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Representations for certain crossed simplicial groups generated by braided Hopf algebras

Gefry Barad

Abstract. We find solutions of a nonlinear equation which provide representations for the new groups R(n) defined in [1]. The groups are crossed simplicial groups in the sense of Loday [8] (Sec.6.3). These solutions are based on the Bulacu and Beattie construction of braided Hopf algebras in the category of Yetter-Drinfeld modules [2]. We discuss the connection of these solutions with Hopf equation in a braided monoidal category (a system of mixed Yang-Baxter type equations presented in [1]). Generalized quantum doubles for pairs of Hopf algebras can afford weak projections.

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1. Introduction

The Drinfeld double for a Hopf algebra plays an important role in solving a non-linear equation (QYBE) which provide representations for the Braid groups of type A. Majid [9]referred to it as a toy model for quantum mechanics and it is an example of algebra factorisation i.e. using certain structures, the multiplication on tensor product of two algebras is modified [14]. Following a similar line of research, we begin with a generalisation of Drinfeld double construction due to Bulacu and Beattie. In same cases, generalized Drinfeld doubles are Hopf algebras with a projection, so there are Radford biproducts with a Hopf algebra in the category of Yetter-Drinfeld modules.

According to Theorem 4.2 [1], any braided Hopf algebra as above has a fusion operator which satisfy the braided Hopf equation, introduced in [1]. The classical Hopf or pentagonal equation was studied by Militaru [6]. Representations for certain groups are given by braided Hopf equation, so the braided Hopf algebras of Bulacu and Beattie give representations for these groups. In the present paper we prove that the same algebraic data, *under some relaxed assumptions*, give different representations. We study the connection between these two solutions of a non-linear equation.

The groups R(n) form the algebraic structure of an operad. Menichi proved [10] that any *cyclic* operad with multiplication gives a cyclic module . Loday and Fiedorowicz introduced the concept of crossed simplicial groups and stated a classification theorem for them in 1991. The groups R(n) are crossed simplicial groups, so it is possible to study them in the context of the classical non-commutative geometry and cyclic homology. As a further line of research, we open the problem to *quantize*

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these groups at the level of Lie algebras, as the Braid groups of type A and B fit into the theory of Drinfeld associators, Malcev completions and Lie algebras [7].

2. Generalized Quantum Doubles which are Hopf algebras with a projection

We review after [2] (Sec.2 and 3) the construction of the generalized guantum double associated with two Hopf algebras in pairing, of Bulacu and Beattie , and the conditions have to be fullfilled for these Hopf algebras to be Hopf algebras with a projection. For convenience, we particularize certain constructions associated with two bialgebras in the case of one Hopf algebra with bijective antipode S. A Hopf algebra endomorphism p is a projection if p(p(x))=p(x) and its image is a Hopf algebra.

Let A be a Hopf algebra. A 2-cocycle is a convolution invertible bilinear form
t: $A \otimes A \to k$ which satisfies for any a,b,c $\in$ A:
$t(x \otimes 1) = \varepsilon(x)$;
$t(a_1 \otimes b_1)t(a_2 b_2 \otimes c) = t(b_1 \otimes c_1)t(a \otimes b_2 c_2)$;
$t(1 \otimes x) = \varepsilon(x)$
Given (A,t) a Hopf algebra with a 2-cocycle, on A a new product $\star$ is defined and the axioms for t imply that the new multiplication is associative and unital:

$$a \star b = t(a_1 \otimes b_1) a_2 b_2 t^{-1}(a_3 \otimes b_3)$$

Let H be a Hopf algebra. A pairing is a convolution invertible bilinear r: $H \otimes H \to k$ which satisfy the following axioms:
$r(x \otimes 1) = \varepsilon(x)$;
$r(xy \otimes z) = \sum r(x \otimes z_{(1)}) r(y \otimes z_{(2)})$;
$r(1 \otimes x) = \varepsilon(x)$;
$r(x \otimes yz) = \sum r(x_{(1)} \otimes y) r(x_{(2)} \otimes z)$; for any $x, y, z \in H$.
In this case, the Hopf algebra A=$H \otimes H$ has a 2-cocycle t defined as:

$$t((a, x) \otimes (b, y)) = \varepsilon(ay) r(b \otimes x)$$

The new bialgebra D(H,r)=(A, $\star$) with the comultiplication of $H \otimes H$ is a Hopf algebra whose antipode is T, $T(a \otimes x) = r(S(a_3), S(x_3)) S(a_2) \otimes S(x_2) r^{-1}(S(a_1), S(x_1))$.

There are two Hopf subalgebras of A, isomorphic with H and given by the canonical inclusions of H in A, $x \to x \otimes 1$ and $x \to 1 \otimes x$. We are interested in the first inclusion i.

Theorem 2.1. *(Proposition 3.1* [2]*)*
There exists a bialgebra projection π from D(H,r) to H that splits i if and only if there is a bialgebra endomorphism ϕ of H which satisfy ((3.1) and (3.3) Remark 3.4 from [2]*):*

$$\sum r(x_{(1)} \otimes y_{(1)}) x_{(2)} \phi(y_{(2)}) = \sum r(x_{(2)} \otimes y_{2)}) \phi(y_{(1)}) x_{(1)}$$

In this case, $\pi(x \otimes y) = x\phi(y)$.

Let C be a strict braided monoidal category, with braiding c (Definition 4.6 [18], [17]). Let B be a braided Hopf algebra in C. So, B is an object in this category, togeter with a comultiplication δ: $B \to B \otimes B$, multiplication m, unit and counit ϵ which are morphisms in C, which satisfy the usual axioms for a Hopf algebra. δ is a braided algebra morphism:

$$\delta(xy) = (m \otimes m)(id \otimes c \otimes id)(\delta(x) \otimes \delta(y))$$

We can define a fusion operator $T : B \otimes B \to B \otimes B$

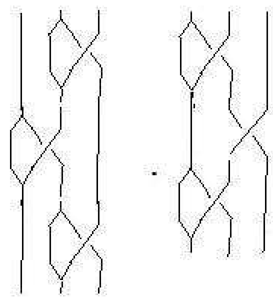
$$T(x \otimes y) = (m \otimes id) \circ (id \otimes c) \circ (\delta \otimes id).$$

We consider the following system. The indices show the positions on tensor product of three objects where the operators act.

$$R(23)R(12)R(23) = R(12)c(23)R(12)$$

$$R(23)c(12)c(23) = c(12)c(23)R(12)$$

The first equation of the system (called [1] the Hopf equation in the braided category C), has the following diagramatic form:



The braiding morphism is represented diagramatically as a crossing; the trivalent graphs represent multiplication or comultiplication. The composition of morphisms are read from top to the bottom of the figure.

Theorem 2.2. *(Theorem 4.2,* [1]*)*
The Hopf equation is verified for (T,c), where c is a braiding of a braided monoidal category, and T is the fusion operator associated with a braided Hopf algebra B in this category.

Theorem 2.3. *(Propositions 3.7 and 3.8* [2]*)*
Given a triple (H,r,ϕ) which satisfy the conditions from the theorem 2.1 above, the vector space H has a structure of a Hopf algebra in the braided monoidal category of left-left Yetter-Drinfeld modules over H.

In the next section, we will relax some of the properties of the triple (H,r,ϕ). In the same way the data above generate Radford biproduct and braided Hopf algebras, this generalization will fit into the theory of Hopf algebras with a weak projection.

3. Hopf algebras with a weak projections

Let H be a Hopf algebra with invertible antipode S. p:$H \to H$ is called a weak projection if p is a coalgebra map, if p(p(x))=p(x), if p(ap(x))=p(a)p(x) and if $p \circ S = S \circ p$. Im(p)=H'is a sub-Hopf algebra of H.

Let B be the set of elements x from H, such that $x_1 \otimes p(x_2) = x \otimes 1$.

We present after Schauenburg [15] and Stefan [16] the following structures of the Hopf algebras with a weak projection (H,p).

Let H'=Im(p) and i is its inclusion in H; let B be the set of elements x $\in$ A such that $x_1 \otimes p(x_2) = x \otimes 1$.

Let q(x)=$x_1 p(S(x_2))$ $\in$ B.

1) a product $\star$ on B, not necessary associative, defined as x $\star$ y =q(xy);

2) a comultiplication on B, $\Delta_R(x) = q(x_1) \otimes q(x_2) = x_{[1]} \otimes x_{[2]}$;

3) a cocycle t: $B \otimes B \to H'$, t(x,y)=p(xy);

4) an action $H' \otimes B \to B$, denoted b $\to$ r = q(br);

5) a map $H' \otimes B \to H'$, b $\leftarrow$ r = p(br);

6) a left coaction $B \to H' \otimes B$, $\rho(x) = p(x_1) \otimes x_2 = x_{(-1)} \otimes x_{(o)}$.

Theorem 3.1. *(Theorem 2.12* [16]*, Theorem 5.1* [15]*)*

On vector space $B \otimes H'$ there are the following multiplication and comultiplication maps, such that the application $B \otimes H' \to H$ given by (b,h)$\to$ bh is a bialgebra isomorphism:

$$(q \otimes y)(p \otimes x) = (q_{[1]} \star (q_{[2](-1)} y_1 \to p_{[1]}) \otimes t(q_{[2](o)}, y_2 \to p_{[2]})(y_3 \leftarrow p_{[3]})x).$$

The comultiplication on $B \otimes H'$ is given by $\Delta(b, h) = \Delta(b, 1)\Delta(1, h)$.

The comultiplication of h is given by the Hopf algebra structure of Im(p) and $\Delta(b, 1) = (b_{[1]}, b_{[2](-1)} \otimes (b_{[2](o)}, 1)$.

There is also a converse of the theorem above, which says that a Hopf algebra H', a coalgebra B and the six maps above generate on the vector space A=$R \otimes B$ a structure of a Hopf algebra with a weak projection onto B if a long list of relations among them is fullfiled.

Remark 3.1. *Schauenburg uses the following definition of a Hopf algebra with a weak projection: p(p(x)y)=p(x)p(y); Stefan proved that the statements above are true in any braided monoidal category. If we work with regular Hopf algebras, (H,p) is a weak algebra as above if and only if (H_{op}, p) is a Hopf algebra with a weak projection in the sense of Schauenburg. B has to be replaced by the set B' of elements $x \in H$ such that $p(x_1) \otimes x_2 = 1 \otimes x$ and H will be isomorphic with $H' \otimes B'$.*

The coaction will be to the right $B \to B \otimes H', \rho(x) = x_1 \otimes p(x_2) = x_{(o)} \otimes x_{(1)}$ and q(x)=$p(S(x_1))x_2 \in B$.

3.1. Applications. Let (H,r) a Hopf algebra with bijective antipode and with a pairing.Let ϕ be a map from H to H. We will use Sweedler notation and for convenience we supress the symbol for tensor product and multiple sums.

Lemma 3.1. *If ϕ is a coalgebra map, then*

$$\pi : D(H, r) \to D(H, r), \pi(x \otimes y) = x\phi(y) \otimes 1$$

is a weak projection: it is a coalgebra endomorphism and $\pi(\pi(x)y) = \pi(x)\pi(y)$.

Proof. π is a coalgebra map because it is a composition of coalgebra maps.

If x =(a,b) and y = (c,d), then π(x)π(y)=($a\phi(b)c\phi(d) \otimes 1$)

$\pi(\pi(x)y) = \pi((a\phi(b) \otimes 1)(c \otimes d))= \pi((a\phi(b)c \otimes d) = a\phi(b)c\phi(d) \otimes 1$ ☐

Lemma 3.2. *If ϕ is a coalgebra map, then $\pi : D(H, r) \to D(H, r), \pi(x \otimes y) = x\phi(y) \otimes 1$ satisfies: π(yπ(x))=π(y)π(x) if and only if*

$$\sum r(x_{(1)} \otimes y_{(1)})x_{(2)}\phi(y_{(2)}) = \sum r(x_{(2)} \otimes y_{2)})\phi(y_{(1)})x_{(1)}$$

for any x and y $\in$ H.

Proof. y=(c,d); x=(a,b)

$\pi(y\pi(x))=\pi(y)\pi(x) \Leftrightarrow \pi[(c,d)(a\phi(b),1)] = (c\phi(d)a\phi(b),1) \Leftrightarrow$
$\pi[cr(a_1\phi(b_1),d_1)a_2\phi(b_2)\otimes d_2r^{-1}(a_3\phi(b_3),d_3)] = (c\phi(d)a\phi(b),1) \Leftrightarrow$
$cr(a_1\phi(b_1),d_1)a_2\phi(b_2)\phi(d_2)r^{-1}(a_3\phi(b_3),d_3) = c\phi(d)a\phi(b) \Leftrightarrow$
$cr(a_1\phi(b_1),d_1)a_2\phi(b_2)\phi(d_2) = c\phi(d_1)a_1\phi(b_1)r(a_2\phi(b_2),d_2)$;

we used the distorted multiplication of D(H,r), as defined in the previous section or formula 2.10 ([2]).

If this equality is true for any a,b,c,d, for c=b=1_H we get the relation stated in the lemma.

If

$$\sum r(x_{(1)}\otimes y_{(1)})x_{(2)}\phi(y_{(2)}) = \sum r(x_{(2)}\otimes y_{2)})\phi(y_{(1)})x_{(1)}$$

then for y=d and x=$a\phi(b)$ we get the equality equivalent to the relation

$$\pi(y\pi(x)) = \pi(y)\pi(x).$$

□

Let D= $D(H,r,\phi)$ a triple where ϕ is a coalgebra map. $D(H,r,\phi)$ is a Hopf algebra with a left weak projection.

We suppose ϕ and the antipode of H commute.

Let B= $\{x\in D|p(x_1)\otimes x_2 = 1\otimes x\} = \{\pi(S(x_1))x_2, x\in D\}$

Let R= $\{x\in D|x_1\otimes p(x_2) = x\otimes 1\} = \{x_1\pi(S(x_2)), x\in D\}$

Remark 3.2. *The antipode of D and its inverse (as any bijective anti-coalgebra map which commute with the projection π) is a bijection between B and R.*

In our special case, a better description is possible, according to Prop. 3.6 ([2]):

B= $\{\phi(S(x_1))\otimes x_2, x\in H\}$

R= $\{\phi(S(x_2))\otimes x_1, x\in H\}$

D will be isomorphic with $H\otimes B$, where the multiplication on $H\otimes B$ is defined as ([15] Thm.5.1):

$$(x\otimes p)(y\otimes q) = (x(p_1\rightarrow y_1)t(p_2\leftarrow y_2, q_{1[o]})\otimes(p_3\leftarrow y_3q_{1[1]})\star q_2).$$

Remark 3.3. *t: $B\otimes B\rightarrow H$, t(x,y)=π(xy). Using the isomorphism above*

$$(1,x)(1,y) = (t(x_1,y_1), x_2\star y_2).$$

In our case, $t(\phi(S(x_1))\otimes x_2, \phi(S(y_1))\otimes y_2) = \pi(\phi(S(x_1))\otimes x_2)$.

$$(\phi(S(y_1)\otimes y_2)) = r(\phi(S(y_3)),x_2)\phi(S(x_1)\phi(S(y_2)\phi(x_3y_4)r^{-1}(\phi(S(y_1)),x_4)$$

The cocycle is trivial i.e. t(x,y)= $\varepsilon(xy)$ if and only if:

$$r(\phi(b_1),d_1)\phi(b_2d_2) = \phi(d_1)\phi(b_1)r(\phi(b_2),d_2).$$

In particular, if H is co-commutative, non-commutative, and ϕ is a Hopf algebra anti-morphism, the relation above is respected and π is not an algebra map.

Remark 3.4. *Let D be a Hopf algebra with a left weak projection as above.*

Let $A = \{h = p[S(c)p(S(x_1b))x_2], b,c\in R, x\in D\}$

$$h = p[S(c)p(S(x_1b))x_2] = p[S(c)p(S(x_1b_1))x_2b_2S(b_3)]$$

If $b \in R$, then $b_1 \otimes b_2 \otimes b_3 \in D \otimes D \otimes R$. Following remark 3.2, h=p(uyz), where u,y,z $\in B$. $u = S(c), y = p(S(x_1 b_1))x_2 b_2, z = S(b_3)$

Following remark 3.3, h=p(uyz) is a product of two cocycles. In general the projection π of a product of elements from B is a product of cocycles.

In particular, if the cocycle is trivial ([15] Sect.6.3-6.6) the elements of A are scalar multiples of 1_D.

Let (D, p) be a Hopf algebra with bijective antipode and left weak projection (p(x)p(y)=p(p(x)y)), and trivial cocycle. For example D=D(H,r, ϕ, coalgebra map) and $r(\phi(b_1), d_1)\phi(b_2 d_2) = \phi(d_1)\phi(b_1)r(\phi(b_2), d_2)$; for any b and d. In the terminology of ([15] Sect.6.3), D is a trivalent product, a concept which unifies Majid's matched pairs of bialgebras, bicrossed products and double crossed products. Any coalgebra map ϕ which is a anti-morphism of algebras for H co-commutative satisfies the condition above.

Let X $=\{x_1 p(S(x_2)), x \in D\}$. Under these assumptions we have the following theorem:

Theorem 3.2. *Consider the following map $R : X \otimes X \to X \otimes X$*

$R(a \otimes b) = a_1 b_1 p(S(a_2 b_2)) \otimes a_3$ *and its inverse*

$T(x \otimes y) = y_3 \otimes S^{-1}(y_2) x_1 p(S(S^{-1}(y_1) x_2))$. *Then R satisfies the following equation on*

$X \otimes X \otimes X \otimes X : R_{34} R_{23} R_{12} R_{34} R_{23} R_{34} T_{23} T_{34} T_{12} T_{23} = R_{23} R_{12} R_{23} T_{12} T_{23} R_{34}$

Proof. The equation above represents the equality of two maps defined $X \otimes X \otimes X \otimes X \to X \otimes X \otimes X \otimes X$.

We evaluate both terms on $\sum a \otimes b \otimes c \otimes d$. We suppress the symbols for tensor products and sums.

For the left hand side:

$$
\begin{gathered}
(a, b, c, d) \to (a, b, c_1 d_1 p(S(c_2 d_2), c_3) \to (a, b_1 c_1 d_1 p(S(b_2 c_2 d_2)), b_3, c_3) \to \\
\to (a_1 b_1 c_1 d_1 p(S(a_2 b_2 c_2 d_2)), a_3, b_3, c_3) \to \\
\to (a_1 b_1 c_1 d_1 p(S(a_2 b_2 c_2 d_2)), a_3, b_3 c_3 p(S(b_4, c_4)), b_5) \to \\
\to (a_1 b_1 c_1 d_1 p(S(a_2 b_2 c_2 d_2)), a_3 b_3 c_3 p(S(a_4 b_4 c_4)), a_5, b_5) \to \\
\to (a_1 b_1 c_1 d_1 p(S(a_2 b_2 c_2 d_2)), a_3 b_3 c_3 p(S(a_4 b_4 c_4)), a_5 b_5 p(S(a_6 b_6)), a_7) \to \\
\to (a_1 b_1 c_1 d_1 p(S(a_2 b_2 c_2 d_2)), a_3 b_3 p(S(a_4 b_4)), p(a_5 b_5) c_3 p(S(a_6 b_6)), a_7) \to \\
\to (a_1 b_1 p(S(a_2 b_2)), p(a_3 b_3) c_1 d_1 p(S(c_2 d_2) p(S(a_4 b_4))), p(a_5 b_5) c_3 p(S(a_6 b_6)), a_7) \to \\
(a_1 b_1 p(S(a_2 b_2)), p(a_3 b_3) c_1 d_1 p(S(c_2 d_2) p(S(a_4 b_4))), a_9, \\
S^{-1}(a_8) p(a_5 b_5) c_3 p(S(c_4) p(S(a_6 b_6)) a_7)) \to
\end{gathered}
$$

$$
(a_1 b_1 p(S(a_2 b_2)), a_{11}, S^{-1}(a_{10}) p(a_3 b_3) c_1 d_1 p(S[p(a_4 b_4) c_2 d_2] a_9), S^{-1}(a_8) p(a_5 b_5) c_3 p(S(c_4) p(S(a_6 b_6)) a_7))
$$

The right hand side is equal to:

$$
\begin{gathered}
(a, b, c, d) \to (a, b_1 c_1 p(S(b_2 c_2), b_3, d) \to (a_1 b_1 c_1 p(S(a_2 b_2 c_2)), a_3, b_3, d) \to \\
\to (a_1 b_1 c_1 p(S(a_2 b_2 c_2)), a_3 b_3 p(S(a_4 b_4)), a_5, d) \to \\
\to (a_1 b_1 p(S(a_2 b_2), p(a_3 b_3) c_1 p[S(c_2) p(S(a_4 b_4))], a_5, d) \to \\
\to (a_1 b_1 p(S(a_2 b_2), a_7, S^{-1}(a_6) p(a_3 b_3) c_1 p(S[p(a_4 b_4) c_2] a_5), d) \to \\
\to (a_1 b_1 p(S(a_2 b_2)), a_{15}, w, S^{-1}(a_1 2) p(a_5 b_5) c_3 p(S(c_4) p(S(a_6 b_6)) a_{11})),
\end{gathered}
$$

where w=

$$
S^{-1}(a_{14}) p(a_3 b_3) c_1 p(S[p(a_8 b_8) c_6] a_9) d_1 p[S(d_2) p(S(a_{10}) S^2[p(a_7 b_7) c_5]) S(c_2) p(S(a_4 b_4)) a_{13}]
$$

If T=$p(S[p(a_7b_7)c_5]a_8)$, then $T_1 \otimes p(WS(T_2)Q)$ appears in the expression above, and does not appear in the left hand side expression.

$$T_1 = p(S[p(a_8b_8)c_6]a_9) \text{ and } S(T_2) = p(S(a_{10})S^2[p(a_7b_7)c_5])$$

T and T_1 are elements of type $p(S[p(x_1b)c]x_2)$ described in Remark 3.4; The cocycle of the Hopf algebra with a weak projection being trivial, these elements are scalar multiples of 1_H. $T_1 \otimes T_2 = \varepsilon(T_1) \otimes T_2 = 1 \otimes T = 1 \otimes \varepsilon(T)$.

The left and the right hand side are equal. $T_1d_1p(S(T_2d_2)Z) = d_1T_1p(S(d_2T_2)Z) = d_1T_1p(S(T_2)S(d_2)Z) = d_1T_1p(S(T_2))p(S(d_2)Z) = d_1p(S(d_2)Z)$.

□

Solutions of the equation above provide representations for a sequence of groups R(n) which appear in [1]. If the weak projection is also an algebra map, then the maps above defined on multiple tensor products of X's can be defined (transposed) on tensor products of B, where B is a braided Hopf algebra, and the Theorem 2.2 in the special case of the braided category of Yetter-Drinfeld modules is recovered (see Theorem 2.2, whose proof is based on the diagramatic calculus in a braided category). In the case above, the proof depends on the concrete, specific calculus above. The operator $R_{12}^{-1}R_{23}R_{12}R_{23}R_{12}^{-1}$ does not satisfy the Braid equation.

3.1.1. *Crossed simplicial groups. Further directions.* We state without proof, which can be easily checked using the results [1], that the groups R(n) introduced in [1] are crossed simplicial groups in the sense of Loday (Section 6.3, [8]). Following the last remark, there is a cohomology theory for Hopf algebras H, having a pairing and a coalgebra map with trivial cocycle as above, similar to the cyclic cohomology for Hopf algebras having a modular pair in involution.

Definition 3.1. *(Definition 6.3.0* [8]*)*

A crossed simplicial group is a family of groups R(n), $n > 0$, such that there exists a category C with objects [n], $n > 0$, containing the simplicial category as a subcategory and such that:

1) the group of automorphisms of [n] is the opposite group of R(n),

2) any morphism from [n] to [m] in C can be uniquely written as the composition between a morphism from the simplicial category and R(n).

Any triple (H, r, ϕ) provides in fact a representation of the operad given by the groups R(n). We state without proof that the category C, which is a semidirect product between the simplicial category and R(n) is represented in this way.

The groups $R(n)$, n=0,1,2... have the following functions, well-defined because of Lemma 2.1 and Lemma 2.2 of [1].

- face-operators d(i):$R(n+1) \to R(n)$, which delete the i-th string.
- doubling- operators s(i):$R(n) \to R(n+1)$, which double the i-th string.
- natural group morphisms b: $R(n) \to S(n)$

The following relations are satisfied:

$$d(i)(xy) = d(i)(x)d(b(x)i)(y)$$
$$s(i)(xy) = s(i)(x)s(b(x)i)(y)$$

The functions above are $crossed{-}group$ morphisms, so the groups R(n) form a system of crossed-simplicial groups, this second definition being equivalent with the definition above [8] [5].

We open the question to find the appropiate Lie algebras L_n such that there are canonical group morphisms from R(n) into S(n) $\rtimes$ exp(L_n), and to check the above equation for a Hopf algebra with a weak projection in a braided category [16].

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(Gefry Barad) Algebra and Number Theory Research Center, University of Bucharest, Department of Mathematics, Str. Academiei nr.14, sector1, Bucuresti, Romania
E-mail address: gbarad@gmail.com