State-morphisms on Hilbert algebras

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Abstract. In this paper (which is a continuation of [2]) we develop a theory of state-morphisms on Hilbert algebras.

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1. Introduction

The notion of state is an analogue of a probability measure and has a very important role in the theory of quantum structures (see [10]). The state on $MV$-algebras was first introduced by Kópka and Chovanek in [13]; the state on $BL$-algebras was introduced by Riečan in [14]. In the case of non-commutative fuzzy structures, these states were introduced by Dvurečenskij for pseudo $MV$-algebras in [9], by Georgescu for pseudo $BL$-algebras in [12], by Dvurecenskij and Rachúnek for bounded non-commutative $Rl$-monoids in [11], and by Ciungu for pseudo $BCK$-algebras in [7].

Hilbert algebras are important tools for certain investigations in algebraic logic since they can be considered as fragments of any propositional logic containing a logical connective implication and the constant 1 which is considered as the logical value "true". The concept of Hilbert algebras was introduced by Henkin and Skolem (under the name implicative models) for investigations in intuitionistic logics and other non-classical logics. Diego in [8] proved that Hilbert algebras form a variety which is locally finite.

This paper is organized as follows:

In Section 2 we recall the basic definitions and put in evidence many rules of calculus in Hilbert algebras which we need in the rest of the paper (especially $c_{10}$ - $c_{28}$). Also we recall some results relative to maximal deductive systems for the case of bounded and unbounded Hilbert algebras (Theorem 2.1 and Corollary 2.1). In Section 3 we recall some results relative to the theory of Bosbach states on Hilbert algebras developed in [2]. In Section 4 we develop a theory of state-morphisms on Hilbert algebras.

2. Preliminaries

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and respectively logical equivalence.
Definition 2.1. ([2]-[6], [8])

A Hilbert algebra is an algebra \((A, \to, 1)\) of type \((2, 0)\) such that the following axioms are fulfilled for every \(x, y, z \in A\):

\begin{align*}
(a_1) & \quad x \to (y \to x) = 1; \\
(a_2) & \quad (x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1; \\
(a_3) & \quad \text{If } x \to y = y \to x = 1, \text{ then } x = y.
\end{align*}

In [8] it is proved that the system of axioms \(\{a_1, a_2, a_3\}\) is equivalent with the system \(\{a_4, a_5, a_6, a_7\}\), where:

\begin{align*}
(a_4) & \quad x \to x = 1; \\
(a_5) & \quad 1 \to x = 1; \\
(a_6) & \quad x \to (y \to z) = (x \to y) \to (x \to z); \\
(a_7) & \quad (x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y).
\end{align*}

For examples of Hilbert algebras see [3]-[6] and [8]. If \(A\) is a Hilbert algebra, then the relation \(\leq\) defined by \(x \leq y\) if \(x \to y = 1\) is a partial order on \(A\) (which will be called the natural ordering); with respect to this ordering 1 is the largest element of \(A\). A bounded Hilbert algebra is a Hilbert algebra with a smallest element 0: in this case for \(x \in A\) we denote \(x^* = x \to 0\).

From [2]-[6], [8] in a Hilbert algebra \(A\) we have the following rules of calculus for \(x, y, z \in A\):

\begin{align*}
(c_1) & \quad x \to 1 = 1; \\
(c_2) & \quad x \leq y \to x; \\
(c_3) & \quad x \leq (x \to y) \to y; \\
(c_4) & \quad ((x \to y) \to y) \to y = x \to y; \\
(c_5) & \quad x \to y \leq (y \to z) \to (x \to z); \\
(c_6) & \quad \text{If } x \leq y, \text{ then } z \to x \leq z \to y \text{ and } y \to z \leq x \to z; \\
(c_7) & \quad x \to (y \to z) = y \to (x \to z).
\end{align*}

For \(x_1, \ldots, x_n \in A\) \((n \geq 1)\) we will define \((x_1, \ldots, x_{n-1}; x_n) = x_n\) if \(n = 1\) and \(x_1 \to (x_2, \ldots, x_{n-1}; x_n)\) if \(n > 1\).

Then we have:

\begin{align*}
(c_8) & \quad \text{If } \sigma \text{ is a permutation of } \{1, 2, \ldots, n-1\} \text{ \((n \geq 2)\), then} \\
& \quad (x_{\sigma(1)}, \ldots, x_{\sigma(n-1)}; x_n) = (x_1, \ldots, x_{n-1}; x_n); \\
(c_9) & \quad x \to (x_1, \ldots, x_{n-1}; x_n) = (x, x_1, \ldots, x_{n-1}; x_n) = (x_1, x, x_2, \ldots, x_{n-1}; x_n) = \ldots = \\
& \quad (x_1, \ldots, x_{n-1}, x; x_n).
\end{align*}

For \(x, y \in A\) we define \(x \cup y = (x \to y) \to ((y \to x) \to x)\). Then we have the following rules of calculus for \(x, y, z \in A\):

\begin{align*}
(c_{10}) & \quad x, y \leq x \cup y \text{ and } x \cup y = y \cup x; \\
(c_{11}) & \quad x \cup x = x, x \cup 1 = 1; \\
(c_{12}) & \quad x \cup (x \to y) = 1; \\
(c_{13}) & \quad (x \to y) \cup (y \to x) = 1; \\
(c_{14}) & \quad x \to (y \to z) = (x \to z) \cup (y \to z); \\
(c_{15}) & \quad x \to (y \cup z) = (x \to y) \cup (x \to z); \\
(c_{16}) & \quad (x \to y) \cup z = x \to (y \cup z).
\end{align*}

If \(A\) is a bounded Hilbert algebra and \(x, y \in A\), then we denote \(x \Uparrow y = x^* \to y\) and \(x \Uparrow y = (x \Uparrow y)^*\). We have the following rules of calculus for \(x, y, z \in A\) (see [2]):

\begin{align*}
(c_{17}) & \quad 0^* = 1, 1^* = 0; \\
(c_{18}) & \quad x \to y^* = y \to x^*; \\
(c_{19}) & \quad x \to x^* = x^*, x^* \to x = x^{**}, (y \to x)^* \leq x \to y;
\end{align*}
(c_{20}) If \( x \leq y \), then \( y^* \leq x^* \);
(c_{21}) \( x, y \leq x \vee y, x \vee y = x^{**}, x \vee 0 = x^{**}, x \vee 1 = 1, x \vee y^* = 1, x \vee (y \rightarrow z) = (x \vee y) \rightarrow (x \vee z) \);
(c_{22}) \( x \vee (y \wedge z) = (x \vee y) \wedge z = y \vee (x \wedge z) \);
(c_{23}) \( x^* \uplus y^* = x \rightarrow y^* \);
(c_{24}) \( (x \rightarrow y)^{**} = x^{**} \rightarrow y^{**} = x \rightarrow y^{**} \).

**Proposition 2.1.** (\cite{2}) Let \( A \) be a bounded Hilbert algebra and \( x, y, z \in A \). Then
\( x \wedge 0 = 0, x \wedge 1 = x^{**}, x \wedge x = x^{**} \);
(c_{25}) \( x \wedge y = y \wedge x \leq x^{**}, y^{**} \);
(c_{26}) \( x \leq y \Rightarrow x \wedge y = x^{**} \);
(c_{27}) \( x \wedge (y \wedge z) = (x \wedge y) \wedge z \).

**Definition 2.2.** If \( A \) is a Hilbert algebra, a subset \( D \) of \( A \) is a deductive system of \( A \) if the following axioms are satisfied:
\( a_8 \) \( 1 \in D \);
\( a_9 \) \( x, x \rightarrow y \in D \), then \( y \in D \).

We denote by \( Ds(A) \) the set of all deductive systems of \( A \).

We say that \( M \in Ds(A), M \neq A \), is maximal if it is a maximal element in the lattice \( (Ds(A), \subseteq) \). Let us denote by \( Max(A) \) the set of all maximal deductive systems of \( A \).

We have the following theorem of characterization for maximal deductive systems:

**Theorem 2.1.** (\cite{15}) Let \( A \) be a Hilbert algebra and \( M \in Ds(A), M \neq A \). The following conditions are equivalent:
(i) \( M \in Max(A) \);
(ii) \( x, y \in A \) and \( x \uplus y \in M \), then \( x \in M \) or \( y \in M \);
(iii) \( x \notin M \), then \( x \rightarrow y \in M \) for every \( y \in A \).

**Corollary 2.1.** If \( A \) is a bounded Hilbert algebra and \( M \in Ds(A), M \neq A \), then the following conditions are equivalent:
(i) \( M \in Max(A) \);
(ii) \( x \notin M \), then \( x^* \in M \).

3. Bosbach states on Hilbert algebras

In this section we recall some results relative to the theory of Bosbach states on a Hilbert algebra \( A \). This concept is obtained by using Bosbach condition \( (1) \).

**Definition 3.1.** (\cite{2}) A Bosbach state on a Hilbert algebra \( A \) is a function \( s : A \rightarrow [0, 1] \) such that the following axioms hold:
\( a_{10} \) \( s(1) = 1 \);
\( a_{11} \) \( s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x), \) for all \( x, y \in A \).

**Remark 3.1.** In \cite{2} I given (following a suggestion of a referee) another definition for a Bosbach state on a Hilbert algebra, namely, a function \( s : A \rightarrow [0, 1] \) is a Bosbach state if verify \( (a_{10}), (a_{11}) \) and
\( a_{12} \) there exists an element \( a \in A \) such that \( s(a) = 0 \).

In this paper we consider the notion of Bosbach state in the sense of Definition 3.1.

**Example 3.1.** The function \( 1 : A \rightarrow [0, 1], 1(x) = 1, \) for every \( x \in A \) is a Bosbach state on \( A \).
Example 3.2. If $M \in \text{Max}(A)$, then $s_M : A \to [0, 1]$, defined by $s_M(x) = 1$ if $x \in M$ and 0 if $x \notin M$ is a Bosbach state on $A$. Indeed, since 1 $\in M$, then $s_M(1) = 1$. Consider $x, y \in A$. If $x, y \in M$, then $x \to y, y \to x \in M$ and the axiom (a_{11}) is verified $(1 + 1 = 1 + 1)$. If $x, y \notin M$, then by Theorem 2.1 we deduce that $x \to y, y \to x \in M$, so the axiom (a_{11}) is also verified $(0 + 1 = 0 + 1)$. If $x \notin M$ and $y \in M$, then $x \to y \in M$ and $y \to x \notin M$, so the axiom (a_{11}) is also verified $(0 + 1 = 1 + 0)$.

Example 3.3. If $s : A \to [0, 1]$ is a Bosbach state, then for every $a \in A$, $s_a : A \to [0, 1], s_a(x) = s(a \to x)$ is also a Bosbach state on $A$. Indeed, $s_a(1) = s(a \to 1) = s(1) = 1$ and for $x, y \in A$, $s_a(x + y) = s(a \to x + y) = s(a \to x) + s(a \to (y \to x)) = s(a \to x) + s((a \to x) \to (a \to y)) = s(a \to y) + s((a \to y) \to (a \to x)) = s(a \to y) + s(a \to (y \to x)) = s_a(y) + s_a(y \to x)$. 

For a Bosbach state $s : A \to [0, 1]$ we define $\text{Ker}(s) = \{x \in A : s(x) = 1\}$.

Proposition 3.1. ([2]) $\text{Ker}(s) \subseteq Ds(A)$.

Proposition 3.2. ([2]) If $s : A \to [0, 1]$ is a Bosbach state on $A$, then for all $x, y \in A$ we have:

(c_{20}) $x \leq y \Rightarrow s(x) \leq s(y)$;

(c_{30}) $s((x \to y) \to y) = s((y \to x) \to x)$.

4. State-morphisms on Hilbert algebras

In this section we develop a theory of state-morphisms on Hilbert algebras. Let us denote by $[0, 1]$ the standard MV-algebra of real unit interval $[0, 1]$, where for $x, y \in [0, 1], x \otimes y = \min\{x + y, 1\}, x \oslash y = \max\{x + y - 1, 0\}, x \sim y = \min\{1 - x + y, 1\}, x \wedge y = \min\{x, y\}$ and $x \lor y = \max\{x, y\}$. Clearly, $x \sim x = 1, x \sim 1 = 1, 1 \sim x = x$ and $x \wedge a \leq y$ if $a \leq x \sim y$ for every $x, y, a \in [0, 1]$. Also, $(x \sim y) \sim y = (y \sim x) \sim x = x \lor y$, for every $x, y \in [0, 1]$.

Definition 4.1. A state-morphism on a Hilbert algebra $A$ is a function $f : A \to [0, 1]$ such that for every $x, y \in A$:

(a_{13}) $f(x \to y) = f(x) \sim f(y)$.

If $A$ is bounded we add the condition

(a_{14}) $f(0) = 0$.

Clearly, $1 : A \to [0, 1], 1(x) = 1$, for every $x \in A$ is a state morphism (called trivial). From (a_{13}) we deduce that $f(1) = f(1 \to 1) = f(1) \sim f(1) = 1$. If $A$ is bounded, then for every $x \in A$ we have $f(x^*) = f(x \to 0) = f(x) \sim f(0) = f(x) \sim 0 = (f(x))^* = 1 - f(x)$.

Proposition 4.1. Let $A$ be a Hilbert algebra and $f : A \to [0, 1]$ a state-morphism.

Then:

(i) $f$ is a Bosbach state;

(ii) If $A$ is a bounded Hilbert algebra, then $f(x \vee y) = f(x) \circ f(y)$, for every $x, y \in A$.

Proof. (i). For every $x, y \in A$ we have $f(x) + f(x \to y) = f(x) + [f(x) \sim f(y)] = f(x) + \min\{1 - f(x) + f(y), 1\} = \min\{1 + f(y), 1 + f(x)\} = f(y) + \min\{1, 1 + f(x) - f(y)\} = f(y) + \min\{1 - f(y) + f(x), 1\} = f(y) + [f(y) \sim f(x)] = f(y) + f(y \to x)$, that is, $f$ is a Bosbach state.

(ii). Suppose $A$ is a bounded Hilbert algebra and consider $x, y \in A$. We have $f(x \land y) = f((x \to y^*))^* = 1 - [f(x) \sim (f(y))^*] = 1 - [f(x) \sim (1 - f(y))] = 1 -
\[
\min\{1 - f(x) + 1 - f(y), 1\} = 1 + \max\{f(x) + f(y) - 2, -1\} = \max\{f(x) + f(y) - 1, 0\} = f(x) \circ f(y).
\]

**Proposition 4.2.** Let \( A \) be a Hilbert algebra and \( s : A \to [0,1] \) a Bosbach state on \( A \). Then the following are equivalent:

(i) \( s \) is a nontrivial state-morphism;

(ii) \( \text{Ker}(s) \in \text{Max}(A) \).

**Proof.** (i) \( \Rightarrow \) (ii). Since \( s \) is a nontrivial state-morphism, then \( \text{Ker}(s) \neq A \). To prove \( \text{Ker}(s) \in \text{Max}(A) \), let \( x, y \in A \) such that \( x \sqcup y \in \text{Ker}(s) \).

Then \( s(x \sqcup y) = 1 \Rightarrow s(x \to y) \sim s(y \to x) = 1 \Rightarrow [s(x) \sim s(y)] \to s((y \to x) \to x) = 1 \).

If \( s(x) \leq s(y) \) then we obtain that \( s((y \to x) \to x) = 1 \Rightarrow [s(y) \sim s(x)] \to s(x) = 1 \Rightarrow s(x) \vee s(y) = 1 \Rightarrow y \in \text{Ker}(s) \).

Analogously, since \( x \sqcup y = y \sqcup x \), if \( s(y) \leq s(x) \), then \( x \in \text{Ker}(s) \), hence by Theorem 2.1 we deduce that \( \text{Ker}(s) \in \text{Max}(A) \).

(ii) \( \Rightarrow \) (i). Suppose \( \text{Ker}(s) \in \text{Max}(A) \) and consider \( x, y \in A \). Since by (c12), \( x \sqcup (x \to y) = 1 \in \text{Ker}(s) \Rightarrow x \in \text{Ker}(s) \) or \( x \to y \in \text{Ker}(s) \).

If \( x \in \text{Ker}(s) \Rightarrow s(x) = 1 \Rightarrow s(x) \sim s(y) = s(y) \). Since \( s \) is supposed Bosbach state, then \( s(x) + s(x \to y) = s(y) + s(y \to x) \). But \( x \in \text{Ker}(s) \Rightarrow y \to x \in \text{Ker}(s) \Rightarrow s(x) = s(y \to x) = 1 \), so we obtain that \( 1 + s(x \to y) = s(y) + 1 \Rightarrow s(x \to y) = s(y) \Rightarrow s(x) \sim s(y) \).

If \( y \to x \in \text{Ker}(s) \Rightarrow s(x \to y) = 1 \). Since \( s(x) + s(x \to y) = s(y) + s(y \to x) \Rightarrow s(x) - s(y) = s(y \to x) - 1 \leq 0 \Rightarrow s(x) - s(y) = s(y \to x) - 1 \leq 0 \Rightarrow s(x) \leq s(y) \Rightarrow s(x) \sim s(y) = 1 = s(x \to y) \), so (a13) is verified.

If \( A \) is a Hilbert algebra and \( D \in Ds(A) \), then for \( x \in A \) we denote by \( x/D \) the equivalence class of \( x \) relative to \( D \) and by \( A/D \) the quotient Hilbert algebra (see [6] and [8]).

We recall that for \( x, y \in A, x/D = y/D \) iff \( x \to y, y \to x \in D \).

**Lemma 4.1.** Let \( A \) be a Hilbert algebra and \( s : A \to [0,1] \) a Bosbach state on \( A \). For all \( x, y \in A \), the following are equivalent:

(i) \( x/\text{Ker}(s) = y/\text{Ker}(s) \);

(ii) \( s(x) = s(y) \).

**Proof.** We have \( x/\text{Ker}(s) = y/\text{Ker}(s) \Leftrightarrow x \to y, y \to x \in \text{Ker}(s) \Leftrightarrow s(x \to y) = s(y \to x) = 1 \).

Since \( s(x) + s(x \to y) = s(y) + s(y \to x) \) we obtain that \( s(x) = s(y) \). \( \square \)

Following Lemma 4.1, if \( A \) is a Hilbert algebra and \( s : A \to [0,1] \) is a Bosbach state on \( A \), then the function \( \tilde{s} : A/\text{Ker}(s) \to [0,1] \), \( \tilde{s}(x/\text{Ker}(s)) = s(x) \), is well-defined.

It follows immediately that \( \tilde{s} \) is a state on \( A/\text{Ker}(s) \).

**Proposition 4.3.** Let \( A \) be a bounded Hilbert algebra and \( s_1, s_2 : A \to [0,1] \) two Bosbach states such that \( s_1 \) is a state-morphism. If \( \text{Ker}(s_1) = \text{Ker}(s_2) \), then \( s_1 = s_2 \).

**Proof.** We denote \( M = \text{Ker}(s_1) \); following Proposition 4.2, \( M \in \text{Max}(A) \). Since \( \text{Ker}(s_2) = M \) we get that \( \text{Ker}(s_2) \) is also maximal.

From Proposition 4.2, it follows that \( s_2 \) is a state-morphism. Clearly, if \( x \in M \), then \( s_1(x) = s_2(x) = 1 \).
Consider now $x \notin M$. Since $M \subseteq \text{Max}(A)$, then $x^* \in M$ (by Corollary 2.1), since $s_1(x^*) = s_2(x^*) = 1 \Leftrightarrow 1 - s_1(x) = 1 - s_2(x) = 1 \Leftrightarrow s_1(x) = s_2(x)$, hence $s_1 = s_2$. \hfill \Box

**Open question 1.** Proposition 4.3 is true if $A$ is unbounded?

**Corollary 4.1.** Let $A$ be a bounded Hilbert algebra and $M \subseteq \text{Max}(A)$. Then there is a unique Bosbach state $s : A \to [0, 1]$ such that $\text{Ker}(s) = M$.

**Proof.** Following Example 3.2, the function $s_M : A \to [0, 1]$, defined by $s_M(x) = 1$ if $x \in M$ and 0 if $x \notin M$ is a Bosbach state on $A$ and $\text{Ker}(s) = M$. Following Proposition 4.3, if $s : A \to [0, 1]$ is another Bosbach state such that $\text{Ker}(s) = M = \text{Ker}(s_M)$, then $s = s_M$. \hfill \Box

**Corollary 4.2.** If $A$ is a bounded Hilbert algebra, then the assignment $s \rightsquigarrow \text{Ker}(s)$ establishes a bijection between the Bosbach states on $A$ and $\text{Max}(A)$.

Let $A$ be a Hilbert algebra and $s : A \to [0, 1]$ a Bosbach state on $A$.

**Definition 4.2.** We say that $s$ is an extremal state if for any $0 < \lambda < 1$ and for any two states $s_1, s_2 : A \to [0, 1]$ such that $s = \lambda s_1 + (1 - \lambda)s_2$, then $s_1 = s_2$.

**Theorem 4.1.** Let $A$ be a Hilbert algebra.

(i) If $s : A \to [0, 1]$ is an extremal state, then $s = 1$;

(ii) If $A$ is bounded and $s$ is a state-morphism then $s$ is an extremal state.

**Proof.** (i). For a fixed element $a \in A$ and $x \in A$ we have $s(x) + s(x \to a) = s(a) + s(a \to x)$. If consider $s_1^*, s_2^* : A \to [0, 1], s_1^*(x) = s(a \to x)$ and $s_2^*(x) = s(x) - s(x \to a) + s(a)$ then $s = \frac{1}{2}s_1^* + \frac{1}{2}s_2^*$. From Example 3.3 we deduce that $s_1^*$ is a Bosbach state on $A$. We will prove that $s_2^*$ is also a Bosbach state on $A$. Indeed, for $x, y \in A$ we have following sequence of equivalences: $s_2^*(x) + s_2^*(x \to y) = s_2^*(y) + s_2^*(y \to x) \Leftrightarrow s(x) - s(x \to a) + s(a) + s(x \to y) - (s(x \to y) \to a) + s(a) =\ldots$

(ii). Assume $s = \lambda s_1 + (1 - \lambda)s_2$, where $0 < \lambda < 1$ and $s_1, s_2$ states on $A$. It is easy to prove that $\text{Ker}(s) = \text{Ker}(s_1) \cap \text{Ker}(s_2)$. But $\text{Ker}(s) \subseteq \text{Max}(A)$, so $\text{Ker}(s) = \text{Ker}(s_1) = \text{Ker}(s_2)$. Apply now Proposition 4.3 we get that $s_1 = s_2$. \hfill \Box

**Open question 2.** If $A$ is a bounded Hilbert algebra, then every extremal state $s : A \to [0, 1]$ is a state-morphism? (as in the case of pseudo-MV algebras and pseudo-BL algebras - see [9] and [12]).
References


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