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On some ${}^{1,3}H_3$ - helicoidal surfaces and their parallel surfaces at a certain distance in 3 - dimensional Minkowski space

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Abstract. The surface obtained by rotating a curve from the plane $(\xi_1\xi_3)$ around the spacelike axis ξ_3, where $\xi_1 = (1, 0, 0)$ and $\xi_3 = (0, 0, 1)$, and simultaneously translating it along that axis is called ${}^{1,3}H_3$ - helicoidal surface. Let S and $\widetilde{S}$ be two surfaces and let δ be a constant positive real number. S and $\widetilde{S}$ are parallel at distance δ if for each point $\widetilde{P} \in \widetilde{S}$ we have $\widetilde{P}(u, v) = P(u, v) + \delta n(u, v)$, where n is the unit normal vector field on S. In this paper we find some properties of some linear ${}^{1,3}H_3$ - helicoidal surfaces and of their parallel surfaces in 3 - dimensional Minkowski space $\mathbb{R}^3_1$.

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1. Introduction

Let $\mathbb{R}^3$ be a 3 - dimensional real vector space.

Definition 1.1. *The 3 - dimensional Minkowski space is the pair* $\left(\mathbb{R}^3, \langle , \rangle_1\right)$, *denoted* $\mathbb{R}^3_1$, *where the pseudo - inner product* $\langle \cdot, \cdot \rangle_1$ *is given by*

$$\langle x, y \rangle_1 = x^t \eta y$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ *and* $\eta = diag(-1, 1, 1)$.

Let $\{\xi_1 = (1, 0, 0),\ \xi_2 = (0, 1, 0),\ \xi_3 = (0, 0, 1)\}$ be an orthonormal base of $\mathbb{R}^3_1$, $\alpha(u) = (a(u), 0, u)$ a curve from the plane $\xi_1\xi_3$ and $\beta(v) = (0, 0, b(v))$ an arbitrary vector. If we rotate the curve around the spacelike axis ξ_3 and simultaneously translating it, we obtain the surface of equation:

$$X(u, v) = (a(u) \cosh v, a(u) \sinh v, u + b(v)), \tag{1}$$

which we have called in [3], ${}^{1,3}H_3$ - helicoidal surface.

In terms of a local parametrization $P(u, v) = X(u, v)$ of surface S, the coefficients $\{E, F, G\}$ of the first and $\{L, M, N\}$ of the second fundamental forms of surface S, are given by

$$E = \langle X_u, X_u \rangle_1, F = \langle X_u, X_v \rangle_1, G = \langle X_v, X_v \rangle_1, \tag{2}$$

$$L = -\langle n_u, X_u \rangle_1, M = -\langle n_u, X_v \rangle_1 = -\langle n_v, X_u \rangle_1, N = -\langle n_v, X_v \rangle_1. \tag{3}$$

Definition 1.2. *A surface on which the Gaussian curvature is everywhere positive (negative) is called synclastic (respectively, anticlastic).*

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Definition 1.3. *Let S be an orientable surface and let n be the unit normal vector field of S. The surface $\widetilde{S}$ is parallel to S at distance δ if the points $\widetilde{P}(u,v) \in \widetilde{S}$ are defined by*

$$\widetilde{P}(u,v) = P(u,v) + \delta n(u,v)$$

where δ is a constant positive real number.

In [4] we have proved:

Theorem 1.1. *Let S be a spacelike orientable surface with Gaussian curvature K and mean curvature H and let δ be a real positive constant such that $1 - 2\delta H - \delta^2 K \neq 0$. Then, the curvatures $\widetilde{K}$ and $\widetilde{H}$ of the surface $\widetilde{S}$ parallel to S at distance δ are given by:*

$$\widetilde{K} = \frac{K}{1 - 2\delta H - \delta^2 K} \text{ and } \widetilde{H} = \frac{H + \delta K}{1 - 2\delta H - \delta^2 K} \tag{4}$$

Theorem 1.2. *Let S be a timelike orientable surface with Gaussian curvature K and mean curvature H and let δ be a real positive constant such that $1 - 2\delta H + \delta^2 K \neq 0$. Then, the curvatures $\widetilde{K}$ and $\widetilde{H}$ of the surface $\widetilde{S}$ parallel to S at distance δ are given by:*

$$\widetilde{K} = \frac{K}{1 - 2\delta H + \delta^2 K} \text{ and } \widetilde{H} = \frac{H - \delta K}{1 - 2\delta H + \delta^2 K} \tag{5}$$

2. Some ${}^{1,3}H_3$ - helicoidal surfaces and their parallel surfaces in $\mathbb{R}^3_1$

For the ${}^{1,3}H_3$ - helicoidal surface given by (1) we have

$$X_u = (a'(u)\cosh v, a'(u)\sinh v, 1) \tag{6}$$

$$X_v = (a(u)\sinh v, a(u)\cosh v, b'(v)) \tag{7}$$

and so

$$X_u \wedge X_v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ a'(u)\cosh v & a'(u)\sinh v & 1 \\ a(u)\sinh v & a(u)\cosh v & b'(v) \end{vmatrix}$$
$$= (a(u)\cosh v - a'(u)b'(v)\sinh v, a(u)\sinh v - a'(u)b'(v)\cosh v, a(u)a'(u)),$$

$$\|X_u \wedge X_v\| = \sqrt{(a'^2(u) - 1)a^2(u) + a'^2(u)b'^2(v)}.$$

We will study only the case:
$(*)$ $a'^2(u) = 1$
where $a(u)$ and $b(v)$ are linear functions.
In the first case:
$(**)$ $a'(u) = 1$ $(a(u) = u + B,\ b(v) = Cv + D)$,
we have successively:

$$X_u = (\cosh v, \sinh v, 1),$$

$$X_v = ((u+B)\sinh v, (u+B)\cosh v, C),$$

$$X_u \wedge X_v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ \cosh v & \sinh v & 1 \\ (u+B)\sinh v & (u+B)\cosh v & C \end{vmatrix}$$
$$= ((u+B)\cosh v - C\sinh v, (u+B)\sinh v - C\sinh v, u+B)$$

and

$$\|X_u \wedge X_v\| = C, \tag{8}$$

from where, the unit normal vector field of this surface is:

$$n(u,v) = \left(\frac{u+B}{C}\cosh v - \sinh v, \frac{u+B}{C}\sinh v - \cosh v, \frac{u+B}{C}\right) \tag{9}$$

Thus:

$$n_u = \left(\frac{\cosh v}{C}, \frac{\sinh v}{C}, \frac{1}{C}\right) \tag{10}$$

$$n_v = \left(\frac{u+B}{C}\sinh v - \cosh v, \frac{u+B}{C}\cosh v - \sinh v, 0\right) \tag{11}$$

and the coefficients of the first fundamental form are:

$$E = 0,\ F = C,\ G = (u+B)^2 + C^2 \tag{12}$$

and those of the second fundamental form are:

$$L = 0,\ M = -1,\ N = \frac{u(u - B\sinh v)}{B\cosh v}. \tag{13}$$

Since

$$\langle n, n\rangle_1 = -\frac{(u+B)^2}{C} + 1 + \frac{(u+B)^2}{C} = 1$$

it follows that n is spacelike and so S is timelike. Using the formulas

$$K = \frac{LN - M^2}{EG - F^2}, H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2} \tag{14}$$

for the Gaussian and mean curvature of surface Swe have:

$$K = \frac{-1}{-C^2} = \frac{1}{C^2} > 0 \tag{15}$$

$$H = \frac{1}{2}\frac{2C}{-C^2} = -\frac{1}{C} \tag{16}$$

From here, the first property of a $^{1,3}H_1$-helicoidal surface:

Proposition 2.1. *In the conditions* (∗∗), *any* $^{1,3}H_3$- *helicoidal surface is umbilical and synclastic.*

Proof. Obviously, from (15) and (16), it follows $H^2 = K$ and $K > 0$, which end the proof. □

Using the definition of the parallel surface we obtain for the parallel surface to S at distance δ the equations $\widetilde{X}(u,v) = (\widetilde{x}(u,v), \widetilde{y}(u,v), \widetilde{z}(u,v))$, where:

$$\begin{cases} \widetilde{x}(u,v) = (u+B)\cosh v + \delta\left(\dfrac{u+B}{C}\cosh v - \sinh v\right) \\ \widetilde{y}(u,v) = (u+B)\sinh v + \delta\left(\dfrac{u+B}{C}\sinh v - \cosh v\right) \\ \widetilde{z}(u,v) = u + Cv + D + \delta\dfrac{u+B}{C} \end{cases} \tag{17}$$

We can compute the Gaussian curvature and the mean curvature of this surface making similar computations as above, but, for simplicity, we will use Theorem 1.2

and we get:

$$\widetilde{K} = \frac{K}{1 - 2\delta H + \delta^2 K} = \frac{\frac{1}{C^2}}{1 + 2\delta \frac{1}{C} + \delta^2 \frac{1}{C^2}}$$

$$= \frac{\frac{1}{C^2}}{\frac{C^2 + 2\delta C + \delta^2}{C^2}} = \frac{1}{C^2 + 2\delta C + \delta^2}$$

from where:

$$\widetilde{K} = \frac{1}{(C+\delta)^2} \tag{18}$$

and

$$\widetilde{H} = \frac{H - \delta K}{1 - 2\delta H + \delta^2 K} = \frac{-\frac{1}{C} - \delta \frac{1}{C^2}}{1 + 2\delta \frac{1}{C} + \delta^2 \frac{1}{C^2}}$$

$$= \frac{-(C+\delta)}{C^2 + 2\delta C + \delta^2},$$

so

$$\widetilde{H} = -\frac{1}{C+\delta} \tag{19}$$

From here, the second property of this surface is:

Proposition 2.2. *The parallel surface to a* $^{1,3}H_3$ *- helicoidal surface, in conditions* $(**)$*, at any distance* δ *with* $1 - 2\delta H + \delta^2 K \neq 0$ *is umbilical and synclastic.*

Proof. $\widetilde{H}^2 - \widetilde{K} = 0$, for every δ, so, $\widetilde{S}$ is umbilical and $\widetilde{K} > 0$, so $\widetilde{S}$ is synclastic. □

For the case
$(***)$ $a'(u) = -1$ $(a(u) = -u + B, b(v) = Cv + D)$,
we have:

$$X = ((-u+B)\cosh v, (-u+B)\sinh v, u + Cv + D),$$

$$X_u = (-\cosh v, -\sinh v, 1),$$

$$X_v = ((-u+B)\sinh v, (-u+B)\cosh v, C),$$

$$X_u \wedge X_v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ -\cosh v & -\sinh v & 1 \\ (-u+B)\sinh v & (-u+B)\cosh v & C \end{vmatrix},$$

from where

$$X_u \wedge X_v = \left(C\sinh v + (-u+B)\cosh v, C\cosh v + (-u+B)\sinh v, -(-u+B)\right), \tag{20}$$

$$\|X_u \wedge X_v\| = C, \tag{21}$$

$$n(u,v) = \left(\sinh v + \frac{-u+B}{C}\cosh v, \cosh v + \frac{-u+B}{C}\sinh v, -\frac{-u+B}{C}\right) \tag{22}$$

$$n_u = \left(-\frac{1}{C}\cosh v, -\frac{1}{C}\sinh v, \frac{1}{C}\right) \tag{23}$$

$$n_v = \left(\cosh v + \frac{-u+B}{C}\sinh v, \sinh v + \frac{-u+B}{C}\cosh v, 0\right) \tag{24}$$

$$E = 0,\ F = C,\ G = C^2 + (-u+B)^2 \tag{25}$$

$$L = 0,\ M = -1,\ N = -\frac{(-u+B)^2}{C} \tag{26}$$

$$\langle n, n\rangle_1 = 1 > 0, \tag{27}$$

so S is timelike.

Using (14) we get:

$$K = \frac{-1}{-C^2} = \frac{1}{C^2} > 0 \tag{28}$$

$$H = \frac{1}{2}\frac{2C}{-C^2} = -\frac{1}{C} \tag{29}$$

and, thus

Proposition 2.3. *Any $^{1,3}H_3$ - helicoidal surface in conditions $(***)$ is umbilical and synclastic.*

The proof is similar to the proof of Proposition 2.2, and thus, is omitted.

The parallel surface to this surface at distance δ will have the equations $\widetilde{X}(u,v) = (\widetilde{x}(u,v), \widetilde{y}(u,v), \widetilde{z}(u,v))$ where:

$$\begin{cases} \widetilde{x}(u,v) = (-u+B)\cosh v + \delta\left(\sinh v + \dfrac{-u+B}{C}\cosh v\right) \\ \widetilde{y}(u,v) = (-u+B)\sinh v + \delta\left(\cosh v + \dfrac{-u+B}{C}\sinh v\right) \\ \widetilde{z}(u,v) = u + Cv + D - \delta\dfrac{-u+B}{C} \end{cases} \tag{30}$$

$$\widetilde{K} = \frac{K}{1-2\delta H+\delta^2 K} = \frac{\frac{1}{C^2}}{1+2\delta\frac{1}{C}+\delta^2\frac{1}{C^2}} = \frac{\frac{1}{C^2}}{\frac{(C+\delta)^2}{C^2}}$$

$$\widetilde{H} = \frac{H-\delta K}{1-2\delta H+\delta^2 K} = \frac{-\frac{1}{C}-\delta\frac{1}{C^2}}{1+2\delta\frac{1}{C}+\delta^2\frac{1}{C^2}} = -\frac{\frac{C+\delta}{C^2}}{\frac{(C+\delta)^2}{C^2}}$$

thus,

$$\widetilde{K} = \frac{1}{(C+\delta)^2}, \widetilde{H} = -\frac{1}{C+\delta},$$

so, obviously,

Proposition 2.4. *The parallel surface to a $^{1,3}H_3$ - helicoidal surface, given by (30), at any distance δ with $1-2\delta H+\delta^2 K \neq 0$ is umbilical..*

From the positivity of the Gaussian curvature of each of these surfaces we have

Proposition 2.5. *Any $^{1,3}H_3$ - helicoidal surface and any parallel surface to it at any distance δ is synclastic.*

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