On some \(1,3H_3\) - helicoidal surfaces and their parallel surfaces at a certain distance in 3 - dimensional Minkowski space

ALINA-MIHAEA PATRICIU

Abstract. The surface obtained by rotating a curve from the plane \((\xi_1; \xi_3)\) around the space-like axis \(\xi_3\), where \(\xi_1 = (1,0,0)\) and \(\xi_3 = (0,0,1)\), and simultaneously translating it along that axis is called \(1,3H_3\) - helicoidal surface. Let \(S\) and \(\tilde{S}\) be two surfaces and let \(\delta\) be a constant positive real number. \(S\) and \(\tilde{S}\) are parallel at distance \(\delta\) if for each point \(\tilde{P}\) in \(\tilde{S}\) we have \(\tilde{P}(u,v) = P(u,v) + \delta n(u,v)\), where \(n\) is the unit normal vector field on \(S\). In this paper we find some properties of some linear \(1,3H_3\) - helicoidal surfaces and of their parallel surfaces in 3 - dimensional Minkowski space \(\mathbb{R}^3_1\).

2010 Mathematics Subject Classification. Primary 53B30; Secondary 53A35.

Key words and phrases. Minkowski space, helicoidal surface, parallel surface.

1. Introduction

Let \(\mathbb{R}^3\) be a 3 - dimensional real vector space.

Definition 1.1. The 3 - dimensional Minkowski space is the pair \((\mathbb{R}^3, \langle , \rangle_1)\), denoted \(\mathbb{R}^3_1\), where the pseudo - inner product \(\langle , \rangle_1\) is given by

\[
\langle x, y \rangle_1 = x^t \eta y
\]

where \(x = (x_1, x_2, x_3)\), \(y = (y_1, y_2, y_3)\) and \(\eta = \text{diag}(-1,1,1)\).

Let \(\{\xi_1 = (1,0,0), \xi_2 = (0,1,0), \xi_3 = (0,0,1)\}\) be an orthonormal base of \(\mathbb{R}^3_1\), \(a(u) = (a(u),0,u)\) a curve from the plane \(\xi_1\xi_3\) and \(\beta(v) = (0,0,b(v))\) an arbitrary vector. If we rotate the curve around the space-like axis \(\xi_3\) and simultaneously translating it, we obtain the surface of equation:

\[
X(u,v) = (a(u) \cosh v, a(u) \sinh v, u + b(v)),
\]

which we have called in [3], \(1,3H_3\) - helicoidal surface.

In terms of a local parametrization \(P(u,v) = X(u,v)\) of surface \(S\), the coefficients \(\{E,F,G\}\) of the first and \(\{L,M,N\}\) of the second fundamental forms of surface \(S\), are given by

\[
E = \langle X_u, X_u \rangle_1, \quad F = \langle X_u, X_v \rangle_1, \quad G = \langle X_v, X_v \rangle_1, \quad L = -\langle n_u, X_u \rangle_1, \quad M = -\langle n_u, X_v \rangle_1, \quad N = -\langle n_v, X_v \rangle_1.
\]

Definition 1.2. A surface on which the Gaussian curvature is everywhere positive (negative) is called synclastic (respectively, anticlastic).

Received August 27, 2010. Revision received November 24, 2010.
The author thanks the anonymous referees for their useful, valuable and well - aimed remarks.
Definition 1.3. Let $S$ be an orientable surface and let $u$ be the unit normal vector field of $S$. The surface $\tilde{S}$ is parallel to $S$ at distance $\delta$ if the points $\tilde{P}(u, v) \in \tilde{S}$ are defined by

$$\tilde{P}(u, v) = P(u, v) + \delta n(u, v)$$

where $\delta$ is a constant positive real number.

In [4] we have proved:

Theorem 1.1. Let $S$ be a spacelike orientable surface with Gaussian curvature $K$ and mean curvature $H$ and let $\delta$ be a real positive constant such that $1 − 2\delta H − \delta^2 K \neq 0$. Then, the curvatures $\tilde{K}$ and $\tilde{H}$ of the surface $\tilde{S}$ parallel to $S$ at distance $\delta$ are given by:

$$\tilde{K} = \frac{K}{1 − 2\delta H − \delta^2 K} \quad \text{and} \quad \tilde{H} = \frac{H + \delta K}{1 − 2\delta H − \delta^2 K} \quad (4)$$

Theorem 1.2. Let $S$ be a timelike orientable surface with Gaussian curvature $K$ and mean curvature $H$ and let $\delta$ be a real positive constant such that $1 − 2\delta H + \delta^2 K \neq 0$. Then, the curvatures $\tilde{K}$ and $\tilde{H}$ of the surface $\tilde{S}$ parallel to $S$ at distance $\delta$ are given by:

$$\tilde{K} = \frac{K}{1 − 2\delta H + \delta^2 K} \quad \text{and} \quad \tilde{H} = \frac{H − \delta K}{1 − 2\delta H + \delta^2 K} \quad (5)$$

2. Some $1,3 H_3$ - helicoidal surfaces and their parallel surfaces in $\mathbb{R}^3$

For the $1,3 H_3$ - helicoidal surface given by (1) we have

$$X_u = (a'(u) \cosh v, a'(u) \sinh v, 1) \quad (6)$$

$$X_v = (a(u) \sinh v, a(u) \cosh v, b'(v)) \quad (7)$$

and so

$$X_u \wedge X_v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ a'(u) \cosh v & a'(u) \sinh v & 1 \\ a(u) \sinh v & a(u) \cosh v & b'(v) \end{vmatrix} = (a(u) \cosh v − a'(u)b'(v) \sinh v, a(u) \sinh v − a'(u)b'(v) \cosh v, a(u)a'(u)),$$

$$||X_u \wedge X_v|| = \sqrt{(a'^2(u) − 1)a^2(u) + a'^2(u)b'^2(v)}.$$
from where, the unit normal vector field of this surface is:

\[ n(u, v) = \left( \frac{u + B}{C} \cosh v - \sinh v, \frac{u + B}{C} \sinh v - \cosh v, u + B \right) \]  

(9)

Thus:

\[ n_u = \left( \frac{1}{C} \cosh v - \frac{1}{C}, \frac{1}{C} \sinh v - \frac{1}{C}, u \right) \]  

(10)

\[ n_v = \left( \frac{u + B}{C} \sinh v - \cosh v, \frac{u + B}{C} \cosh v - \sinh v, 0 \right) \]  

(11)

and the coefficients of the first fundamental form are:

\[ E = 0, \quad F = C, \quad G = \left( u + B \right)^2 + C^2 \]  

(12)

and those of the second fundamental form are:

\[ L = 0, \quad M = -1, \quad N = \frac{u(u - B \sinh v)}{B \cosh v} \]  

(13)

Since

\[ \langle n, n \rangle_1 = -\frac{(u + B)^2}{C} + 1 + \frac{(u + B)^2}{C} = 1 \]

it follows that \( n \) is spacelike and so \( S \) is timelike. Using the formulas

\[ K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{1}{2} \frac{EN - 2FM + GL}{EG - F^2} \]  

(14)

for the Gaussian and mean curvature of surface \( S \) we have:

\[ K = -\frac{1}{C^2} = \frac{1}{C^2} > 0 \]  

(15)

\[ H = \frac{1}{2} \frac{2C}{-C^2} = -\frac{1}{C} \]  

(16)

From here, the first property of a \( 1.3H_1 \)-helicoidal surface:

**Proposition 2.1.** In the conditions \( (** \))\), any \( 1.3H_3 \)-helicoidal surface is umbilical and synclastic.

**Proof.** Obviously, from (15) and (16), it follows \( H^2 = K \) and \( K > 0 \), which end the proof. \( \square \)

Using the definition of the parallel surface we obtain for the parallel surface to \( S \) at distance \( \delta \) the equations \( \tilde{X}(u, v) = (\tilde{x}(u, v), \tilde{y}(u, v), \tilde{z}(u, v)) \), where:

\[
\begin{align*}
\tilde{x}(u, v) &= (u + B) \cosh v + \delta \left( \frac{u + B}{C} \cosh v - \sinh v \right) \\
\tilde{y}(u, v) &= (u + B) \sinh v + \delta \left( \frac{u + B}{C} \sinh v - \cosh v \right) \\
\tilde{z}(u, v) &= u + Cv + D + \delta \frac{u + B}{C}
\end{align*}
\]

(17)

We can compute the Gaussian curvature and the mean curvature of this surface making similar computations as above, but, for simplicity, we will use Theorem 1.2
and we get:

\[
\tilde{K} = \frac{K}{1 - 2\delta H + \delta^2 K} = \frac{1}{C^2} \frac{1}{1 + 2\delta \frac{1}{C} + \delta^2 \frac{1}{C^2}}
\]

\[
= \frac{1}{C^2 + 2\delta C + \delta^2} = \frac{1}{C^2 + 2\delta C + \delta^2}
\]

from where:

\[
\tilde{K} = \frac{1}{(C + \delta)^2}
\]

(18)

and

\[
\tilde{H} = \frac{H - \delta K}{1 - 2\delta H + \delta^2 K} = \frac{-1}{C} - \frac{\delta}{C^2} \frac{1}{1 + 2\delta \frac{1}{C} + \delta^2 \frac{1}{C^2}}
\]

\[
= \frac{-(C + \delta)}{C^2 + 2\delta C + \delta^2},
\]

so

\[
\tilde{H} = -\frac{1}{C + \delta}
\]

(19)

From here, the second property of this surface is:

**Proposition 2.2.** The parallel surface to a $1^{1,3}H_3$ - helicoidal surface, in conditions (∗∗), at any distance $\delta$ with $1 - 2\delta H + \delta^2 K \neq 0$ is umbilical and synclastic.

**Proof.** $\tilde{H}^2 - \tilde{K} = 0$, for every $\delta$, so, $\tilde{S}$ is umbilical and $\tilde{K} > 0$, so $\tilde{S}$ is synclastic. □

For the case

\[
\text{(***)} \quad a'(u) = -1 \quad (a(u) = -u + B, b(v) = Cv + D),
\]

we have:

\[
X = ((-u + B) \cosh v, (-u + B) \sinh v, u + Cv + D),
\]

\[
X_u = (-\cosh v, -\sinh v, 1),
\]

\[
X_v = ((-u + B) \sinh v, (-u + B) \cosh v, C),
\]

\[
X_u \wedge X_v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ -\cosh v & -\sinh v & 1 \\ (-u + B) \sinh v & (-u + B) \cosh v & C \end{vmatrix},
\]

from where

\[
X_u \wedge X_v = (C \sinh v + (-u + B) \cosh v, C \cosh v + (-u + B) \sinh v, -(-u + B)),
\]

(20)

\[
||X_u \wedge X_v|| = C,
\]

(21)

\[
n(u, v) = \left( \sinh v + \frac{-u + B}{C} \cosh v, \cosh v + \frac{-u + B}{C} \sinh v, -\frac{-u + B}{C} \right)
\]

(22)

\[
n_u = \left( -\frac{1}{C} \cosh v, -\frac{1}{C} \sinh v, \frac{1}{C} \right)
\]

(23)

\[
n_v = \left( \cosh v + \frac{-u + B}{C} \sinh v, \sinh v + \frac{-u + B}{C} \cosh v, 0 \right)
\]

(24)

\[
E = 0, \quad F = C, \quad G = C^2 + (-u + B)^2
\]

(25)
$L = 0, \ M = -1, \ N = -\frac{(-u + B)^2}{C}$

$$\langle n, n \rangle_1 = 1 > 0,$$

so $S$ is timelike.

Using (14) we get:

$$K = \frac{-1}{C^2} = \frac{1}{C^2} > 0$$

and, thus

**Proposition 2.3.** Any $1.3H_3$ - helicoidal surface in conditions $(\ast\ast\ast)$ is umbilical and synclastic.

The proof is similar to the proof of Proposition 2.2, and thus, is omitted.

The parallel surface to this surface at distance $\delta$ will have the equations $\tilde{X}(u, v) = (\tilde{x}(u, v), \tilde{y}(u, v), \tilde{z}(u, v))$ where:

$$\left\{ \begin{array}{l}
\tilde{x}(u, v) = (-u + B) \cosh v + \delta \left( \sinh v + \frac{-u + B}{C} \cosh v \right)
\\
\tilde{y}(u, v) = (-u + B) \sinh v + \delta \left( \cosh v + \frac{-u + B}{C} \sinh v \right)
\\
\tilde{z}(u, v) = u + C v + D - \delta \frac{-u + B}{C}
\end{array} \right.$$

$$\tilde{K} = \frac{K}{1 - 2\delta H + \delta^2 K} = \frac{1}{C^2} \frac{1}{1 + 2\delta \frac{1}{C} + \delta^2 \frac{1}{C^2}} = \frac{1}{C^2} \frac{(C + \delta)^2}{C^2}$$

$$\tilde{H} = \frac{H - \delta K}{1 - 2\delta H + \delta^2 K} = \frac{-\frac{1}{C} - \delta \frac{1}{C^2}}{1 + 2\delta \frac{1}{C} + \delta^2 \frac{1}{C^2}} = -\frac{C + \delta}{(C + \delta)^2}$$

thus,

$$\tilde{K} = \frac{1}{(C + \delta)^2}, \tilde{H} = -\frac{1}{C + \delta}$$

so, obviously,

**Proposition 2.4.** The parallel surface to a $1.3H_3$ - helicoidal surface, given by (30), at any distance $\delta$ with $1 - 2\delta H + \delta^2 K \neq 0$ is umbilical.

From the positivity of the Gaussian curvature of each of these surfaces we have

**Proposition 2.5.** Any $1.3H_3$ - helicoidal surface and any parallel surface to it at any distance $\delta$ is synclastic.
References

[3] A.-M. Patriciu, Some minimal helicoidal surfaces in Minkowski space \( \mathbb{R}^3_1 \), *submitted*.
[4] A.-M. Patriciu, Parallel surfaces in 3-dimensional Minkowski space \( \mathbb{R}^3_1 \), *submitted*.

(Alina-Mihaela Patriciu) Department of Mathematics and Informatics, Faculty of Sciences, University "Vasile Alecsandri" of Bacău, 157 Calea Mărăşeşti, Bacău, 600115, Romania
E-mail address: alina.patriciu@ub.ro