A theorem of representation for Hilbert algebras

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Abstract. The main scope of this paper is to prove the following theorem of representation for a Hilbert algebra $A$: There exist a complete residuated lattice $L_r(A)$ which is a $G-$ algebra and an injective morphism of Hilbert algebras $i_A : A \rightarrow L_r(A)$. As a consequence, we deduce that the free Hertz algebra $H_A$ over $A$ (see [15]) is isomorphic with a Hertz subalgebra of $L_r(A)$. Also, I give the description of the elements of $L_r(A)$.

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1. Introduction

The concept of Hilbert algebras was introduced in the 50’s by Henkin and Skolem for investigations in intuitionistic and other non-classical logics, as an algebraic counterpart of Hilbert’s positive implicative propositional calculus ([16]). Hilbert algebras were intensively studied by A. Diego ([5]) and this theory was further developed by Busneag ([3]). $BCK$ algebras were introduced by Iséki in 1966 ([9], [11]) to give an algebraic framework for Meredith’s implicational logic $BCK$. Since Iséki’s definition, these algebras have been studied by several authors. For further information see for example [2], [4], [7], [10], [12] and the references given there.

The paper is organized as follows: In Section 2 we recall the basic definitions and some results relative to $BCK$ algebras; also we put in evidence some rules of calculus in Hilbert and $BCK$ algebras (which we need in Section 3). In the final of Section 2 we put in evidence a theorem of embedding for Hilbert algebras into complete integral residuated lattices which is $G-$ algebra (Theorem 2.2).

In Section 3 we give a characterization of the elements of the complete integral residuated lattice $L_r(A)$ from Section 2.

2. Preliminaries

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and respectively logical equivalence.

Definition 2.1. ([4], [10]) A $BCK$ algebra is an algebra $(A, \rightarrow, 1)$ of type $(2,0)$ such that the following axioms are verified for every $x, y, z \in A$:

$(a_1)$ $x \rightarrow x = 1$;

$(a_2)$ If $x \rightarrow y = y \rightarrow x = 1$, then $x = y$;

$(B)$ $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;

$(C)$ $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

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(K) \( x \rightarrow (y \rightarrow x) = 1. \)

The relation \( a \leq b \) iff \( a \rightarrow b = 1 \) is a partial order on \( A \) (called the natural order on \( A \)); with respect to this order 1 is the largest element of \( A \).

For examples of BCK algebras see [4] and [10].

A Hilbert algebra \([3, 5, 10]\) is a BCK algebra \((A, \rightarrow, 1)\) which verifies one of the following equivalent conditions for all \( x, y \in A \):

\((a_1)\): \( x \rightarrow (x \rightarrow y) = x \rightarrow y; \)
\((a_2)\): \( (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y). \)

In a BCK algebra we have \(([4, 7, 10, 12])\) the following rules of calculus for \( x, y, z \in A \):

\((c_1)\): \( x \leq y \rightarrow x; \)
\((c_2)\): \( x \leq (x \rightarrow y) \rightarrow y; \)
\((c_3)\): \( (x \rightarrow y) \rightarrow y \leq x \rightarrow y; \)
\((c_4)\): \( x \rightarrow y \leq (z \rightarrow y) \rightarrow (z \rightarrow x) \leq x \rightarrow (x \rightarrow y); \)
\((c_5)\): \( x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z). \)

If \( A \) is a Hilbert algebra, then

\((c_6)\): \( x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z). \)

If \( A \) is a BCK algebra and \( x_1, \ldots, x_n, x \in A (n \geq 1) \) we define \((x_1, \ldots, x_n; x) = x_1 \rightarrow (x_2 \rightarrow \ldots (x_n \rightarrow x) \ldots)). \)

Following (C) we deduce that if \( \sigma \) is permutation of \((1, 2, \ldots, n)\), then for every \( x, y, x_1, \ldots, x_n \in A \):

\((c_8)\): \( (x_{\sigma(1)}, \ldots, x_{\sigma(n)}; x) = (x_1, \ldots, x_n; x); \)
\((c_9)\): \( (x_1, \ldots, x_n; x \rightarrow y) = x \rightarrow (x_1, \ldots, x_n; y). \)

If \( A \) is a Hilbert algebra then:

\((c_{10})\): \( (x_1, \ldots, x_n; x \rightarrow y) = (x_1, \ldots, x_n; x) \rightarrow (x_1, \ldots, x_n; y). \)

For a BCK algebra \( A \), two elements \( x, y \in A \) and a natural number \( n \geq 1 \) we denote \( x \rightarrow_n y = (x, x, \ldots, x; y) \), where \( n \) indicates the number of occurrences of \( x \). Clearly, if \( A \) is a Hilbert algebra, then \( x \rightarrow_n y = x \rightarrow y \), for every \( n \geq 1 \).

A deductive system (or i-filter) of a BCK algebra \( A \) is a nonempty subset \( D \subseteq A \) such that:

\((a_0)\): \( \{1\} \in D; \)
\((a_1)\): \( \text{If } x, x \rightarrow y \in D, \text{ then } y \in D. \)

It is clear that if \( D \) is a deductive system, \( a \leq b \) and \( a \in D \), then \( y \in D \) (that is, \( D \) is increasing subset of \( A \)).

We denote by \( Ds(A) \) the set of all deductive systems of \( A \) (clearly, \( \{\} \), \( A \in Ds(A) \)).

For a nonempty subset \( X \subseteq A \), the deductive system generated by \( X \) will be denoted by \([X]\). It is known \((7, 12)\) that \([X] = \{x \in A : (x_1, \ldots, x_n; x) = 1, \text{ for some } x_1, \ldots, x_n \in X\}\). In particular for \( a \in A, \{a\} \equiv [a] = \{x \in A : a \rightarrow_n x = 1, \text{ for some } n \geq 1\}. \)

If \( D \in Ds(A) \) and \( a \in A \setminus D \), then \( \{D \cup \{a\} \equiv D(a) = \{x \in A : a \rightarrow_n x \in D, \text{ for some } n \geq 1\}. \)

In particular, if \( A \) is a Hilbert algebra, then for \( X = \{x_1, \ldots, x_n\}, [X] = \{x \in A : (x_1, \ldots, x_n; x) = 1\} \) and if \( D \in Ds(A) \) and \( a \in A \setminus D \), then \( D(a) = \{x \in A : a \rightarrow x \in D\}. \)

Remark 2.1. If \( A \) is a Hilbert algebra, then if \( X = \{x_1, \ldots, x_m\} \) and \( Y = \{y_1, \ldots, y_n\}, [X \cup Y] = [X(y_1, \ldots, y_n)] = [Y(x_1, \ldots, x_m)] \)

(where \([X](y_1, \ldots, y_n) = \ldots (\ldots [X](y_1))(y_2)\ldots)(y_n)\).
For a $BCK$ algebra $A$ we let $W(A)$ denote the set of all words $X = x_1x_2...x_n$ ($n \geq 1$) over $A$.

For any word $W = x_1x_2...x_n \in W(A)$ and an element $a \in A$, we shall write $W \to a = (x_1, x_2, ..., x_n; a) \in A$.

Remark 2.2. If $W \in W(A)$, then $W \to a = 1 \Rightarrow a \in |W|$. If $A$ is a Hilbert algebra, then $W \to a = 1 \Leftrightarrow a \in |W|$.

From (C) we deduce that for $X,Y \in W(A)$ and $a \in A$, then:

$(c_{12}) \ X \to (Y \to a) = Y \to (X \to a) = (X \to a) \to a$, where $X,Y \in W(A)$ stand for concatenation of $X$ and $Y$.

Let $Fin(W(A))$ be the set of all finite non-empty subsets of $W(A)$.

One readily sees ([13]) that the relation $\rho_A$ defined on $Fin(W(A))$ by the stipulation $\{X_1, ..., X_n\} \rho_A \{Y_1, ..., Y_n\}$ iff for all $W \in W(A)$ and $a \in A$ we have $W \to (X_i \to a) = 1$ for all $i = 1, 2, ..., m$ iff $W \to (Y_j \to a) = 1$ for all $j = 1, 2, ..., n$, is an equivalence on $Fin(W(A))$; the $\rho_A$-class of $\{X_1, ..., X_n\}$ will be briefly denoted as $< X_1, ..., X_n >$. Further, we equip the quotient set $M_A = Fin(W(A))/\rho_A$ with two binary operations $\cap$ and $\ast$, as follows:

$$< X_1, ..., X_m > \cap < Y_1, ..., Y_n > = < X_1, ..., X_m, Y_1, ..., Y_n >,$$

$$< X_1, ..., X_m > \ast < Y_1, ..., Y_n > = < X_iY_j : i = 1, 2, ..., m, j = 1, 2, ..., n >.$$  

Definition 2.2. By a meet-semilattice-ordered monoid we mean an algebra $(M, \land, \bullet, e)$ such that :

$(a_7) \ (M, \land)$ is a meet-semilattice;

$(a_8) \ (M, \bullet, e)$ is a monoid;

$(a_9) \ (x \land y) \bullet z = (x \bullet z) \land (y \bullet z)$ and $z \bullet (x \land y) = (z \bullet x) \land (z \bullet y)$ for every $x, y, z \in A$.

If the identity element $e$ is the least element of $M$ (that is, $e$ play the role of 0), then $M$ is called dually integral.

In [13] it is proved the following result:

Proposition 2.1. For every $BCK$ algebra $A$, the structure $(M_A, \cap, \ast, < 1 >)$ is a dually integral meet-semilattice-ordered monoid.

Remark 2.3. In [13], the above result is obtained for the case of a pseudo $BCK$ algebra $A$; if $A$ is a $BCK$ algebra, then the operation $\ast$ is commutative. Indeed, if $\alpha = < X_1, ..., X_m >, \beta = < Y_1, ..., Y_n > \in M_A$, then $\alpha \ast \beta = < X_iY_j : i = 1, 2, ..., m, j = 1, 2, ..., n >$. If $W \in W(A)$ and $a \in A$, then $W \to (X_iY_j \to a) = 1$ iff $W \to (X_i \to (Y_j \to a)) = 1$ iff $W \to (Y_j \to (X_i \to a)) = 1$, for all $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, so $\alpha \ast \beta = \beta \ast \alpha$.

Lemma 2.1. Let $(M, \land, \bullet, e)$ a dually integral meet-semilattice-ordered (commutative) monoid. Then for every $x, y \in M$ :

$(c_{13}) \ x \leq x \bullet y, y \leq x \bullet y$;

$(c_{14}) \ x \leq x \bullet x$.

Proof. $(c_{13})$. We have $x \bullet (y \land e) = (x \bullet y) \land (x \bullet e) \Rightarrow x \bullet e = (x \bullet y) \land x \Rightarrow x = (x \bullet y) \land x \Rightarrow x \leq x \bullet y$.

$(c_{14})$. Clearly. \qed
Remark 2.4. It is worth noticing that the partial order \( \subseteq \) associated with the meet operation \( \sqcap \) on \( M_A \) we have \( < X_1, ..., X_m > \subseteq < Y_1, ..., Y_n > \) iff for all \( W \in W(A) \) and \( a \in A, W \rightarrow (X_i \rightarrow a) = 1 \) for all \( i = 1, 2, ..., m \), then \( W \rightarrow (Y_j \rightarrow a) = 1 \) for all \( j = 1, 2, ..., n \).

Corollary 2.1. If \( A \) is a Hilbert algebra, then \( \alpha \ast \alpha = \alpha \) for every \( \alpha \in M_A \).

Proof. By \((c_{14})\) we deduce that \( \alpha \subseteq \alpha \ast \alpha \). To prove that \( \alpha \ast \alpha \subseteq \alpha \),

let \( \alpha = < X_1, ..., X_m > \in M_A, W \in W(A) \) and \( \alpha \in A \) such that \( W \rightarrow (\alpha \ast \alpha \rightarrow a) = 1 \). Since \( \alpha \ast \alpha = < X_1X_1, X_1X_2, ..., X_2X_2, ..., X_{n-1}X_{n-1}, X_nX_n > \), then in particular we have \( W \rightarrow (X_iX_i \rightarrow a) = 1 \) for all \( i = 1, 2, ..., m \).

Since \( A \) is a Hilbert algebra, then for all \( i = 1, 2, ..., m \) we have \( W \rightarrow (X_i \rightarrow (X_i \rightarrow a)) = 1 \Rightarrow W \rightarrow (X_i \rightarrow a) = 1 \), hence \( \alpha \ast \alpha \subseteq \alpha \), so \( \alpha \ast \alpha = \alpha \).

We recall that if \((M, \wedge)\) is a meet-semilattice, then \( F \subseteq M \) is a filter \((\lbrack 1 \rbrack)\) if \( x, y \in F \Rightarrow x \wedge y \in F \) and if \( x \leq y \) and \( x \in F \Rightarrow y \in F \).

For \((M, \wedge, \bullet, e)\) a dually integral meet-semilattice-ordered monoid, let \( \mathcal{F}(M) \) the set of all filters of \((M, \wedge)\) augmented by \( \emptyset \).

Let us introduce the following notation for \( F, G \in \mathcal{F}(M) : \)

\[ F \lor G = \text{the filter generated by } F \cup G = \{a \in M : x \wedge y \leq a \text{ for some } x, y \in F \cup G\}, \]

\[ F \circ G = \{a \in M : x \bullet y \leq a \text{ for some } x \in F \text{ and } y \in G\}, \]

\[ F \rightarrow G = \{a \in M : \{a\} \bullet F \subseteq G\} = \{a \in M : x \geq a \bullet f \text{ with } f \in F, \text{ then } x \in G\}. \]

We recall \((\lbrack 6 \rbrack, \lbrack 14 \rbrack)\) that an integral residuated lattice is an algebra \((L, \lor, \wedge, \circ, 0, 1)\) such that \((L, \lor, \wedge, 0, 1)\) is a bounded lattice, \((L, \circ, 1)\) is a (commutative) monoid whose identity 1 is the greatest element of the lattice and \( x \circ a \leq y \) iff \( a \leq x \rightarrow y \) for all \( a, x, y \in L \).

Remark 2.5. \((\lbrack 6 \rbrack, \lbrack 14 \rbrack)\) If \((L, \lor, \wedge, \circ, 0, 1)\) is an integral residuated lattice then \((L, \rightarrow, 1)\) is a BCK algebra.

In \([13]\) it is proved the following result:

Lemma 2.2. If \( A \) is a BCK algebra, then \((\mathcal{F}(M_A), \lor, \cap, \circ, O, M_A)\) is a complete integral residuated lattice.

For \( a \in A, \) we put \( i_A(a) = \{< X_1, ..., X_m > \in M_A : X_i \rightarrow a = 1, \text{ for all } i = 1, 2, ..., m\}. \)

In \([13]\) it is proved the following result:

Theorem 2.1. If \( A \) is a BCK algebra, then the map \( i_A : A \rightarrow L_r(A) = \mathcal{F}(M_A) \) is an injective morphism of BCK algebras. Moreover, if for \( a, b \in A \) there exists \( a \lor b \) in \( A \), then \( i_A(a \lor b) = i_A(a) \lor i_A(b) \).

Taking as guide-line the case of BL algebras \((\text{see } [8], \text{ Definition } 4.2.12)\), an integral residuated lattice \( L \) is a \( G \)-algebra if \( x \circ x = x \) for every \( x \in L \).

We have the following results:

Proposition 2.2. \((\lbrack 14 \rbrack)\) Let \((L, \lor, \wedge, \circ, 0, 1)\) is an integral residuated lattice. Then the following are equivalent:

(i): \( L \) is a \( G \)-algebra;

(ii): \( x \circ y = x \wedge y \), for every \( x, y \in L \);

(iii): \( x \circ (x \rightarrow y) = x \wedge y \), for every \( x, y \in L \).
Proposition 2.3. ([14]) For an integral residuated lattice \((L, \vee, \wedge, \odot, \rightarrow, 0, 1)\) the following are equivalent:

(i): \((L, \rightarrow, 1)\) is a Hilbert algebra;
(ii): \((L, \vee, \wedge, \odot, \rightarrow, 0, 1)\) is a \(G\)-algebra.

Lemma 2.3. If \(A\) is a Hilbert algebra, then the integral residuated lattice \(L_r(A)\) is a \(G\)-algebra.

Proof. We must prove that for \(F \in L_r(A)\), \(F \odot F = F\). Since \(L_r(A)\) is an integral residuated lattice, then \(F \odot F \subseteq F\) ([6], [14]). If \(\alpha \in F\), by Corollary 2.1, \(\alpha = \alpha \ast \alpha\), hence \(\alpha \in F \odot F \Rightarrow F \subseteq F \odot F\), so \(F = F \odot F\).

From Lemma 2.3 and Theorem 2.1 we obtain the following theorem of representation for Hilbert algebras:

Theorem 2.2. If \(A\) is a Hilbert algebra, then there exist a complete integral residuated lattice \(L_r(A)\) which is a \(G\) - algebra and an injective morphism of Hilbert algebras \(i_A : A \rightarrow L_r(A)\). Moreover, if for \(a, b \in A\) there exists \(a \vee b\) in \(A\), then \(i_A(a \vee b) = i_A(a) \vee i_A(b)\).

Remark 2.6. For others theorems of representation for Hilbert algebras, see [3], [5].

3. A characterization of the elements of \(L_r(A)\)

If \((S, \wedge)\) is a meet-semilattice, for a nonempty subset \(M \subseteq S\), by \([M]\) we denote the filter of \(S\) generated by \(M\).

We have ([1]): \([M]\) = \{\(x \in S : x_1 \wedge \ldots \wedge x_n \leq x\) for some \(x_1, \ldots, x_n \in M\}\}. In particular, if \(M = \{a\}\), \([a]_{\text{nat}} = \{a \in S : a \leq x\}\).

Remark 3.1. We recall ([1]) that if \((S, \wedge)\) is a meet-semilattice then:

(i): If \(a, b \in S\) and \(a \leq b \Rightarrow [b] \subseteq [a]\);
(ii): If \(a_1, a_2, \ldots, a_n \in S\) then \([a_1 \wedge a_2 \wedge \ldots \wedge a_n] = [a_1] \vee [a_2] \vee \ldots \vee [a_n]\).

Lemma 3.1. If \(A\) is a BCK algebra, then for every \(a \in A\), \(i_A(a) = [a \rightarrow a]\).

Proof. If \(<X_1, \ldots, X_n> \in [a \rightarrow a]\) \(\Rightarrow< a > \subseteq< X_1, \ldots, X_n >\). Since \(1 \rightarrow (a \rightarrow a) = 1 \Rightarrow 1 \rightarrow (X_1 \rightarrow a) = 1\), for \(i = 1, 2, \ldots, n \Rightarrow i A_i (a) \Rightarrow [a \rightarrow a] \subseteq i_A(a)\).

Conversely, let \(<X_1, \ldots, X_n> \in i_A(a)\), that is, \(X_i \rightarrow a = 1\), for \(i = 1, 2, \ldots, n\). To prove \(<a> \subseteq< X_1, \ldots, X_n >\), let \(W = a_1 a_2 \ldots a_m \in W(A)\) and \(x \in A\) such that \(W \rightarrow (a \rightarrow x) = 1\).

For \(i \in \{1, 2, \ldots, n\}\) consider \(X_i = x_1 \ldots x_i \in W(A)\). From \(W \rightarrow (a \rightarrow x) = 1 \Rightarrow a \rightarrow (W \rightarrow x) = 1 \Rightarrow a \leq (a_1, \ldots, a_m; x)\) implies \((x_1, \ldots, x_i; a) \leq (x_1, \ldots, x_i, a_1, \ldots, a_m; x)\) \\
\(X_i \rightarrow (W \rightarrow a) = 1 \Rightarrow W \rightarrow (X_i \rightarrow a) = 1\), for \(i = 1, 2, \ldots, n \Rightarrow< X_1, \ldots, X_n > \in [a \rightarrow a] \Rightarrow i_A(a) = [a \rightarrow a]\).

Lemma 3.2. Let \(A\) be a BCK algebra and \(<X_1, \ldots, X_n> \in M_A\). Then
\(<X_1, \ldots, X_n > = [X_1 > \vee \ldots \vee [X_n >).

Proof. We have \([<X_1, \ldots, X_n>] = [X_1 > \cap \ldots \cap [X_n >] \Rightarrow M_{\text{nat}} = [X_1 > \vee \ldots \vee [X_n >].\]

Lemma 3.3. If \(A\) is a Hilbert algebra and \(a_1, a_2, \ldots, a_n \in A\), then \([<a_1 a_2 \ldots a_n>] = [a_1 > \cap \ldots \cap [a_n >].\)
Proof. It is suffice to prove that for two elements \( x, y \in A \), we have the equality \([ab] = [a] \cap [b]\).

Indeed, \( <a> \land <b> = <ab> \) and since \( <a> \land <b> = <ab> \) we deduce that \([ab] \subseteq [a] \land [b] \). So \([ab] \subseteq [a] \land [b] \).

To prove the converse inclusion, let \( X_1, \ldots, X_n \in [a] \land [b] \). Then \( <a> \land <b> = <X_1, \ldots, X_n> \).

Consider \( W \subseteq W(A) \) and let \( \forall x \in X \subseteq X_n \). We deduce that \([W] \subseteq [X_1, \ldots, X_n] \) and since \( [W] \subseteq [X_1, \ldots, X_n] \) we have \([W] = [X_1, \ldots, X_n] \).

From the above results we obtain the following theorem of characterization for the elements of \( L_r(A) \) when \( A \) is a Hilbert algebra:

Theorem 3.1. Let \( A \) be a Hilbert algebra. Then for \( F \subseteq L_r(A) = \mathcal{F}(MA) \) we have \( F = \bigvee_{X_1, \ldots, X_n \in F} \bigl[ \bigvee_{x \in X_1} i_A(x) \bigr] \cdots \bigvee_{x \in X_n} i_A(x) \bigr] \).

Proof. For \( F \subseteq L_r(A) = \mathcal{F}(MA) \) we have \( F = \bigvee_{X_1, \ldots, X_n \in F} \bigl[ \bigvee_{x \in X_1} i_A(x) \bigr] \cdots \bigvee_{x \in X_n} i_A(x) \bigr] \).

Definition 3.1. A Hertz algebra is a Hilbert algebra \( A \) with the property that for every \( x, y \in A \), the infimum \( x \cap y \) (relative to the natural ordering) exists in \( A \) (that is, \( A \) is meet-semilattice relative to the natural order) and for every \( x, y \in A \) we have the relation:

\((P)\): \( x \rightarrow (y \rightarrow (x \land y)) = 1 \).

In [15] it is proved the equivalence as above definition with:

Definition 3.2. A Hertz algebra is an algebra \((A, \rightarrow, \land)\) of type \((2, 2)\) satisfying the following axioms:

\((a_{10})\): \( x \rightarrow x = y \rightarrow y \);
\((a_{11})\): \( (x \rightarrow y) \land y = y \);
\((a_{12})\): \( x \land (x \rightarrow y) = x \land y \);
\((a_{13})\): \( x \rightarrow (y \land z) = (x \rightarrow y) \land (x \rightarrow z) \).

Definition 3.3. If \( A \) is a Hilbert algebra, a Hertz algebra \( H_A \) (together with an injective morphism of Hilbert algebras \( \varphi_A : A \rightarrow H_A \)) is said to be free over \( A \) if:

\((a_{14})\): \( H_A \) is generated (as a Hertz algebra) by \( \varphi_A(A) \);
\((a_{15})\): For every Hertz algebra \( H \) and every morphism of Hilbert algebras \( f : A \rightarrow H \), there exists a unique morphism of Hertz algebras \( f' : H_A \rightarrow H \) such that \( f' \circ \varphi_A = f \).
Theorem 3.2. ([15]) For every Hilbert algebra $A$, there exists the free Hertz algebra $H_A$ over $A$, unique up to an isomorphism of Hertz algebras.

In what follows we only recall the construction of the Hertz algebra $H_A$ (using the model and notations from [15]).

Let $\mathcal{F}(A)$ the set of all finite and nonempty subsets of $A$ and $I = \{1\}$.

For $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, ..., y_n\} \in \mathcal{F}(A)$ we define

$$X \rightarrow Y = \bigcup_{1 \leq i \leq m} \{(x_1, x_2, ..., x_m; y_i)\}$$

and $X \wedge Y = X \cup Y$.

Consider the relation $\theta_A$ on $\mathcal{F}(A)$ defined for $X, Y \in \mathcal{F}(A)$ by

$$X \theta_A Y \text{ if and only if } X \rightarrow Y = Y \rightarrow X = I.$$

Then $\theta_A$ is an equivalence relation on $\mathcal{F}(A)$ compatible with the operations $\rightarrow$ and $\wedge$.

For $X \in \mathcal{F}(A)$ we denote by $[X]_{\theta_A}$ the equivalence class of $X$ modulo $\theta_A$ and by $H_A = \mathcal{F}(A)/\theta_A$.

For $a \in A$ we define $\varphi_A : A \rightarrow H_A, \varphi_A(a) = [\{a\}]_{\theta_A}$. Then $(H_A, \rightarrow, 1)$ is the free Hertz algebra over $A$ (where for $X, Y \in \mathcal{F}(A), [X]_{\theta_A} \rightarrow [Y]_{\theta_A} = [X \rightarrow Y]_{\theta_A}$).

Suppose that $X \rightarrow Y = 1$ for every $i = 1, 2, ..., m, j = 1, 2, ..., n \in X \subseteq [Y]$ and $y_i \in [X]$ for every $i = 1, 2, ..., m, j = 1, 2, ..., n \in Y \subseteq [X]$.

If $H$ is a Hertz algebra and $f : A \rightarrow H$ is a morphism of Hilbert algebras, then $f' : H_A \rightarrow H, f'([X]_{\theta_A}) = \bigwedge_{i=1}^{m} f(x_i) \ (X = \{x_1, x_2, ..., x_m\})$ is the unique morphism of Hertz algebras such that $f' \circ \varphi_A = f$.

For a Hilbert algebra $A$ I want to re-write the relation $\theta_A$ using the notation from Section 2.

So, we can consider an element $X = \{x_1, x_2, ..., x_m\} \in \mathcal{F}(A)$ as the word $X = x_1 x_2 ... x_n \in W(A)$ and for $a \in A, X \leftrightarrow a = (x_1, x_2, ..., x_n; a) \in A$.

Lemma 3.4. If $A$ is a Hilbert algebra, then $\rho_A = \theta_A$.

Proof. Clearly, for $X = \{x_1, x_2, ..., x_m\}, Y = \{y_1, ..., y_n\} \in \mathcal{F}(A), X \theta_A Y \text{ if and only if } X \rightarrow Y = Y \rightarrow X = 1$ for every $i = 1, 2, ..., m, j = 1, 2, ..., n \in X \subseteq [Y]$ and $y_i \in [X]$ for every $i = 1, 2, ..., m, j = 1, 2, ..., n \in Y \subseteq [X]$.

Theorem 3.3. If $A$ is a Hilbert algebra, then there exist an injective morphism of Hertz algebras $\Psi_A : H_A \rightarrow L_\varphi(A)$ such that $\Psi_A \circ \varphi_A = i_A$.

Proof. The existence of $\Psi_A : H_A \rightarrow L_\varphi(A)$ is assured by Theorem 3.2 and for $X = \{x_1, x_2, ..., x_m\} \in \mathcal{F}(A), \Psi_A([X]_{\theta_A}) = \bigwedge_{i=1}^{m} i_A(x_i)$.

To prove the injectivity of $\Psi_A$, consider $Y = \{y_1, ..., y_n\} \in \mathcal{F}(A)$ such that $\Psi_A([X]_{\theta_A}) = \Psi_A([Y]_{\theta_A}) \iff \bigwedge_{i=1}^{m} i_A(x_i) = \bigwedge_{j=1}^{n} i_A(y_j).$ Then for every $j = 1, 2, ..., n$:
A THEOREM OF REPRESENTATION FOR HILBERT ALGEBRAS

\[ \bigwedge_{i=1}^{m} i_A(x_i) \leq i_A(y_j) \Rightarrow (\bigwedge_{i=1}^{m} i_A(x_i)) \Rightarrow i_A(y_j) = 1 \Rightarrow (i_A(x_1), ..., i_A(x_m); i_A(y_j)) = 1 \Rightarrow i_A((x_1, ..., x_m); y_j) = 1 \Rightarrow [Y] \subseteq [X] \text{ and analogously} [X] \subseteq [Y], \text{ hence } [X] = [Y], \text{ that is, } \Psi_A \text{ is injective.} \]

Corollary 3.3. If \( A \) is a Hilbert algebra, then the free Hertz algebra \( H_A \) over \( A \) is isomorphic with a Hertz subalgebra of \( L_r(A) \).

References


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