Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 37(4), 2010, Pages 58–64
ISSN: 1223-6934

State-morphisms on Hilbert algebras

Cătălin Buşneag

Abstract. In this paper (which is a continuation of [2]) we develop a theory of state-morphisms on Hilbert algebras.

2010 Mathematics Subject Classification. Primary 03G25; Secondary 28A60.
Key words and phrases. Hilbert algebra, deductive system, Bosbach states, Riečan states, state-morphism.

1. Introduction

The notion of *state* is an analogue of a probability measure and has a very important role in the theory of quantum structures (see [10]). The state on MV-algebras was first introduced by Kôpka and Chovanek in [13]; the state on $BL-$algebras was introduced by Riečan in [14]. In the case of non-commutative fuzzy structures, these states were introduced by Dvurecenskij for pseudo MV-algebras in [9], by Georgescu for pseudo BL-algebras in [12], by Dvurecenskij and Rachünek for bounded non-commutative Rl-monoids in [11], and by Ciungu for pseudo BCK-algebras in [7].

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "true". The concept of Hilbert algebras was introduced by Henkin and Skolem (under the name *implicative models*) for investigations in intuitionistic logics and other non-classical logics. Diego in [8] proved that Hilbert algebras form a variety which is locally finite.

This paper is organized as follows:

In Section 2 we recall the basic definitions and put in evidence many rules of calculus in Hilbert algebras which we need in the rest of paper (especially $c_{10} - c_{28}$). Also we recall some results relative to maximal deductive systems for the case of bounded and unbounded Hilbert algebras (Theorem 2.1 and Corollary 2.1). In Section 3 we recall some results relative to the theory of Bosbach states on Hilbert algebras developed in [2]. In Section 4 we develop a theory of state-morphisms on Hilbert algebras.

2. Preliminaries

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and respectively logical equivalence.

Received August 19, 2010. Revision received October 29, 2010.

Definition 2.1. *(*[2]-[6], [8]*)*

A Hilbert algebra is an algebra $(A, \to, 1)$ *of type (2,0) such that the following axioms are fulfilled for every* $x, y, z \in A$*:*

(a_1) $x \to (y \to x) = 1$*;*

(a_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;

(a_3) *If* $x \to y = y \to x = 1$, *then* $x = y$.

In [8] it is proved that the system of axioms $\{a_1, a_2, a_3\}$ is equivalent with the system $\{a_4, a_5, a_6, a_7\}$, where:

(a_4)**:** $x \to x = 1$;

(a_5)**:** $1 \to x = 1$;

(a_6)**:** $x \to (y \to z) = (x \to y) \to (x \to z)$;

(a_7)**:** $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

For examples of Hilbert algebras see [3]-[6] and [8]. If A is a Hilbert algebra, then the relation $\leq$ defined by $x \leq y$ iff $x \to y = 1$ is a partial order on A (which will be called the *natural ordering*); with respect to this ordering 1 is the largest element of A. A *bounded* Hilbert algebra is a Hilbert algebra with a smallest element 0; in this case for $x \in A$ we denote $x^* = x \to 0$.

From [2]-[6], [8] in a Hilbert algebra A we have the following rules of calculus for $x, y, z \in A$:

(c_1) $x \to 1 = 1$;

(c_2) $x \leq y \to x$;

(c_3) $x \leq (x \to y) \to y$;

(c_4) $((x \to y) \to y) \to y = x \to y$;

(c_5) $x \to y \leq (y \to z) \to (x \to z)$;

(c_6) If $x \leq y$, then $z \to x \leq z \to y$ and $y \to z \leq x \to z$;

(c_7) $x \to (y \to z) = y \to (x \to z)$;

For $x_1, ..., x_n \in A$ $(n \geq 1)$ we will define $(x_1, ..., x_{n-1}; x_n) = x_n$ if $n = 1$ and $x_1 \to (x_2, ..., x_{n-1}; x_n)$ if $n > 1$.

Then we have:

(c_8) If σ is a permutation of $\{1, 2, ..., n-1\}$ $(n \geq 2)$, then

$$(x_{\sigma(1)}, ..., x_{\sigma(n-1)}; x_n) = (x_1, ..., x_{n-1}; x_n);$$

(c_9) $x \to (x_1, ..., x_{n-1}; x_n) = (x, x_1, ..., x_{n-1}; x_n) = (x_1, x, x_2, ..., x_{n-1}; x_n) = ... = (x_1, ..., x_{n-1}, x; x_n)$.

For $x, y \in A$ we define $x \sqcup y = (x \to y) \to ((y \to x) \to x)$. Then we have the following rules of calculus for $x, y, z \in A$:

(c_{10}) $x, y \leq x \sqcup y$ and $x \sqcup y = y \sqcup x$;

(c_{11}) $x \sqcup x = x, x \sqcup 1 = 1$;

(c_{12}) $x \sqcup (x \to y) = 1$;

(c_{13}) $(x \to y) \sqcup (y \to x) = 1$;

(c_{14}) $x \to (y \to z) = (x \to z) \sqcup (y \to z)$;

(c_{15}) $x \to (y \sqcup z) = (x \to y) \sqcup (x \to z)$;

(c_{16}) $(x \to y) \sqcup z = x \to (y \sqcup z)$.

If A is a bounded Hilbert algebra and $x, y \in A$, then we denote $x \curlyvee y = x^* \to y$ and $x \curlywedge y = (x \to y^*)^*$. We have the following rules of calculus for $x, y, z \in A$ (see [2]):

(c_{17}) $0^* = 1, 1^* = 0$;

(c_{18}) $x \to y^* = y \to x^*$;

(c_{19}) $x \to x^* = x^*, x^* \to x = x^{**}, (y \to x)^* \leq x \to y$;

(c_{20}) If $x \leq y$, then $y^* \leq x^*$;
(c_{21}) $x, y \leq x \curlyvee y, x \curlyvee x = x^{**}, x \curlyvee 0 = x^{**}, x \curlyvee 1 = 1, x \curlyvee x^* = 1, x \curlyvee (y \to z) = (x \curlyvee y) \to (x \curlyvee z)$;
(c_{22}) $x \curlyvee (y \curlyvee z) = (x \curlyvee y) \curlyvee z = y \curlyvee (x \curlyvee z)$;
(c_{23}) $x^* \sqcup y^* = x \to y^*$;
(c_{24}) $(x \to y)^{**} = x^{**} \to y^{**} = x \to y^{**}$.

Proposition 2.1. *([2]) Let A be a bounded Hilbert algebra and $x, y, z \in A$. Then*
(c_{25}) $x \curlywedge 0 = 0, x \curlywedge 1 = x^{**}, x \curlywedge x = x^{**}$;
(c_{26}) $x \curlywedge y = y \curlywedge x \leq x^{**}, y^{**}$;
(c_{27}) $x \leq y \Rightarrow x \curlywedge y = x^{**}$;
(c_{28}) $x \curlywedge (y \curlywedge z) = (x \curlywedge y) \curlywedge z$.

Definition 2.2. *If A is a Hilbert algebra, a subset D of A is a* deductive system *of A if the following axioms are satisfied:*
(a_8) $1 \in D$;
(a_9) *If $x, x \to y \in D$, then $y \in D$.*

We denote by $Ds(A)$ the set of all deductive systems of A.

We say that $M \in Ds(A), M \neq A$, is *maximal* if it is a maximal element in the lattice $(Ds(A), \subseteq)$. Let us denote by $Max(A)$ the set of all maximal deductive systems of A. We have the following theorem of characterization for maximal deductive systems:

Theorem 2.1. *([15]) Let A be a Hilbert algebra and $M \in Ds(A), M \neq A$. The following conditions are equivalent:*
(i) $M \in Max(A)$;
(ii) *If $x, y \in A$ and $x \sqcup y \in M$, then $x \in M$ or $y \in M$;*
(iii) *If $x \notin M$, then $x \to y \in M$ for every $y \in A$.*

Corollary 2.1. *If A is a bounded Hilbert algebra and $M \in Ds(A), M \neq A$, then the following conditions are equivalent:*
(i) $M \in Max(A)$;
(ii) *If $x \notin M$, then $x^* \in M$.*

3. Bosbach states on Hilbert algebras

In this section we recall some results relative to the theory of Bosbach states on a Hilbert algebra A. This concept is obtained by using Bosbach condition ([1]).

Definition 3.1. *([2]) A* Bosbach state *on a Hilbert algebra A is a function $s : A \to [0, 1]$ such that the following axioms hold:*
(a_{10}) $s(1) = 1$;
(a_{11}) $s(x) + s(x \to y) = s(y) + s(y \to x)$, *for all $x, y \in A$.*

Remark 3.1. *In [2] I given (following a suggestion of a referee) another definition for a Bosbach state on a Hilbert algebra, namely, a function, $s : A \to [0, 1]$ is a Bosbach state if verify (a_{10}), (a_{11}) and*
(a_{12}) *there exists an element $a \in A$ such that $s(a) = 0$.*

In this paper we consider the notion of Bosbach state in the sense of Definition 3.1.

Example 3.1. *The function $\mathbf{1} : A \to [0, 1], \mathbf{1}(x) = 1$, for every $x \in A$ is a Bosbach state on A.*

Example 3.2. *If $M \in Max(A)$, then $s_M : A \to [0,1]$, defined by $s_M(x) = 1$ if $x \in M$ and 0 if $x \notin M$ is a Bosbach state on A. Indeed, since $1 \in M$, then $s_M(1) = 1$. Consider $x, y \in A$. If $x, y \in M$, then $x \to y, y \to x \in M$ and the axiom (a_{11}) is verified ($1+1 = 1+1$). If $x, y \notin M$, then by Theorem 2.1 we deduce that $x \to y, y \to x \in M$, so the axiom (a_{11}) is also verified ($0+1 = 0+1$). If $x \notin M$ and $y \in M$, then $x \to y \in M$ and $y \to x \notin M$, so the axiom (a_{11}) is also verified ($0+1 = 1+0$).*

Example 3.3. *If $s : A \to [0,1]$ is a Bosbach state, then for every $a \in A$, $s_a : A \to [0,1], s_a(x) = s(a \to x)$ is also a Bosbach state on A. Indeed, $s_a(1) = s(a \to 1) = s(1) = 1$ and for $x, y \in A$, $s_a(x) + s_a(x \to y) = s(a \to x) + s(a \to (x \to y)) = s(a \to x) + s((a \to x) \to (a \to y)) = s(a \to y) + s((a \to y) \to (a \to x)) = s(a \to y) + s(a \to (y \to x)) = s_a(y) + s_a(y \to x)$.*

For a Bosbach state $s : A \to [0,1]$ we define $Ker(s) = \{x \in A : s(x) = 1\}$.

Proposition 3.1. *([2]) $Ker(s) \in Ds(A)$.*

Proposition 3.2. *([2]) If $s : A \to [0,1]$ is a Bosbach state on A, then for all $x, y \in A$ we have:*
(c_{29}) $x \leq y \Rightarrow s(x) \leq s(y)$;
(c_{30}) $s((x \to y) \to y) = s((y \to x) \to x)$.

4. State-morphisms on Hilbert algebras

In this section we develop a theory of state-morphisms on Hilbert algebras. Let us denote by $[\mathbf{0},\mathbf{1}]$ the standard $MV-$algebra of real unit interval $[0,1]$, where for $x, y \in [0,1], x \oplus y = \min\{x+y, 1\}, x \odot y = \max\{x+y-1, 0\}, x \rightsquigarrow y = \min\{1 - x + y, 1\}, x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Clearly, $x \rightsquigarrow x = 1, x \rightsquigarrow 1 = 1, 1 \rightsquigarrow x = x$ and $x \wedge a \leq y$ iff $a \leq x \rightsquigarrow y$ for every $x, y, a \in [0,1]$. Also, $(x \rightsquigarrow y) \rightsquigarrow y = (y \rightsquigarrow x) \rightsquigarrow x = x \vee y$, for every $x, y \in [0,1]$.

Definition 4.1. *A* state-morphism *on a Hilbert algebra A is a function $f : A \to [0,1]$ such that for every $x, y \in A$:*
(a_{13}) $f(x \to y) = f(x) \rightsquigarrow f(y)$.
If A is bounded we add the condition
(a_{14}) $f(0) = 0$.

Clearly, $\mathbf{1} : A \to [0,1], \mathbf{1}(x) = 1$, for every $x \in A$ is a state morphism (called *trivial*). From (a_{13}) we deduce that $f(1) = f(1 \to 1) = f(1) \rightsquigarrow f(1) = 1$. If A is bounded, then for every $x \in A$ we have $f(x^*) = f(x \to 0) = f(x) \rightsquigarrow f(0) = f(x) \rightsquigarrow 0 = (f(x))^* = 1 - f(x)$.

Proposition 4.1. *Let A be a Hilbert algebra and $f : A \to [0,1]$ a state-morphism. Then:*
(i) *f is a Bosbach state*;
(ii) *If A is a bounded Hilbert algebra, then $f(x \curlywedge y) = f(x) \odot f(y)$, for every $x, y \in A$.*

Proof. (i). For every $x, y \in A$ we have $f(x) + f(x \to y) = f(x) + [f(x) \rightsquigarrow f(y)] = f(x) + \min\{1 - f(x) + f(y), 1\} = \min\{1 + f(y), 1 + f(x)\} = f(y) + \min\{1, 1 + f(x) - f(y)\} = f(y) + \min\{1 - f(y) + f(x), 1\} = f(y) + [f(y) \rightsquigarrow f(x)] = f(y) + f(y \to x)$, that is, f is a Bosbach state.

(ii). Suppose A is a bounded Hilbert algebra and consider $x, y \in A$. We have $f(x \curlywedge y) = f((x \to y^*)^*) = 1 - [f(x) \rightsquigarrow (f(y))^*] = 1 - [f(x) \rightsquigarrow (1 - f(y))] = 1 -$

$\min\{1-f(x)+1-f(y),1\} = 1+\max\{f(x)+f(y)-2,-1\} = \max\{f(x)+f(y)-1,0\} = f(x) \odot f(y)$.

□

Proposition 4.2. *Let A be a Hilbert algebra and $s : A \to [0,1]$ a Bosbach state on A. Then the following are equivalent:*
(i) *s is a nontrivial state-morphism;*
(ii) *$Ker(s) \in Max(A)$.*

Proof. $(i) \Rightarrow (ii)$. Since s is a nontrivial state-morphism, then $Ker(s) \neq A$. To prove $Ker(s) \in Max(A)$, let $x, y \in A$ such that $x \sqcup y \in Ker(s)$.

Then $s(x \sqcup y) = 1 \Rightarrow s(x \to y) \rightsquigarrow s((y \to x) \to x) = 1 \Rightarrow [s(x) \rightsquigarrow s(y)] \rightsquigarrow s((y \to x) \to x) = 1$.

If $s(x) \leq s(y)$ then we obtain that $s((y \to x) \to x) = 1 \Rightarrow [s(y) \rightsquigarrow s(x)] \rightsquigarrow s(x) = 1 \Rightarrow s(x) \vee s(y) = 1 \Rightarrow s(y) = 1 \Rightarrow y \in Ker(s)$.

Analogously, since $x \sqcup y = y \sqcup x$, if $s(y) \leq s(x)$, then $x \in Ker(s)$, hence by Theorem 2.1 we deduce that $Ker(s) \in Max(A)$.

$(ii) \Rightarrow (i)$. Suppose $Ker(s) \in Max(A)$ and consider $x, y \in A$. Since by (c_{12}), $x \sqcup (x \to y) = 1 \in Ker(s) \Rightarrow x \in Ker(s)$ or $x \to y \in Ker(s)$.

If $x \in Ker(s) \Rightarrow s(x) = 1 \Rightarrow s(x) \rightsquigarrow s(y) = s(y)$. Since s is supposed Bosbach state, then $s(x)+s(x \to y) = s(y)+s(y \to x)$. But $x \in Ker(s) \Rightarrow y \to x \in Ker(s) \Rightarrow s(x) = s(y \to x) = 1$, so we obtain that $1+s(x \to y) = s(y)+1 \Rightarrow s(x \to y) = s(y) = s(x) \rightsquigarrow s(y)$.

If $x \to y \in Ker(s) \Rightarrow s(x \to y) = 1$. Since $s(x) + s(x \to y) = s(y) + s(y \to x) \Rightarrow s(x) - s(y) = s(y \to x) - 1 \leq 0 \Rightarrow$

$s(x) - s(y) = s(y \to x) - 1 \leq 0 \Rightarrow s(x) \leq s(y) \Rightarrow s(x) \rightsquigarrow s(y) = 1 = s(x \to y)$, so (a_{13}) is verified. □

If A is a Hilbert algebra and $D \in Ds(A)$, then for $x \in A$ we denote by x/D the equivalence class of x relative to D and by A/D the quotient Hilbert algebra (see [6] and [8]).

We recall that for $x, y \in A, x/D = y/D$ iff $x \to y, y \to x \in D$.

Lemma 4.1. *Let A be a Hilbert algebra and $s : A \to [0,1]$ a Bosbach state on A. For all $x, y \in A$, the following are equivalent:*
(i) *$x/Ker(s) = y/Ker(s)$;*
(ii) *$s(x) = s(y)$.*

Proof. We have $x/Ker(s) = y/Ker(s) \Leftrightarrow x \to y, y \to x \in Ker(s) \Leftrightarrow s(x \to y) = s(y \to x) = 1$.

Since $s(x) + s(x \to y) = s(y) + s(y \to x)$ we obtain that $s(x) = s(y)$. □

Following Lemma 4.1, if A is a Hilbert algebra and $s : A \to [0,1]$ is a Bosbach state on A, then the function $\widehat{s} : A/Ker(s) \to [0,1]$, $\widehat{s}(x/Ker(s)) = s(x)$, is well-defined.

It follows immediately that $\widehat{s}$ is a state on $A/Ker(s)$.

Proposition 4.3. *Let A be a bounded Hilbert algebra and $s_1, s_2 : A \to [0,1]$ two Bosbach states such that s_1 is a state-morphism. If $Ker(s_1) = Ker(s_2)$, then $s_1 = s_2$.*

Proof. We denote $M = Ker(s_1)$; following Proposition 4.2, $M \in Max(A)$. Since $Ker(s_2) = M$ we get that $Ker(s_2)$ is also maximal.

From Proposition 4.2, it follows that s_2 is a state-morphism. Clearly, if $x \in M$, then $s_1(x) = s_2(x) = 1$.

Consider now $x \notin M$. Since $M \in Max(A)$,then $x^* \in M$ (by Corollary 2.1), since $s_1(x^*) = s_2(x^*) = 1 \Leftrightarrow 1 - s_1(x) = 1 - s_2(x) = 1 \Leftrightarrow s_1(x) = s_2(x)$, hence $s_1 = s_2$. □

Open question 1. Proposition 4.3 is true if A is unbounded?

Corollary 4.1. *Let A be a bounded Hilbert algebra and $M \in Max(A)$. Then there is a unique Bosbach state $s : A \to [0,1]$ such that $Ker(s) = M$.*

Proof. Following Example 3.2, the function $s_M : A \to [0,1]$, defined by $s_M(x) = 1$ if $x \in M$ and 0 if $x \notin M$ is a Bosbach state on A and $Ker(s) = M$. Following Proposition 4.3, if $s : A \to [0,1]$ is another Bosbach state such that $Ker(s) = M = Ker(s_M)$, then $s = s_M$. □

Corollary 4.2. *If A is a bounded Hilbert algebra, then the assignment $s \rightsquigarrow Ker(s)$ establishes a bijection between the Bosbach states on A and $Max(A)$.*

Let A be a Hilbert algebra and $s : A \to [0,1]$ a Bosbach state on A.

Definition 4.2. *We say that s is an* extremal state *if for any $0 < \lambda < 1$ and for any two states $s_1, s_2 : A \to [0,1]$ such that $s = \lambda s_1 + (1-\lambda)s_2$,then $s_1 = s_2$.*

Theorem 4.1. *Let A be a Hilbert algebra.*
(i) If $s : A \to [0,1]$ is an extremal state, then $s = \mathbf{1}$;
(ii) If A is bounded and s is a state-morphism then s is an extremal state.

Proof. (i). For a fixed element $a \in A$ and $x \in A$ we have $s(x)+s(x \to a) = s(a)+s(a \to x)$. If consider $s_1^a, s_2^a : A \to [0,1], s_1^a(x) = s(a \to x)$ and $s_2^a(x) = s(x)-s(x \to a)+s(a)$ then $s = \frac{1}{2}s_1^a + \frac{1}{2}s_2^a$. From Example 3.3 we deduce that s_1^a is a Bosbach state on A. We will prove that s_2^a is also a Bosbach state on A. Indeed, for $x, y \in A$ we have following sequence of equivalences: $s_2^a(x) + s_2^a(x \to y) = s_2^a(y) + s_2^a(y \to x) \Leftrightarrow s(x) - s(x \to a) + s(a) + s(x \to y) - s((x \to y) \to a) + s(a) =$
$s(y) - s(y \to a) + s(a) + s(y \to x) - s((y \to x) \to a) + s(a) \Leftrightarrow$
$s(y \to a) + s((y \to x) \to a) = s(x \to a) + s((x \to y) \to a) \Leftrightarrow s(y \to a) + s(y \to x) + s((y \to x) \to a) + s(x \to y) = s(x \to a) + s(x \to y) + s((x \to y) \to a) + s(y \to x)$
$\Leftrightarrow s(y \to a) + s(a) + s(a \to (y \to x)) + s(x \to y) = s(x \to a) + s(a) + s(a \to (x \to y)) + s(y \to x) \Leftrightarrow s((a \to y) \to (a \to x)) - s((a \to x) \to (a \to y)) =$
$s(x \to a) + s(y \to x) - s(y \to a) - s(x \to y) \Leftrightarrow s(a \to x) - s(a \to y) = s(x \to a) + s(y \to x) - s(y \to a) - s(x \to y) \Leftrightarrow$
$[s(a \to x) - s(x \to a)] + [s(y \to a) - s(a \to y)] + [s(x \to y) - s(y \to x)] = 0 \Leftrightarrow$
$[s(x) - s(a)] + [s(a) - s(y)] + [s(y) - s(x)] = 0 \Leftrightarrow 0 = 0.$

If s is supposed extremal, from $s = \frac{1}{2}s_1^a + \frac{1}{2}s_2^a \Rightarrow s_1^a = s_2^a$. Then for every $x \in A$ we have $s(a \to x) = s(x) - s(x \to a) + s(a) \Rightarrow s(a \to x) - s(x) = s(a) - s(x \to a)$.

Since $x \leq a \to x$ and $a \leq x \to a$ we deduce that $s(a \to x) - s(x) = s(a) - s(x \to a) = 0$.

So $s(a \to x) = s(x)$, for every $a, x \in A$. In particular for $a = x$ we obtain $s(x) = s(1) = 1$, for every $x \in A$, hence $s = \mathbf{1}$.

(ii). Assume $s = \lambda s_1 + (1-\lambda)s_2$, which $0 < \lambda < 1$ and s_1, s_2 states on A. It is easy to prove that $Ker(s) = Ker(s_1) \cap Ker(s_2)$. But $Ker(s) \in Max(A)$, so $Ker(s) = Ker(s_1) = Ker(s_2)$. Apply now Proposition 4.3 we get that $s_1 = s_2$. □

Open question 2. If A is a bounded Hilbert algebra, then every extremal state $s : A \to [0,1]$ is a state-morphism? (as in the case of pseudo-MV algebras and pseudo-BL algebras - see [9] and [12]).

References

[1] B. Bosbach, Residuation groupoids, *Resultate der Mathematik* **5** (1982), 107–122.
[2] C. Buşneag, States on Hilbert algebrs, *Studia Logica* **94** (2010), no. 2, 177–188.
[3] D. Buşneag, *Contributions to the study of Hilbert algebras* (in romanian), Ph. D Thesis, Univ. of Bucharest, 1985.
[4] D. Buşneag, Hilbert algebras of fractions and maximal Hilbert algebras of quotients, *Kobe J. Math.* **5** (1988), 161–172.
[5] D. Buşneag, On extensions of pseudo-valuation on Hilbert algebra, *Discrete Mathematics* **263** (2003), 11–24.
[6] D. Buşneag, *Categories of algebraic logic*, Editura Academiei Romane, Bucharest, 2006.
[7] L. Ciungu, States on pseudo-BCK algebras, *Math. Reports* **10** (2008), no. 60, 17–36.
[8] A. Diego, Sur les algèbres de Hilbert, *Coll. Logique Math.*, Serie A (1966), no. 21, 1–54.
[9] A. Dvurecenskij, States on pseudo-MV algebras, *Studia Logica* **68** (2001), 301–327.
[10] A. Dvurecenskij and S. Pulmannova, *New trends in quantum structures*, Kluver & Ister Science, Bratislava, 2000.
[11] A. Dvurecenskij and J. Rachünek, On Riečan and Bosbach states for bounded non-commutative RL-monoids, *Math. Slovaca* **5** (2006), 487–500.
[12] G. Georgescu, Bosbach states of fuzzy structures, *Soft Computing* **8** (2004), 217–230.
[13] F. Kôpka and F. Chovanec, D-posets, *Math. Slovaca* **44** (1994), 21–34.
[14] B. Riečan, On the probability on BL-algebras, *Acta Math. Nitra* **4** (2000), 3–13.
[15] D.D.Taşcău, Some properties of the operation $x \sqcup y = (x \to y) \to ((y \to x) \to x)$ in a Hilbert algebra, *Anals of the University of Craiova, Math. Comp. Sci. Ser.* **34** (2007), no. 1, 79–82.

(Cătălin Buşneag) Faculty of Mathematics and Computer Science, University of Craiova, 13, Al.I. Cuza st., 200585, Craiova, Romania
E-mail address: `catalinbusneag@yahoo.com, catalin.busneag@pirelli.com`