A Artin Braid groups B(n), generated by s_i, 1 ≤ i ≤ n with relations: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$. s_i and s_j commutes if i and j are not consecutive.

Received August 17, 2010. Revision received December 27, 2010.
This paper is suported by the Sectorial Operational Programme Human Resources Development (SOP HRD), financed from the European Social Fund and by the Romanian Government under the contract number SOP HRD/89/1.5/S/62988.

We will use Sweedler notation and the multi-linearity of the involved operators. We will use group-like notation for functions: ABC means $C \circ (B \circ A)$.

Solutions for QYBE are given by quasitriangular bialgebras (H, R), where the element R from $H \otimes H$ satisfies:

$\triangle \otimes id(R) = R_{13}R_{23}$.

$id \otimes \triangle$(R)$= R_{13}R_{12}$.

$id \otimes \epsilon$(R)$= \epsilon \otimes id$(R)=1.

R$\triangle(h) = \triangle^{cop}(h)R$, for any h from H.

If H is a Hopf algebra, R is invertible and induces on any left H-module a solution for QYBE. The dual notion is that of a co-quasitriangular bialgebra (H, σ), where $\sigma : H \otimes H \to k$ verifies natural axioms such that on any right H-comodule M, the function $S(x \otimes y) = \sum x_1 \otimes y_1 \sigma(x_2 \otimes y_2)$ is a solution for QYBE. These axioms are:

$\sigma(x \otimes 1) = \varepsilon(x)$;

$\sigma(xy \otimes z) = \sum \sigma(x \otimes z_{(1)})\sigma(y \otimes z_{(2)})$;

$\sigma(1 \otimes x) = \varepsilon(x)$;

$\sigma(x \otimes yz) = \sum \sigma(x_{(1)} \otimes z)\sigma(x_{(2)} \otimes y)$; for any $x, y, z \in H$.

$\sum \sigma(x_{(1)} \otimes y_{(1)})y_{(2)}x_{(2)} = \sum \sigma(x_{(2)} \otimes y_{2)})x_{(1)}y_{(1)}$,

The Yetter-Drinfeld modules $(M, ., f)$ also generates solutions for QYBE. M is a left H-module and a right H-comodule with the following compatibility relation:

$h_1 m_0 \otimes h_2 m_1 = (h_2 m)_0 \otimes (h_2 m)_1 h_1$, for any h in H and m in M.

In this case, the operator $R(m \otimes n) = n_1 m \otimes n_0$ is a solution for QYBE on $M \otimes M$.

Let R be a QYBE operator. The bialgebra A(R), generated by e^i_j, $1 \leq$ i,j $\leq$ dim_C(M), modulo the relations: $\sum_{k,l} R^{kl}_{ij} e^a_k e^b_l = \sum_{k,l} R^{ba}_{kl} e^k_i e^l_j$ is cobraided (Chap.VIII [10]).

Definition 2.1. *([12], [3])$R \in End(M \otimes M)$ satisfies the Pentagonal equation if $R_{23}R_{12} = R_{12}R_{13}R_{23}$. The Hopf equation is given by: $R_{12}R_{23} = R_{23}R_{13}R_{12}$.*

Lemma 2.1. *([12]) R is a solution of the Hopf equation if and only if $Flip \circ R \circ Flip$ is a solution of the pentagonal equation.*

2.1. A unifying equation for QYBE and the Hopf equation.

Let R: $M \otimes M \to M \otimes M$ be an invertible linear operator. We introduce the following equation (UE):

$B_{123}R_{34}D_{124}R_{24}R_{14}D_{123} = R_{34}R_{24}R_{14}$, where D is the inverse of the "'Yang-Baxter operator" B : $H \otimes H \otimes H \to H \otimes H \otimes H$. B=$(R_{23}R_{13}R_{12})(R_{12}R_{13}R_{23})^{-1}$

The equation represents the equality of two operators, built using R and defined from $M \otimes M \otimes M \otimes M \to M \otimes M \otimes M \otimes M$. The indices attached to R,D,B show the positions from the tensor product where these operators act, on the remaining positions the action is given by identity.

Theorem 2.1. *Any operator R which satisfy QYBE is a solution to this equation. Any operator which satisfy the Hopf equation is a solution to this equation.*

Proof. If R satisfy QYBE, then B=D is the identity operator, so the equation UE is satisfied. If R satisfy the Hopf equation, then B= $R_{12}R_{13}^{-1}R_{12}^{-1}$ and an easy calculation shows the R satisfy also UE. Let S be the inverse of R.

$R_{34}R_{24}R_{14} = R_{12}R_{34}R_{24}S_{12}$

The UE equation is given by:

$R_{34}R_{24}R_{14} = R_{23}R_{13}R_{12}S_{23}S_{13}S_{12}R_{34}R_{12}R_{14}R_{24}R_{13}R_{23}S_{12}S_{13}S_{23}$

$R_{23}R_{13}R_{12} = R_{12}R_{23}$, so the right hand side is equal to

A=$R_{12}S_{13}S_{12}R_{34}R_{12}R_{14}R_{24}R_{13}R_{23}S_{12}S_{13}S_{23}$

$R_{34}R_{12} = R_{12}R_{34}$ and $R_{24}R_{13} = R_{13}R_{24}$ so
A=$R_{12}S_{13}[R_{34}R_{14}R_{13}]R_{24}R_{23}S_{12}S_{13}S_{23}$=
=$R_{12}S_{13}[R_{13}R_{34}]R_{24}R_{23}S_{12}S_{13}S_{23}$=$R_{12}R_{34}R_{24}[R_{23}S_{12}S_{13}S_{23}]$=
=$R_{12}R_{34}R_{24}S_{12}$, so the left hand side of the UE equation is equal to A.

□

2.2. A system of mixed Yang-Baxter type equations. Other solutions for the UE equation. We consider the following system. R,S: $M \otimes M \to M \otimes M$

$R(1,2)S(1,3)R(2,3) = R(2,3)R(1,3)R(1,2)$

$S(1,2)S(1,3)R(2,3) = R(2,3)S(1,3)S(1,2)$

It is a unifying system for the QYBE (if R=S) and for the Hopf equation (if S is the identity operator). We recall the following theorem from [2] (Theorem 4.1).

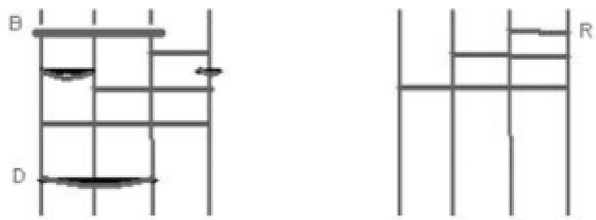
Theorem 2.2. *If the pair (R,S) is a solution for the system above, then R verifies the UE. In particular, if R is a solution of the pentagonal equation, hen R satisfy the UE equation.*

Lemma 2.2. *Flip$\circ$R verifies the UE equation:* $B_{123}R_{34}D_{124}R_{24}R_{14}D_{123}$=$R_{34}R_{24}R_{14}$ *if and only if R verifies the following equation:* $B_{123}R_{34}D_{123}R_{23}R_{12}D_{234}$=$R_{34}R_{23}R_{12}$, *called R- equation, where D is the inverse of the ”‘braid operator”*

$B : H \otimes H \otimes H \to H \otimes H \otimes H$. B=$(R_{23}R_{12}R_{23})(R_{12}R_{23}R_{12})^{-1}$

The $\mathbf{R-equation}$ *represents the equality of two operators, built using R and defined from* $M \otimes M \otimes M \otimes M \to M \otimes M \otimes M \otimes M$. *The indices attached to R,D,B show the positions from the tensor product where these operators act, on the remaining positions the action is given by identity.*

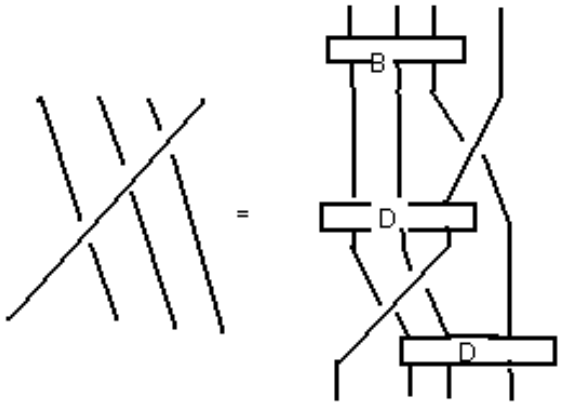
Proof. Graphically,the equation $B_{123}R_{34}D_{124}R_{24}R_{14}D_{123} = R_{34}R_{24}R_{14}$ is written as:



R-equation has the following graphical representation.

Similar relations between the QYBE and Braid operators were found in [7] (Prop. 114 and 124, Dfn. 12). Flip composition and the inverses of the operators connect the Hopf and the pentagonal equation. Unlike other non-linear equations, if R is a solution of UE, its inverse or their ”flip” compositions do not seem to provide other solutions of UE.

The R- equation is an equation in the strict symmetric (so braided), abelian category of the vector spaces $Vect$. R is a sum of operators $f \otimes g$ and the Flip (the symmetry of the braided category Vect) is a natural transformation. So, we can push the operators $f \otimes g$ above the crossings among strings from the second figure above: the result will be the UE equation, followed by a composition of braid operators (flips). □



Theorem 2.3. *A solution of the following system generates an operator R, which verifies the UE equation* $R_{34}R_{24}R_{14} = R_{23}R_{13}R_{12}S_{23}S_{13}R_{34}R_{14}R_{24}R_{13}R_{23}S_{12}S_{13}S_{23}$*, where S is the inverse of R.*

R,A,B,C,X,Y: $M \otimes M \to M \otimes M$

$R_{23}R_{13}R_{12} = A_{12}B_{13}R_{23}$

$R_{23}R_{13}A_{12} = A_{12}C_{13}R_{23}$

$R_{23}C_{13}B_{12} = B_{12}X_{13}Y_{23}$

$R_{23}R_{13}R_{12} = R_{12}X_{13}Y_{23}$

C=R; X=B; Y=A, or other imposed conditions (B = Id and A=R, in which case we have the Hopf equation) simplify the form of the system above.

Proof. Let (R,A,B,C,X,Y) be a solution to the system above. We will show that R verifies the UE equation.

$R_{23}R_{13}R_{12}S_{23}S_{13}R_{34}R_{14}R_{24}R_{13}R_{23}S_{12}S_{13}S_{23}=$
$A_{12}B_{13}R_{23}S_{23}S_{13}R_{34}R_{14}R_{24}R_{13}R_{23}S_{12}S_{13}S_{23}=$
$A_{12}B_{13}S_{13}R_{34}R_{14}R_{13}R_{24}R_{23}S_{12}S_{13}S_{23} = A_{12}B_{13}S_{13}R_{13}X_{14}Y_{34}R_{24}R_{23}S_{12}S_{13}S_{23}$
$=A_{12}B_{13}X_{14}Y_{34}R_{24}R_{23}S_{12}S_{13}S_{23} = A_{12}R_{34}C_{14}B_{13}R_{24}R_{23}S_{12}S_{13}S_{23}=$
$R_{34}A_{12}C_{14}R_{24}B_{13}R_{23}S_{12}S_{13}S_{23}=R_{34}R_{24}R_{14}A_{12}B_{13}R_{23}S_{12}S_{13}S_{23} = R_{34}R_{24}R_{14}$ □

2.3. Hopf algebras with a projection. Generalized Yetter-Drinfeld modules. Panaite and Staic [15] introduced the concept of (a,b)-Yetter-Drinfeld module, for a Hopf algebra H with bijective antipode and a, b automorphisms of H.

M is a left-right (a,b)-Yetter-Drinfeld module if it is a left module and a right comodule over H, and the following compatibility relation between the action and the coaction is satisfied:

$(x_1 \triangleright y_1) \otimes (b(x_2)y_2) = (x_2 \triangleright y)_1 \otimes (x_2 \triangleright y)_2 a(x_1)$

We prove that a solution for the system:

$R(1,2)S(1,3)R(2,3) = R(2,3)R(1,3)R(1,2)$

$S(1,2)S(1,3)R(2,3) = R(2,3)S(1,3)S(1,2)$

is given by any (p, id)- module over a Hopf algebra with a projection p.

A projection p for a Hopf algebra H is a Hopf algebra endomorphism which satisfies p(p(x))=p(x) for any x from H. Any quasitriangular Hopf algebra H (for example the Drinfeld double of a finite dimensional Hopf algebra A) has its Drinfeld double a

Hopf algebra with a projection. Also any braided Hopf algebra (a Hopf algebra in the category of Yetter-Drinfeld modules over H) provides using Radford biproduct a Hopf algebra with a projection [4], [11] Thm.9.4.12.

Theorem 2.4. *Let M be a (p, id)- Yetter - Drinfeld module over a Hopf algebra with a projection p. We define:*
$R(m \otimes n) = m_1 \otimes m_2 \triangleright n$
$S(m \otimes n) = m_1 \otimes p(m_2) \triangleright n$
Then,(R,S) is a solution of the system above.

Proof. $R_{23}R_{13}R_{12}(m \otimes n \otimes w) = R_{12}(R_{13}(m \otimes n_1 \otimes n_2 \triangleright w))=$
$R_{12}(m_1 \otimes n_1 \otimes m_2 \triangleright n_2 \triangleright w)) = m_1 \otimes m_2 \triangleright n_1 \otimes m_3 n_2 \triangleright w.$
$R_{12}S_{13}R_{23}(m \otimes n \otimes w) = R_{23}(S_{13}(m_1 \otimes m_2 \triangleright n \otimes w)) = R_{23}(m_1 \otimes m_3 \triangleright n \otimes p(m_2) \triangleright w).$
The first term m_1 is the same in both expresions we want to prove they are equal, so it is enough to prove that: $R(m_2 \triangleright n \otimes p(m_1) \triangleright w)) = m_1 \triangleright n_1 \otimes m_2 n_2 \triangleright w$
$\Leftrightarrow (m_2 \triangleright n)_1 \otimes (m_2 \triangleright n)_2 p(m_1) \triangleright w = m_1 \triangleright n_1 \otimes m_2 n_2 \triangleright w$
M is a (p,id) Yetter-Drinfeld module , so the equality is true (both elements are the image of the defining relation for M through $id \otimes - \triangleright w$)
$R_{23}S_{13}S_{12}(m \otimes n \otimes w) = S_{12}(S_{13}(m \otimes n_1 \otimes n_2 \triangleright w))=$
$S_{12}(m_1 \otimes n_1 \otimes p(m_2) \triangleright n_2 \triangleright w)) = m_1 \otimes p(m_2) \triangleright n_1 \otimes p(m_3) n_2 \triangleright w.$
$S_{12}S_{13}R_{23}(m \otimes n \otimes w) = R_{23}(S_{13}(m_1 \otimes p(m_2) \triangleright n \otimes w)) =$
$R_{23}(m_1 \otimes p(m_3) \triangleright n \otimes p(m_2) \triangleright w))$
It is enough to prove that: $R(p(m_2) \triangleright n \otimes p(m_1) \triangleright w)) = p(m_1) \triangleright n_1 \otimes p(m_2) n_2 \triangleright w$, which is (p,id) Yetter -Drinfeld condition for h=p(m) in the image of p. ∎

2.4. Braided Hopf algebras.

In this section, we re-prove a particular case of Theorem 4.2 [2], which states the Hopf equation in braided monoidal category, satisfied by the fusion operator of a Hopf algebra in a braided monoidal category.

Let H be a Hopf algebra with bijective antipode. Let B be a Hopf algebra in the strict braided monoidal category of left-right Yetter-Drinfeld modules over H. So, B is an object in this category, togeter with a comultiplication $\delta : B \to B \otimes B$, multiplication m, unit and counit ϵ morphisms which satisfy the usual axioms for a Hopf algebra. δ is a braided algebra morphism:
$\delta(xy) = (m \otimes m)(id \otimes c \otimes id)(\delta(x) \otimes \delta(y))$
c is the braiding on B induced by the Yetter-Drinfeld structure (instead of the regular flip).

We find a special class of solutions to the system associated to these data. The proof uses classic facts about braided geometry [11].

Lemma 2.3. *(*[11] *and Sect.2.2* [4]*) On B⊗H there is a natural Hopf algebra structure (Radford's biproduct $B \rtimes H$), given by:*
$(b, h)(c, k) = (b(h_1 c), h_2 k)$
$\Delta(b, h) = (b^1, b^2_1 h_1) \otimes (b^2_0, h_2)$
The subscript denotes the Yetter-Drinfeld coaction and the suprascript denotes δ Sweedler notation. H acts by its left Yetter- Drinfeld action on B. The multiplication from B appears also in the definition of the product. B and H are subalgebras of $B \rtimes H$. There is a projection $p : B \rtimes H \to H$, given by $p(b, h) = \epsilon(b)h$. Any Hopf algebra with a projection is isomorphic with a Radford's biproduct.

To our knowledge, the following structural result concerning a braided Hopf algebra is new. We provide B with the following module and comodule structure maps over the Hopf algebra $(B \rtimes H)^{cop}$: $B \rtimes H \otimes B \to B$ $(b,h) \rhd m = b(hm)$, where h acts on m using the Yetter -Drinfeld action of H on B, and the second operation is the braided Hopf algebra multiplication on B.

$\rho : B \to B \otimes B \otimes H, b \to b_0^2 \otimes b^1 \otimes b_1^2 = \Delta^{op}(b,1) = b_{[o]} \otimes b_{[1]}$

Note: the restriction of the action to H, and the composition of the coaction by $id \otimes p = id \otimes \epsilon_B id_H$ recovers the Yetter Drinfeld module structure of B.

Given a Hopf algebra with a projection (H, p, H_1), there is a standard way (Sect.2.2 [4]) to recover B as the set of coinvariants with respect to $(id \otimes p)\Delta$. Similarly we can define on B an H^{cop} module and comodule structure and we conjecture that B is a (p,id) Yetter-Drinfeld module.

Let R and S be the following maps:

R: $B \otimes B \to B \otimes B$, $R(b \otimes c) = b_0^2 \otimes b^1(b_1^2 c) = b_{[o]} \otimes b_{[1]} \rhd c$

S: $B \otimes B \to B \otimes B$, $S(b \otimes c) = b_0 \otimes (b_1 c)$

S = $flip \circ c$, where c is the braiding on the Yetter-Drinfeld module B.

(R,S) is a solution for the mixed Yang-Baxter system if and only if

(T=$flip \circ R$,c) satisfies the following sistem :

$T(23)T(12)T(23) = T(12)c(23)T(12)$

$T(23)c(12)c(23) = c(12)c(23)T(12)$

Theorem 2.5. *B is a (p,id) Yetter-Drinfeld module over* $(B \rtimes H)^{cop}$*.*

Proof. Let $H = (m, h)$. m is in B, and h in Im(p).

$\Delta(m,h) = (m_0^2, h_2) \otimes (m^1, m_1^2 h_1) = H_1 \otimes H_2$

$\Delta(m,1) = (m_0^2, 1) \otimes (m^1, m_1^2 1) = D_1 \otimes D_2$

$\rho((m^1, m_1^2 h_1) \triangleright b)\epsilon(m_0^2)h_2 = \rho(\epsilon(m_0^2)(m^1, m_1^2) \triangleright (h_1 \triangleright b)h_2 = \rho(\epsilon(D_1)D_2 \triangleright (h_1 \triangleright b)h_2 =$
$\rho((m,1) \triangleright (h_1 \triangleright b)h_2 = \rho((m.(h_1 \triangleright b)h_2 = \Delta(m,1)\Delta(h_1 b)h_2 = \Delta(m,1)\Delta((1,h)\rho(b)$,
where we used the B is a Yetter-Drinfeld module over Im(p)=H.

$\Delta(m,1)\Delta((1,h)\rho(b) = \Delta(m,h)\rho(b)$, which is the definition of a (p,id)-Yetter - Drinfeld module; we used the multiplicativity of the comultiplication in the "'big" Hopf algebra, and the definition of the coaction.

As a corollary, we apply Theorem 4 to this (p,id)- Yetter-Drinfeld module to prove the (T,c) verifies the Hopf equation associated with the braided Hopf algebra B. □

3. Generalizations. Hopf algebras with a weak projections, entwined modules and bilinear forms.

Let H be a Hopf algebra. We investigate the sufficient conditions satisfied by a bilinear form $\sigma : H \otimes H \to k$ such that on any right H-comodule M, the function $R(x \otimes y) = \sum y_1 \otimes x_1 \sigma(x_2 \otimes y_2)$ is a solution for the R- equation. We suppose σ is convolution invertible, with inverse r and H has invertible antipode. The inverse of R is $S(x \otimes y) = \sum y_1 \otimes x_1 r(y_2 \otimes x_2)$.

Remark 3.1. *We write the R-equation in the following form:*

$R_{34}R_{23}R_{12}R_{34}R_{23}R_{34}S_{23}S_{34} = R_{23}R_{12}R_{23}S_{12}S_{23}R_{34}R_{23}R_{12}$

If R and S are induced on $M \otimes M$ *by bilinear forms above, the equality is written, when evaluated on* $x \otimes y \otimes z \otimes t$ *as:*

$t_1 \otimes y_1 \otimes x_1 \otimes z_1 r(x_2 \otimes z_2) r(y_2 \otimes z_3)\sigma(x_3 \otimes y_3)\sigma(x_4 \otimes z_4)\sigma(x_5 \otimes t_2)\sigma(y_4 \otimes z_5)\sigma(y_5 \otimes t_3)\sigma(z_6 \otimes t_4)$

$= t_1 \otimes y_1 \otimes x_1 \otimes z_1 \sigma(y_2 \otimes t_2)\sigma(x_2 \otimes t_3)\sigma(z_2 \otimes t_4) r(x_3 \otimes z_3) r(y_3 \otimes z_4)\sigma(x_4 \otimes y_4)\sigma(x_5 \otimes z_5)\sigma(y_5 \otimes z_6)$.

An immediate consequence is the following: if H is co-commutative , any bilinear σ induces on any H-comodule a solution for the R-equation.

Let M be a H-comodule.

Lemma 3.1. *(a)If* $\sigma(x \otimes yz) = \sum \sigma(x_{(1)} \otimes y)\sigma(x_{(2)} \otimes z)$, $\sigma(x \otimes 1) = \varepsilon(x)$, *and* $F(F(x, y), z) = F(x, y_1)S(y_2)F(y_3, z)$ *for any x,y,z from H, where S is the antipode of H and F is the twisting of the multiplication of H by the inverse of* σ*:*

$$\sum \sigma(x_{(1)} \otimes y_{(1)})F(y_{(2)}, x_{(2)}) = \sum \sigma(x_{(2)} \otimes y_{2)})x_{(1)}y_{(1)},$$

then $R(x \otimes y) = \sum y_1 \otimes x_1\sigma(x_2 \otimes y_2)$ *is a solution of the R-equation.*

(b)If $\sigma(xy \otimes z) = \sum \sigma(y \otimes z_{(1)})\sigma(x \otimes z_{(2)})$, $\sigma(1 \otimes x) = \varepsilon(x)$, *and* $F(F(x, y), z) = F(xy, z)$ *for any x,y,z from H, where F is the twisting of the opposite multiplication of H by* σ*:*

$$\sum \sigma(x_{(2)} \otimes y_{(2)})F(x_{(1)}, y_{(1)}) = \sum \sigma(x_{(1)} \otimes y_{1)})y_{(2)}x_{(2)},$$

the R defined as above using the bilinear and M is a solution of the R-equation.

Note: if F is an associative product, and in the settings of (b), a sufficient condition is: there is an action of the Hopf algebra H on the vector space H, given by

F: $H \otimes H \to H$.

An example of such an F which satisfies both conditions above is given by p(x)y, where p is a Hopf algebra projection. If p=id or p=$\varepsilon(x)$, we get H a coquasitriangular Hopf algebra, or a Hopf algebra with a Hopf bilinear, studied by Militaru. Given a projection p for a Hopf algebra, the fact that F (and the given associated p) is induced by a bilinear σ requires further restrictions, which will be studied elsewhere. For example, if p=$\varepsilon(x)$, then σ has to be defined on $C \otimes H$, where C is a sub-coalgebra not containing the unit of H.(Remark 2.2 [13])

Proof. We will use the remark 3.1 above.

(a) $r(x, y) = \sigma(x, S(y))$ Using the right multiplicativity of σ and grouping the terms which contain x's, a sufficient condition to satisfy the R-equation is:

(we decrease by 1 the indices from the Sweedler notation) For any y,z,t from H:

$S(z_1)y_2z_3t_1\sigma(y_1 \otimes S(z_2))\sigma(y_3 \otimes z_4)\sigma(y_4 \otimes t_2)\sigma(z_5 \otimes t_3) =$

$\sigma(y_1 \otimes t_1)t_2\sigma(z_1 \otimes t_3)S(z_2)\sigma(y_2 \otimes S(z_3))y_3z_4\sigma(y_4 \otimes z_5)$

$\Leftrightarrow S(z_1)F(z_2, y_1)t_1\sigma(z_3 \otimes t_3)\sigma(y_2 \otimes t_2)= t_2S(z_2)F(z_3, y_2)\sigma(y_1 \otimes t_1)\sigma(z_1 \otimes t_3)$.

We apply to $F(z_2, y_1)t_1\sigma(y_2 \otimes t_2)$ the following formula:

$F(cd_2, b_2)\sigma(b_1 \otimes d_1) = F(c, b_1)d_1\sigma(b_2 \otimes d_2)$, which can be easily proved using the definition of F. So the equality above is equivalent to: $S(z_1)F(z_2t_2, y_2)\sigma(z_3 \otimes t_3)\sigma(y_1 \otimes t_1)= t_2S(z_2)F(z_3, y_2)\sigma(y_1 \otimes t_1)\sigma(z_1 \otimes t_3)$

We multiply by $z_0r(y_o \otimes t_o)$, we apply

$\sum \sigma(x_{(1)} \otimes y_{(1)})F(y_{(2)}, x_{(2)}) = \sum \sigma(x_{(2)} \otimes y_{2)})x_{(1)}y_{(1)}$,

we get:$F(F(t, z), y) = F(t, z_1)S(z_2)F(z_3, y)$

(b)Let V be the inverse of the antipode S of H. $r(x, y) = \sigma(V(x), y)$

Using the left multiplicativity of σ and grouping the terms which contain t's and z's, a sufficient condition to satisfy the R-equation is:(after we decrease the indices from the Sweedler notation by 1)

$\sigma(y_3x_3V(x_1y_1) \otimes z_1)\sigma(x_2 \otimes y_2)z_2y_4x_4 = z_1x_1y_1\sigma(y_4x_4V(x_2y_2) \otimes z_2)\sigma(x_3 \otimes y_3)$

$y_3x_3\sigma(x_2 \otimes y_2) = F(x_2, y_2)\sigma(x_3 \otimes y_3)$

$y_4x_4\sigma(x_3 \otimes y_3) = F(x_3, y_3)\sigma(x_4 \otimes y_4)$

we apply these relations in the first term of the equality above, and we get:

$\sigma(F(x_2,y_2)V(x_1y_1)\otimes z_1)z_2F(x_3y_3) = z_1x_1y_1\sigma(F(x_3,y_3)V(x_2y_2)\otimes z_2)$
$\sigma(F(x_2,y_2)V(x_1y_1)\otimes z_1)z_2F(x_3y_3) = \sigma(V(x_1y_1)\otimes z_1)\sigma(F(x_2,y_2)\otimes z_2)z_3F(x_3y_3)$
$= \sigma(F(x_3,y_3)\otimes z_3)F(F(x_2y_2),z_2)\sigma(V(x_1y_1)\otimes z_1)$
$z_1x_1y_1\sigma(F(x_3,y_3)V(x_2y_2)\otimes z_2)$=$z_1x_1y_1\sigma(V(x_2y_2)\otimes z_2)\sigma(F(x_3,y_3)\otimes z_3)$
The R-equation become: $F(F(x_2y_2),z_2)\sigma(V(x_1y_1)\otimes z_1)$ =$z_1x_1y_1\sigma(V(x_2y_2)\otimes z_2)$.
After multiplication with $\sigma(x_oy_o\otimes z_o)$, the equality become F(F(x,y),z)=F(xy, z).
□

3.1. Entwining modules. Let H be a Hopf algebra with invertible antipode S. Let M be a left H-module and a right H-comodule with the following compatibility relation between module and comodule structures, given by a map f: $H\otimes H\to H\otimes H$.

$(a\triangleright m)_0\otimes(a\triangleright m)_1 = a_{(f)}\triangleright m_0\otimes m_{1(f)}$

where $f(x\otimes y)$ is denoted by $\sum x_{(f)}\otimes y_{(f)}$.

The associativity and the coassociativity of the operations of H, as well as the definitions of the action and coaction, impose several relations for f which form in fact the definition of an entwining map. Applying the compatibility relation above for ab, a, $b\triangleright m$ and m, we obtain the following relations for f:

$f(ab\otimes x) = a_{(g)}b_{(f)}\otimes(x_f)_g$, where f=g

$a_{(f)}\otimes\Delta x_{(f)} = f(a_f\otimes x_1)\otimes(x_2)_{(f)}$

f(1,x)=x and $(id\otimes\epsilon)(f(h\otimes 1) = h$

Theorem 3.1. *If a right-right entwining associated with a Hopf algebra H satisfies the following relation for any a, b and c from H:*

$(a_2)_{(f)}\otimes f(S(a_1)S((b_1)_{(f)})a_3b_2\otimes c) = (a_2)_{(f)}\otimes d_1\otimes d_2c$, *where*

$d = S(a_1)S((b_1)_{(f)})a_3b_2$, *then the canonical map* $R(m\otimes n) = m_1\triangleright n\otimes m_0$.

R: $M\otimes M\to M\otimes M$ *is a solution for the R-equation, for any entwinned module M associated with (H,H,f).*

The condition from the hypothesis implies that f satisfies the Hopf module compatibility $f(d\otimes x) = d_1\otimes d_2x$, when the element from the first position,

d= $\epsilon((a_2)_{(f)})S(a_1)S((b_1)_{(f)})a_3b_2$.

An easy computation shows that $f(h\otimes x) = h_2\otimes h_3xS^{-1}(p(h_1))$ is an entwining map for a Hopf algebra H with bijective antipode S, where p is a Hopf algebra projection. An f-entwined module is a (p,id)-Yetter Drinfeld module as in section 2.3. Also, f satisfies the relation prescribed by the theorem above. This is a third reason the map R associated with a braided Hopf algebra in the category of Yetter-Drinfeld module satisfies the R-equation. The first two proofs were given by the mixed system of Yang-Baxter type equations from sect. 2.3, and by the general theorem 4.2 [2].

Proof. The inverse of the canonical application is $S(a\otimes b) = b_0\otimes S(b_1)\triangleright a$

For convenience we will supress the symbol for the tensor product from sums of monomials, and the symbol for the action $\triangleright$, writing hm , where h is in H and m is a vector from M. We write the R-equation in the following way, and we evaluate both terms on $a\otimes b\otimes c\otimes d$:

$R_{34}R_{23}R_{12}R_{34}R_{23}R_{34}S_{23}S_{34}S_{12}S_{23}$=$R_{23}R_{12}R_{23}S_{12}S_{23}R_{34}$

We succesively apply the operators in the left to the right order (group-like notation). The fundamental monomials from the sequences below are separated by arrows-the places where we apply the operators in the order given by the equation above. For the left hand side:

$$(a,b,c,d) \to (a,b,c_2d,c_1) \to (a,b_2c_2d,b_1,c_1) \to (a_2b_2c_2d,a_1,b_1,c_1) \to$$
$$(a_2b_3c_2d,a_1,b_2c_1,b_1)$$
$$\to (a_3b_3c_2d,a_2b_2c_1,a_1,b_1) \to (a_4b_3c_2d,a_3b_2c_1,a_2b_1,a_1) \to$$
$$(a_4b_4c_2d,(a_2)_{(f)}b_1,S((b_2)_{(f)})a_3b_3c_1,a_1)$$
$$\to (a_5b_4c_2d,(a_3)_{(f)}b_1,a_1,S(a_2)S((b_2)_{(f)})a_4b_3c_1) \to$$
$$((a_3)_{(f)(f)}b_1,S((b_2)_{(f)})a_5b_5c_2d,a_1,S(a_2)S((b_3)_{(f)})a_4b_4c_1)$$
$$\to ((a_4)_{(f)(f)}b_1,a_1,S(a_2)S((b_2)_{(f)})a_5b_5c_2d,S(a_3)S((b_3)_{(f)})a_4b_4c_1)$$

For the right hand side:
$(a,b,c,d) \to (a,b_2c,b_1,d) \to (a_2b_2c,a_1,b_1,d) \to (a_3b_2c,a_2b_1,a_1,d) \to$
$((a_2)_{(f)}b_1,S((b_2)_f)a_3b_3c,a_1,d) \to ((a_3)_{(f)}b_1,a_1,S(a_2)S((b_2)_f)a_4b_3c,d) \to$

$$((a_3)_{(f)}b_1,a_1,R(S(a_2)S((b_2)_f)a_4b_3c,d)) = ((a_3)_{(f)}b_1,a_1,(c_2)_{f'}d,[S(a_2)S((b_2)_f)a_4b_3]_{f'}c_1)$$

The equality between the left and the right sides is equivalent to an equality of elements from $H \otimes H \otimes H$, which is exactly the condition stated in the theorem. ∎

Examples of entwined maps which satisfy the condition above are not trivial to be found. Among (a,b)- Yetter-Drinfeld modules, only (p=projection,id) Yetter Drinfeld modules satisfy it. The theory of Hopf algebras with a *weak* projection will provide an example of a special solution to the R-equation .

3.2. Hopf algebras with a weak projection. Let H be a Hopf algebra with invertible antipode S. p:$H \to H$ is called a weak projection if p is a coalgebra map, if p(p(x))=p(x) , if p(p(x)y)=p(x)p(y) and if $p \circ S = S \circ p$. Im(p)=H'is a sub-Hopf algebra of H

Notes:

A left or right multiplicative bilinear form on $H \otimes H$ can define an action on any H-comodule. In the cases above, further conditions have to be imposed in such a way the action and the coaction were given by an entwining structure which satisfy the conditions from the previous theorem.

A weak projection p does not generate an entwinning structure like (p,id) Yetter Drinfeld modules, unless it is a Hopf algebra projection.

We generalize the action of H on B from the previous section.

Let B be the set of elements x from H, such that $x_1 \otimes p(x_2) = x \otimes 1$.

There is an action of H on B given by $x \triangleright b = x_1 b_1 p(S(x_2 y_2))$.

In general, this action and the coaction given by comultiplication are not connected by an entwinning structure, as for p an algebra map. B has a coalgebra structure.

Theorem 3.2. *Consider $R(a \otimes b) = a_1 \triangleright b \otimes a_2$ and its inverse $T(x \otimes y) = y_2 \otimes S^{-1}(y_1) \triangleright x$. R satisfies the R-equation on multiple tensor products of vector spaces B's if, for any b,c from B, and any x from H, the element $p[S(c)p(S(x_1b)x_2]$ commutes with any element from B.*

We will review the structural results of Schauenburg and Stefan about the theory of Hopf algebras with a weak projection in the last section. Various conditions on ingredients could imply the condition above.

Proof. The R-equation is written as:
$R_{34}R_{23}R_{12}R_{34}R_{23}R_{34}S_{23}S_{34}S_{12}S_{23} = R_{23}R_{12}R_{23}S_{12}S_{23}R_{34}$
and we evaluate both terms on $a \otimes b \otimes c \otimes d$:

The sequence of arrows below shows the application of the operators R's in the order prescribed by the equation above.

For the left hand side:

$$(a,b,c,d) \to (a,b,c_1d_1p(S(c_2d_2),c_3) \to (a,b_1c_1d_1p(S(b_2c_2d_2)),b_3,c_3)$$
$$\to (a_1b_1c_1d_1p(S(a_2b_2c_2d_2)),a_3,b_3,c_3) \to$$
$$(a_1b_1c_1d_1p(S(a_2b_2c_2d_2)),a_3,b_3c_3p(S(b_4,c_4)),b_5) \to$$
$$(a_1b_1c_1d_1p(S(a_2b_2c_2d_2)),a_3b_3c_3p(S(a_4b_4c_4)),a_5,b_5)$$
$$\to (a_1b_1c_1d_1p(S(a_2b_2c_2d_2)),a_3b_3c_3p(S(a_4b_4c_4)),a_5b_5p(S(a_6b_6)),a_7) \to$$
$$(a_1b_1c_1d_1p(S(a_2b_2c_2d_2)),a_3b_3p(S(a_4b_4)),p(a_5b_5)c_3p(S(a_6b_6)),a_7) \to$$
$$(a_1b_1p(S(a_2b_2)),p(a_3b_3)c_1d_1p(S(c_2d_2)p(S(a_4b_4))),p(a_5b_5)c_3p(S(a_6b_6)),a_7) \to$$

$$(a_1b_1p(S(a_2b_2)),p(a_3b_3)c_1d_1p(S(c_2d_2)p(S(a_4b_4))),a_9,S^{-1}(a_8)p(a_5b_5)c_3p(S(c_4)p(S(a_6b_6))a_7)) \to$$

$$(a_1b_1p(S(a_2b_2)),a_{11},S^{-1}(a_{10})p(a_3b_3)c_1d_1p(S[p(a_4b_4)c_2d_2]a_9),S^{-1}(a_8)p(a_5b_5)c_3p(S(c_4)p(S(a_6b_6))a_7))$$

The right hand side is equal to:

$$(a,b,c,d) \to (a,b_1c_1p(S(b_2c_2),b_3,d) \to (a_1b_1c_1p(S(a_2b_2c_2)),a_3,b_3,d) \to$$
$$(a_1b_1c_1p(S(a_2b_2c_2)),a_3b_3p(S(a_4b_4)),a_5,d)$$
$$\to (a_1b_1p(S(a_2b_2),p(a_3b_3)c_1p[S(c_2)p(S(a_4b_4))],a_5,d) \to$$
$$(a_1b_1p(S(a_2b_2),a_7,S^{-1}(a_6)p(a_3b_3)c_1p(S[p(a_4b_4)c_2]a_5),d) \to$$
$$(a_1b_1p(S(a_2b_2)),a_{15},x,S^{-1}(a_12)p(a_5b_5)c_3p(S(c_4)p(S(a_6b_6))a_{11})), \text{ where x=}$$

$$S^{-1}(a_{14})p(a_3b_3)c_1p(S[p(a_8b_8)c_6]a_9)d_1p[S(d_2)p(S(a_{10})S^2[p(a_7b_7)c_5])S(c_2)p(S(a_4b_4))a_{13}]$$

If T=$p(S[p(a_7b_7)c_5]a_8)$, then $T_1 \otimes p(WS(T_2)Q)$ appears in the expression above, and does not appear in the left hand side expression.

$T_1 = p(S[p(a_8b_8)c_6]a_9)$ and $S(T_2) = p(S(a_{10})S^2[p(a_7b_7)c_5])$

The left and the right hand side are equal if any element from B commutes with elements $T = p(S[p(x_1b)c]x_2)$.

In our case, the element from B is d, $x = a_7, b := b_7, c := c_5$.

Td=dT implies $T_1d_1p(S(T_2d_2)Z) = d_1T_1p(S(d_2T_2)Z) = d_1T_1p(S(T_2)S(d_2)Z)$
$= d_1T_1p(S(T_2))p(S(d_2)Z) = d_1p(S(d_2)Z)$. We used that T is in Im(p) and p is a weak projection. ∎

4. Factorisation structures.

Let R: $M \otimes M \to M \otimes M$ be a solution of the R-equation.

According to the Theorem 3.1 [2], R provides a representation for the group R(N), for any N, which means:

There are operators R(x,y): $M^{\otimes N} \to M^{\otimes N}$, which act as identity on the tensor products of M's outside the range x,x+1...y-1,y; In this range, there are well defined using the formula R(x,t)R(x,y)=R(t,y)R(x,t)R(x+1,t+1), according to the Theorem 3.1 [2].

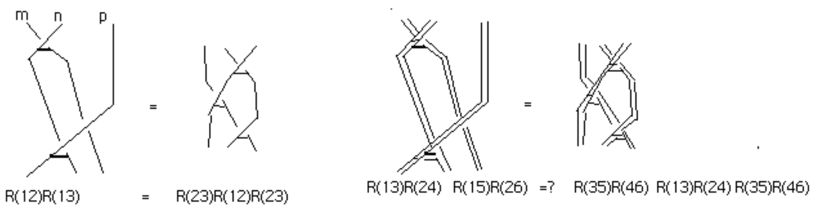
There is also a cabling procedure (Theorem 2.1, [2]), which allow us to define the operators T(m,n):$M^{\otimes m+n} \to M^{\otimes m+n}$, T(m,n)=$R_{1,m+1}R_{2,m+2}...R_{n,m+n}$

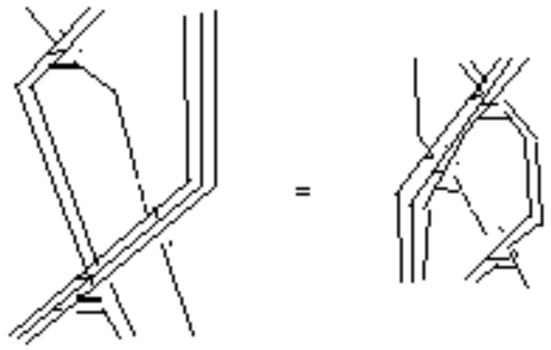
We consider the tensor algebra T(M). For $x \in T^m(M)$ and $y \in T^n(M)$, we consider the following binary operation: $x \star y = T(m,n)(x \otimes y)$

Lemma 4.1. $\star$ *define an associative product on the tensor algebra T(M):*

Proof. Let $x \in T^m(M), y \in T^n(M), z \in T^p(M)$.
$(x \star y) \star z = x \star (y \star z)$ if and only if
$T(m+n,p)(T(m,n)(x \otimes y) \otimes z) = T(m, n+p)(x \otimes T(n,p)(y \otimes z))$
This is in fact the (m,n,p) cabling of the fundamental relation
$R(1,2)R(1,3) = R(2,3)R(1,2)R(2,3)$ (Theorem 2.1, Lemma 2.2 from [2]).

□





Remark 4.1. *$\star = m \circ T'$, where m is the multiplication of T(M) given by the tensor product, and T': $T(M) \otimes T(M) \to T(M) \otimes T(M)$ is obtained from*
T:$T(M) \otimes T(M) \to T(M)$ by a spliting isomorphism p(m,n):$T^{m+n} \to T^n \otimes T^m$. composed with T(m,n).
T'satisfy, like R, the R-equation, so the construction above can be iterated.

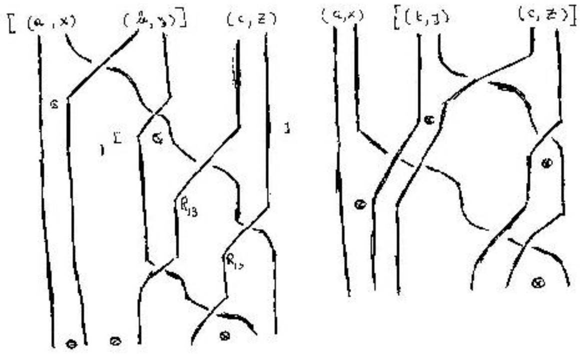
Definition 4.1. *Let A and B be two associative algebras. R: $B \otimes A \to A \otimes B$ is called a factorisation map if the following product $\circ$ defined on $A \otimes B$ is associative:*
$(a,b) \circ (c,p) = ac_R \otimes b_R p$.
We used the following notation $R(x \otimes y) = y_R \otimes x_R$. The multiplication of A, and respectively B were used on the first and the last positions.

If a map R satisfies certain axioms (the naturality with respect to multiplications from A and B) diagramatically presented in the last figure, then R is a factorisation map. A factorisation map replaces the regular flip.

T' is not a factorisation map for $(T(M), \otimes) \otimes$ (T(M),⊗). More precisely, if T'is such a map, then T' is a solution for the Braid equation, which forces
R: $M \otimes M \to M \otimes M$ to be also solution for the Braid equation. Instead:

Lemma 4.2. *T' define a factorisation structure for X=$(T(V), \otimes) \otimes (T(V), \star)$*

Proof.



Let (a,x), (b,y) and (c,z) elements from the algebra X.

The two figures above show the computation of the products [(a,x)(b,y)](c,z) and (a,x)[(b,y)(c,z)]. For all 6 elements from T(M) having degree 1 ($\in M$), we have to check the equality: R(2,3)R(3,4)R(3,5)R(4,6)=R(2,3)R(4,5)R(3,4)R(5,6)R(4,5)R(5,6) in the group R(n).

Recall the groups R(n) is given by generators and relations: generators: $R_{x,y}$, where 1≤x<y≤n and relations: $R_{x,y}R_{x,z} = R_{y,z}R_{x,y}R_{x+1,y+1}$ if x<y<z

$R_{x,y}R_{z,t} = R_{z,t}R_{x,y}$ if x<y<z<t and $R_{a,y}R_{x,z} = R_{x,z}R_{a+1,y+1}$, where x<a<y<z

There are exactly the relations satisfied by insertion permutations. These groups form the algebraic structure of the operad under a cabling operation. (Theorem 2.1 [2]). They admit the following presentation (Theorem 3.1 [2]), which show that a map R solution of the R-equation gives a representation of R(n) on M^n:

R(n) is generated by R(i,i+1), where 1 ≤ i ≤ n-1; R(i,i+1) satisfy the relations:

1)$[R(i, i+1), R(j, j+1)]$=0 if $|i-j|$ is not equal to 1.

2)For any i, R(i,i+1), R(i+1,i+2) and R(i+2,i+3) satisfy the relation prescribed by the R-equation : $R_{23}R_{12}R_{23}S_{12}S_{23}R_{34}R_{23}R_{12}R_{34}R_{23}S_{34}S_{23}S_{34}$=$R_{34}R_{23}R_{12}$

(indices are shifted by i).

R(2,3)R(3,4)R(3,5)R(4,6)=R(2,3)R(4,5)R(3,4)R(5,6)R(4,5)R(5,6) iff

R(3,4)R(3,5)R(4,6)=R(4,5)R(3,4)R(5,6)R(4,5)R(5,6) iff

R(4,5)R(3,4)R(4,5)R(4,6)=R(4,5)R(3,4)R(5,6)R(4,5)R(5,6) iff

R(4,5)R(4,6)=R(5,6)R(4,5)R(5,6) which is a defining relation for the group R(n). If all six elements above a,x...c,z have different degrees in T(M), the cabling of the 6 strings with the degrees of a,b,c,x,y,z, from the previous equality between two elements of R(n) generate two equal elements in R(n) (Thm 2.1 and Lemma 2.2 [2]), so the equality between the two figures representing the products is still valid, so X is an associative algebra.

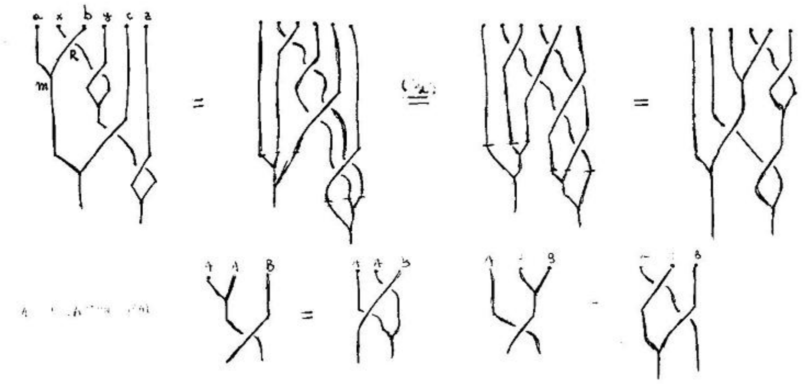
∎

Definition 4.2. *Let A be an associative algebra with multiplication m.*

R: $A \otimes A \to A \otimes A$ is called a twisted factorisation structure if R is a factorisation map for $(A, m) \otimes (A, m \circ R)$

Remark 4.2. *1. $m \circ R$ is not necessarly an associative product. The definition above says that the product built on $A \otimes A$, as in the definition of a factorisation map, is associative.*

2. If R is a factorisation map for $(A, m) \otimes (A, m)$ and R satisfies the following equation $R_{12}R_{23}R_{34}R_{12}R_{23}=R_{23}R_{12}R_{34}R_{23}R_{34}$, then R is a twisted factorisation structure. In particular, R is not necessarly a twistor or an R-matrix (Theorems 3.1 and 3.2 [14]*).*



The axioms for a factorisation map (naturality with respect to the products from A and B); the diagramatic proof of remark 2, using the naturality of R with respect to the multiplication and the new stated equation.

3.In our case T' satisfies the R-equation. T'satisfies the equation above if and only if T' is a Braid operator.

5. Co-quasi Hopf algebras and crossed products.Braided Hopf algebras with a weak projection.

5.1. Co-quasi Hopf algebras and crossed products. We would like to discuss the context of Lemma 2 b). F is a twist of the multiplication of A=H^{op} by a convolution invertible bilinear σ, with inverse r.

Define $\omega(a \otimes b \otimes c) = \sigma(a_1, b_1)\sigma(a_2b_2, c_1)r(a_3, b_3c_2)r(b_4, c_3)$

F is an associative product if $\omega(a \otimes b \otimes c)$ is cocentral which by definition means:

$\omega(a_1 \otimes b_1 \otimes c_1)a_2 \otimes b_2 \otimes c_2 = \omega(a_2 \otimes b_2 \otimes c_2)a_1 \otimes b_1 \otimes c_1$

Let A the quasi-bialgebra above. Let R be an associative algebra. We consider the following algebraic data:

1) a weak action of H on R: . : $H \otimes R \to R$

2) a linear map $\sigma : H \otimes H \to R$

and the following product on $R \otimes H$: (r,h) $\star$ (s,g)=$(r(h_1.s)\sigma(h_2, g_1), h_3g_2)$

The product is called ω-associative if

$[(r, h)(s, g)](t, k) = (r, h_1)[(s, g_1)(t, k_1)]\omega(h_2 \otimes g_2 \otimes k_2)$

Theorem 5.1. *(Theorem 9* [1]*) If the following conditions are satisfied, the product above is ω-associative:*
$1_H.r = r$
$\sigma(h,1) = \sigma(1,h) = \epsilon(h)1_R$
$[h_1.(g_1.r)]\sigma(h_2,g_2) = \sigma(h_1,g_1)[(h_2g_2).r]$
$[h_1.\sigma(a_1,b_1)]\sigma(h_2,a_2b_2) = \sigma(h_1,a_1)\sigma(h_2a_2,b_1)\omega^{-1}(h_3,a_3,b_2)$

5.2. Hopf algebras with a weak projection. We present after Schauenburg [17] and Stefan [18]the following structures of the Hopf algebras with a weak projection (A,p), as defined in subsection 3.2

Let B=Im(p) and i is its inclusion in A; Let R be the set of elements x $\in$ A such that $x_1 \otimes p(x_2) = x \otimes 1$. Let q(x)=$x_1p(S(x_2)) \in$ R

1) a product $\star$ on R, not necessarily associative, defined as x $\star$ y =q(xy)
2) a comultiplication on R, $\Delta_R(x) = q(x_1) \otimes q(x_2) = x_{[1]} \otimes x_{[2]}$
3) a cocycle t: $R \otimes R \to B$, t(x,y)=p(xy)
4) an action $B \otimes R \to R$, denoted b $\to$ r = q(br)
5) a map $B \otimes R \to B$, b $\leftarrow$ r = p(br)
6) a left coaction $R \to B \otimes R$ $\rho(x) = p(x_1) \otimes x_2 = x_{(-1)} \otimes x_{(o)}$

Theorem 2.12 ([18]) Theorem 5.1([17]): On vector space $R \otimes B$ there are the following multiplication and comultiplication maps, such that the application $R \otimes B \to A$ given by (r,b)$\to$ rb is a bialgebra isomorphism:

$(q \otimes y)(p \otimes x) = (q_{[1]} \star (q_{[2](-1)}y_1 \to p_{[1]}) \otimes t(q_{[2](o)}, y_2 \to p_{[2]})(y_3 \leftarrow p_{[3]})x)$

The comultiplication on $R \otimes B$ is given by $\Delta(r,h) = \Delta(r,1)\Delta(1,h)$.

The comultiplication of h is give by the Hopf algebra structure of Im(p), and
$\Delta(r,1) = (r_{[1]}, r_{[2](-1)} \otimes (r_{[2](o)}, 1)$.

There is also a converse of the theorem above, which says that a Hopf algebra B, a coalgebra R and the six maps above generate on the vector space A=$R \otimes B$ a structure of a Hopf algebra with a weak projection onto B if a long list of relations among them is fullfiled.

Let us go back to theorem 7 sect.3.2.
(r,1)(1,h)=(r,h)
(1,h)(r,1)=$(h_1 \to r_1, h_2 \leftarrow r_2)$

For any p $\in$R and h=$p[S(c)p(S(x_1b)x_2]$, we want ph=hp. We apply ϵ on the left and the right positions of the both expressions above, and we get:

$h \leftarrow r = \epsilon(r)h$ for any h of the special form, and any r in R.
$h \to r = \epsilon(h)r$ for any h of the special form, and any r in R.

p satisfies p(p(x)y)=p(xp(y))=p(x)p(y) if and only if $h \leftarrow r = \epsilon(r)h$ for any h in B, and any r in R, so in this case the first condition is automatically satisfied. (section 6.1 [17]). In this special case, (1,h)(r,1)=$(h_1 \to r, h_2)$ and h_1 is also on the form described by $p[S(c)p(S(x_1b)x_2]$. If the action $h \to r$ is also trivial for any h, (and in particular it is trivial for any h= $p[S(c)p(S(x_1b)x_2])$ then any element from R commutes with any element from B. The Hopf algebra H is a crossed product by a cocycle, as described in section 5.0.1, when the weak action is trivial.

5.3. Entwinning and factorisation structures. Let A be a unital algebra and C a coalgebra. f:$A \otimes C \to A \otimes C$ is denoted $f(a \otimes c) = a_{(f)} \otimes c_{(f)}$

(A,C,f) is an entwining structure if the following conditions hold:
$f(ab \otimes c) = a_{(f')}b_{(f)} \otimes c_{(ff')}$
$f(1_A \otimes c) = 1_A \otimes c$
$(id \otimes \Delta)f(a \otimes c) = a_{(f)f'} \otimes (c_1)_{f'} \otimes (c_2)_f$

$(id \otimes \epsilon)f(a \otimes c) = \epsilon(c)a$

M is an f-module if it is a left A-module and a right C-comodule, such that $\rho(am) = a_f m_1 \otimes (m_2)_f$.

We recall from [12] (Thm.8)the connection between the notions of entwining and factorisations.

Theorem: Let (A,C,f) be an entwining structure, where C is finite dimensional. Let B be the algebra C^*. e_j and e^j dual bases in C and B

Let R :$B \otimes A \to A \otimes B$ be the map $R(k \otimes a) = \sum k((e_j)_{(f)})a_{(f)} \otimes e^j$. Then R is a factorisation map in the sense of Defn. 4.1.Conversely, f can be recovered from R associated with A and C^* Tensor algebra and universal differential calculus associated with an entwining structure, and iterated tensor product of algebras were studied in [9], [16].

Lemma 10.3 [5] states that for a Yetter -Drinfeld datum (H,A,C) and C finite dimensional, we have a functor G from the category of left modules over $C^* \otimes_{smash} A$ and (H, A, C)-Yetter -Drinfeld modules , given by: G(M)=M

u.m $= (\epsilon \otimes_{smash} u) \triangleright m$

$\rho : M \to M \otimes C, m \to \sum (E_j^* \otimes 1) \triangleright m \otimes E_j,$

where E_i is a basis for C, and E_j^* is the corresponding dual basis for C.

So, the coaction is given by a canonical element.

$C^* \otimes_{smash} A$ is a generalized Drinfeld double of a Yetter -Drinfeld datum (H, A, C), defined in [8]. It is an associative algebra built using a factorisation map associated with the entwining structure, as described by the theorem above. Among particular cases of this construction there are Drinfeld double and Heisenberg double of a Hopf algebra. Compatibility of the action and the coaction for a comodule in the presence of a bilinear, in the case of Lemma 2, as well as the factorisation structures induced by the entwining maps of Thm. 6 and their canonical elements as above will be studied in a further project.

References

[1] A. Balan, Crossed products and cleft extensions for coquasi-Hopf algebras, *The Arabian Journal for Science and Engineering*, Volume **33**, Number 2C

[2] G. Barad, On some groups related to the Braid Groups of type A, Volume **37**, No. 2 (June 2010) of *"Annals of the University of Craiova, Mathematics and Computer Science Series"*

[3] J. Bichon and R. Street, Militaru's D-equation in monoidal categories, *Applied Categorical Structures.* Volume **11**, Number 4, August 2003 , pp. 337-357(21)

[4] Beattie, Bulacu, Braided Hopf Algebras Obtained from Coquasitriangular Hopf Algebras, *Communications in Mathematical Physics*, Volume **282**, Number 1, August 2008 , pp. 115-160(46)

[5] D. Bulacu, Fl.Panaite, F.Van Oystaeyen, Generalized diagonal crossed products and smash products for quasi-Hopf algebras. Applications. *Comm.Math.Phys.*, **266** (2006), nr.2, 355-399

[6] S. Caenepeel, G Militaru and Z Shenglin, Crossed modules and Doi-Hopf modules, *Israel Journal of Mathematics*, **100**(1997), 221-247.

[7] S. Caenepeel, G. Militaru, S. Zhu. *Frobenius and separable functors for generalized modules categories and nonlinear equations.* Springer, 2002.

[8] S. Caenepeel, G Militaru, Z Shenglin, Crossed modules and Doi-Hopf modules, *Israel Journal of Mathematics*, **100**, Number 1, 1997.

[9] PM Hajac, E. Wagner, The Pullbacks of Principal Coactions, arXiv:1001.0075v1 [math.KT 2009]

[10] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics 155, Springer-Verlag, 1995.

[11] Majid, *Foundations of Quantum Group Theory*, Cambridge University Press 2000.

[12] G. Militaru, The Hopf modules category and the Hopf equation. *Communications in Algebra*, **26**, no.10, pp. 3071–3097, 1998.

[13] Militaru, G. New types of bialgebras arising from the Hopf equation, *Communications in Algebra*, **10**(1998), 3099-3117

[14] Javier L. Pena, Florin Panaite, Freddy Van Oystaeyen, General twisting of algebras, *Adv. Math.* **212** (1), 315-337 (2007).

[15] F.Panaite, M. Staic - Generalized (anti) Yetter-Drinfeld modules as components of a braided T-category, *Israel J. Math.* **158**, 349-366 (2007).

[16] F.Panaite, On iterated twisted tensor products of algebras, *Internat. J. Math.* **19** (9), 1053-1101 (2008) (with P. Jara, J. Lopez and F. Van Oystaeyen).

[17] P Schauenburg, The Structure of Hopf Algebras with a Weak Projection, *Algebras and Representation Theory* **3** (1999), 187-211.

[18] D.Stefan ,Weak Projections onto a Braided Hopf Algebra, (with A.Ardizzoni, C.Menini), *J. Algebra* **318** (2007), 180-201.

(Gefry Barad) Algebra and Number Theory Research Center, University of Bucharest, Department of Mathematics, Str. Academiei nr.14, sector1, Bucuresti, Romania
E-mail address: gbarad@gmail.com