A note on *BL*-algebras with internal state

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**Abstract.** The scope of this paper is to put in evidence some properties of the *BL*-algebras with internal state. I introduce the concepts of prime and maximal state-filters, I prove a Prime state-filter theorem 4.7 and I characterize the set $\text{Rad}_\sigma(A)$, which represents the intersection of all maximal state-filters of a state *BL*-algebra $(A,\sigma)$. Also, I introduce the concepts of simple, semisimple and local state *BL*-algebras relative to its state-filter set.

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1. Introduction

The concept of state *MV*-algebras was firstly introduced by Flaminio and Montagna in [4] and [5] as a *MV*-algebra endowed with a unary operation $\sigma$ (called a state-operator), which preserves the usual properties of states. Di Nola and Dvurečenskij presented in [6] a stronger version of states *MV*-algebras namely state-morphism *MV*-algebras. Afterwards Ciungu, Dvurečenskij and Hyčko extended in [2] the concept of state (morphism) *MV*-algebra and in the case of *BL*-algebras and they extended the properties of a state-operator. The present article is structured into five sections.

In Section 2, basic properties regarding the concepts of *MV*-algebra, *BL*-algebra are being presented, as well as some basic properties of the operations defined on these algebras, which are to be used afterwards. The concept of state (morphism) --operator on a *BL*-algebra also belongs to this section, as well some of its properties.

In Section 3 some examples of state *BL*-algebras are presented. In Section 4 the concept of state-filter on a state *BL*-algebra is introduced. There are presented some examples of filters and state-filters, as well as the concepts of maximal state-filter, prime state-filter, some of their characteristics and, if the state-operator $\sigma$ is a morphism, the set $\text{Rad}_\sigma(A)$ is characterised, in which $\text{Rad}_\sigma(A)$ represents the intersection of all maximal state-filters of a state *BL*-algebra $(A,\sigma)$.

In Section 5, there are presented some classes of *BL*-algebras such as simple, semisimple and local as well as simple, semisimple and local state *BL*-algebras. There are introduced the concepts of simple, semisimple and local state *BL*-algebras relative to its state-filters set and there are established relations between these structures in certain conditions imposed to the state-operator $\sigma$.

2. Preliminaries

**Definition 2.1.** An algebra $(A,\land,\lor,\circ,\rightarrow,0,1)$ of the type $(2,2,2,2,0,0)$ is called a *BL*-algebra if satisfies the following axioms:
(1) \((A, \wedge, \lor, 0, 1)\) is a bounded lattice;
(2) \((A, \circ, 1)\) is a commutative monoid;
(3) \(x \circ y \leq z\) if and only if \(x \leq y \rightarrow z\);
(4) \(x \wedge y = x \circ (x \rightarrow y)\);
(5) \((x \rightarrow y) \lor (y \rightarrow x) = 1\);
for every \(x, y, z \in A\).

We will denote \(x^* = x \rightarrow 0\), \(x, y \in A\). If \(x \in A\), we define \(x^0 = 1\) and for \(n \geq 1\) we define \(x^n = x^{n-1} \circ x\).

**Definition 2.2.** Let \(A\) be a BL–algebra and \(x \in A\). If there exists the least number \(n \in \mathbb{N}^*\) such that \(x^n = 0\), then we set \(\text{ord}(x) = n\). If there is no such a number (that is, \(x^n > 0\) for every \(n \geq 0\)), then we set \(\text{ord}(x) = \infty\).

We recall some results relative to BL–algebras:

**Proposition 2.1.** Let \(A\) be a BL–algebra. Then:
(1) if \(a \leq b\) and \(c \leq d\) then \(a \circ c \leq b \circ d\);
(2) \(a \circ (b \lor c) = (a \circ b) \lor (a \circ c)\);
(3) \(a \lor (b \circ c) \geq (a \lor b) \circ (a \lor c)\);
(4) \(a^m \lor b^n \geq (a \lor b)^{mn}, m, n \in \mathbb{N}\);
(5) \((a \circ b)^* = a \rightarrow b^*\);
(6) \(a \circ (a \rightarrow (a \circ b)) = a \circ b\); for every \(a, b, c \in A\).

**Definition 2.3.** An algebra \((A, \oplus, 0)\) of the type \((2, 1, 0)\) is called a MV–algebra if satisfies the following axioms:
(1) \((A, \oplus, 0)\) is a commutative monoid;
(2) \(x^{**} = x\), for every \(x \in A\);
(3) \(x \oplus 0^* = 0^*, for every x \in A\);
(4) \((x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x\), for every \(x, y \in A\).

On a BL–algebra \((A, \land, \lor, \circ, \rightarrow, 0, 1)\) we define the operation \(\oplus\) on \(A\) by \(x \oplus y = (x^* \land y^*)^*, x, y \in A\). If \(x^{**} = x\), for every \(x \in A\), then \((A, \oplus, 0)\) becomes a MV–algebra. We are now defining the concept of state-operator on a BL–algebra.

**Definition 2.4.** [2] Let \(A\) be a BL–algebra. An application \(\sigma : A \rightarrow A\) which verifies the properties:
(1) \(_{BL} \sigma(0) = 0\);
(2) \(_{BL} \sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(x \land y)\);
(3) \(_{BL} \sigma(x \circ y) = \sigma(x) \circ \sigma(x \rightarrow x \circ y)\);
(4) \(_{BL} \sigma(\sigma(x) \circ \sigma(y)) = \sigma(x) \circ \sigma(y)\);
(5) \(_{BL} \sigma(\sigma(x) \rightarrow \sigma(y)) = \sigma(x) \rightarrow \sigma(y)\);
for every \(x, y \in A\), is called state-operator on \(A\), and the pair \((A, \sigma)\) is called a state BL–algebra or, more precisely, a BL–algebra with internal state.

Some examples of state-operators will be presented in Section 3.

**Proposition 2.2.** [2] In a state BL–algebra \((A, \sigma)\) the following hold:
(a) \(\sigma(1) = 1\);
(b) \(\sigma(x^*) = \sigma(x)^*\), for every \(x \in A\);
(c) if \(x, y \in A\) and \(x \leq y\) then \(\sigma(x) \leq \sigma(y)\);
(d) \(\sigma(x \circ y) \geq \sigma(x) \circ \sigma(y)\), for every \(x, y \in A\);
(e) \(\sigma(x \rightarrow y) \leq \sigma(x) \rightarrow \sigma(y)\), for every \(x, y \in A\);
(f) $\sigma (\sigma (x)) = \sigma (x)$, for every $x \in A$;

(g) $\sigma (A)$ is a $BL$-subalgebra of $A$ and $\sigma (A) = \{ x \in A \mid \sigma (x) = x \}$.

**Definition 2.5.** [2] A state-morphism operator on a $BL$-algebra $A$ is an application $\sigma : A \to A$ which verifies $(1)_{BL}$, $(2)_{BL}$, $(4)_{BL}$, $(5)_{BL}$ and $(6)_{BL}$ $\sigma (x \circ y) = \sigma (x) \circ \sigma (y)$, for every $x, y \in A$.

**Remark 2.1.** Any state-morphism operator $\sigma$ on a $BL$-algebra $A$ is a state-operator on $A$. Indeed, by using $(6)_{BL}$ we have:

$\sigma (x) \circ \sigma (x \circ y) = \sigma (x \circ (x \circ y)) = \sigma (x \circ y)$ , according to Proposition 2.1.

If $\sigma$ is a state-operator on $A$, we define $\ker (\sigma) = \{ x \in A \mid \sigma (x) = 1 \}$.

**Definition 2.6.** A state-operator $\sigma : A \to A$ is called faithful iff $\ker (\sigma) = 1$.

3. Examples of state-operators on $BL$-algebras

**Example 3.1.** If $A$ is a $BL$-algebra, then $\sigma : A \to A$, defined by $\sigma (x) = x$, for every $x \in A$, is a state-operator on $A$, called the identity state-operator on $A$. Thus $(A, id_A)$ is a state $BL$-algebra.

**Example 3.2.** [2] Let $A = \{0, a, b, 1\}$ be with $0 < a < b < 1$.

Then $(A, \wedge, \vee, \circ, \to, 0, 1)$ with the following operations:

\[
\begin{array}{c|cccc}
\circ & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & a \\
b & 0 & 0 & b & b \\
c & 0 & a & 0 & a \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\to & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & a & 1 & 1 & 1 \\
b & b & 0 & a & 1 \\
c & 1 & 0 & a & 1 \\
\end{array}
\]

it becomes a $BL$-algebra, but not a $MV$-algebra (since $b^{**} = 1 \neq b$).

The fact that $\sigma : A \to A$, given by $\sigma (0) = 0, \sigma (a) = a, \sigma (b) = \sigma (1) = 1$, is a state-operator on $A$, is verified. Moreover, $(6)_{BL}$ holds, so $\sigma$ is a state-morphism operator on $A$.

**Example 3.3.** [3] Let $A = \{0, a, b, c, d, 1\}$, with the operations $\circ$ and $\to$ given by the following tables:

\[
\begin{array}{c|cccc}
\circ & 0 & a & b & c & d & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & a & 0 & a \\
b & 0 & 0 & 0 & b & b & b \\
c & 0 & a & 0 & a & b & c \\
d & 0 & b & b & d & d & d \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\to & 0 & a & b & c & d & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
a & 0 & a & c & c & c & 1 \\
b & 0 & a & c & c & c & 1 \\
c & 0 & a & c & c & c & 1 \\
d & 0 & a & c & c & c & 1 \\
\end{array}
\]

Then the $BL$-algebra $(A, \wedge, \vee, \circ, \to, 0, 1)$ is a $MV$-algebra.

We will determine the state-operators on $A$. Let $\sigma : A \to A$ be a state-operator.

From $(1)_{BL}$ we have $\sigma (c \to a) = \sigma (c) \to \sigma (a)$, so

$\sigma (c) = \sigma (c) \to \sigma (a)$. From the table of the operation $\to$ we deduce that the equation $x = x \to y$ has only the solutions $x = c, y = a$ and $x = y = 1$.

In the first case we have $\sigma (c) = c$ and $\sigma (a) = a$ and then

$\sigma (d) = \sigma (a^*) = \sigma (a)^* (\text{according to the Proposition 2.2, (b)}) = a^* = d$, and $\sigma (b) = \sigma (c^*) = \sigma (c)^* = b$, so $\sigma = id_A$. In the second case we have $\sigma (c) = \sigma (a) = 1$, and then $\sigma (d) = \sigma (a)^* = 0, \sigma (b) = \sigma (c)^* = 0$, thus
\(\sigma(0) = \sigma(b) = \sigma(d) = 0\) and \(\sigma(c) = \sigma(a) = 1\), which verifies \((1)_{BL} - (6)_{BL}\), so this is also a state-morphism operator.

**Example 3.4.** [3] Let \(A = \{0, a, b, c, d, 1\}\), with the following tables of operations:

<table>
<thead>
<tr>
<th>(\odot)</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>a</td>
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<td>b</td>
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<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

Then it becomes a BL-algebra, which is a MV-algebra. Let \(\sigma : A \rightarrow A\) be a state-operator. As in the Example 3.3 we have

\(\sigma(d \rightarrow c) = \sigma(d) \rightarrow \sigma(d \land c) = \sigma(d) \rightarrow \sigma(c) = \sigma(c)\). Since the equation \(x = x \rightarrow y\) has only the solutions \(x = d\), \(y = c\) and \(x = y = 1\) we obtain \(\sigma(d) = d, \sigma(c) = c\) or \(\sigma(d) = \sigma(c) = 1\). In the first case we have \(\sigma = \text{id}_A\), and in the second case we have \(\sigma(a) = \sigma(b) = \sigma(0) = 0\) and \(\sigma(c) = \sigma(d) = \sigma(1) = 1\), both operators being state-morphism operators.

**Example 3.5.** [3] Let \(A = \{0, c, a, b, 1\}\), in which \(0 < c < a, b < 1\) and \(a, b\) are incomparable, with the following tables of operations:

<table>
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<tr>
<th>(\odot)</th>
<th>0</th>
<th>c</th>
<th>a</th>
<th>b</th>
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<td>0</td>
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</tbody>
</table>

The application \(\sigma : A \rightarrow A\), given by \(\sigma(0) = 0\) and \(\sigma(x) = 1\) otherwise, is a state-morphism operator.

We recall that a \(t\)-norm is a function \(t : [0,1] \times [0,1] \rightarrow [0,1]\), which verifies the conditions:

1. \(t(x, y) = t(y, x)\), for every \(x, y \in [0,1]\);
2. \(t(t(x, y), z) = t(x, t(y, z))\), for every \(x, y, z \in [0,1]\);
3. \(t(x, 1) = x\), for every \(x \in [0,1]\);
4. \(t(x, y) \leq \min\{x, y\}, x \in [0,1]\).

If \(t\) is continuous, we define \(x \odot_t y = t(x, y)\) and \(x \rightarrow_t y = \sup\{z \in [0,1] \mid t(x, z) \leq y\}\), for \(x, y \in [0,1]\). In these conditions \(I_t = ([0,1], \min, \max, \odot_t, \rightarrow_t, 0, 1)\) is a BL-algebra. Moreover, according to [1], the variety of BL-algebras is generated by all the \(I_t\), with a continuous norm \(t\). There are three basic continuous \(t\)-norms on \([0,1]\):

1. \(\text{Łukasiewicz}: L(x, y) = \max\{x + y - 1, 0\}\), with \(x \rightarrow_L y = \min\{1 - x + y, 1\}\);
2. \(\text{Gödel}: G(x, y) = \min\{x, y\}\), with \(x \rightarrow_G y = 1\) if \(x \leq y\) and \(x \rightarrow_G y = y\) otherwise;
3. \(\text{product}: P(x, y) = xy\), with \(x \rightarrow_P y = 1\) if \(x \leq y\) and \(x \rightarrow_P y = \frac{y}{x}\) otherwise.

Then we have:

**Proposition 3.1.** [2]

1. If \(\sigma\) is a state-operator on \(I_t\), then \(\sigma(x) = x\), for every \(x \in [0,1]\).
2. Let \(a \in [0,1]\) and we define \(\sigma_a(x) = x\) if \(x \leq a\) and \(\sigma_a(x) = 1\) otherwise. For \(a \in [0,1]\) we define the application \(\sigma_a(x) = x\) if \(x < a\) and \(\sigma_a(x) = 1\) otherwise.
Let $\sigma_a$ be state-morphism operators on $I_G$ and, if $\sigma$ is a state-operator on $I_G$, then $\sigma = \sigma_a$ or $\sigma = \sigma^a$ for a certain $a \in [0, 1]$. 

(3) If $\sigma$ is a state-operator $1_P$, then $\sigma(x) = x$, for every $x \in [0, 1]$ or $\sigma(x) = 1$, for every $x > 0$.

**Proposition 3.2.** [2] Let $A$ be a finite linear Gödel $BL$-algebra, that is, $x^2 = x$, for every $x \in A$. Then, with the notations from Proposition 3.1 $\sigma^a$ and $\sigma_0$ are state-morphism operators, and any state-operator on $A$ is of the form $\sigma^a$ or $\sigma_0$, for a certain $a \in [0, 1]$.

Actually we have the following more general result:

**Proposition 3.3.** Let $A$ be a linear Gödel $BL$-algebra and $B \subseteq A$ such that $0 \in B, 1 \notin B$ and, if $x \in B$, $y \in A \setminus B$, then $x < y$. Then the application $\sigma_B : [0, 1] \rightarrow [0, 1]$, given by $\sigma_B(x) = x$ if $x \in B$ and $\sigma_B(x) = 1$ otherwise, is a state-morphism operator on $A$, and, conversely, any state-operator on $A$ is of such a form.

**Proof.** Firstly we observe that, if $x, y \in A$ then

\[ x \circ y \geq x \circ (x \land y) = x \circ (x \land (x \lor y)) = x \circ (x \lor y) = x \circ (x \land y) = x \circ (x \land y) \]

\[ x \lor y \geq x \land y, \text{ so } x \land y = x \land y = \min \{x, y\}, \text{ for every } x, y \in A. \]

Then $x \lor y = \sup \{z \in A \mid x \land z \leq y\} = \sup \{z \in A \mid x \land z \leq y\}.$

If $x < y$, then $x \lor y = 1.$

If $x > y$, then $\sup \{z \in A \mid x \land z \leq y\} = \sup \{z \in A \mid x \land z \leq y\} = y.$

We will verify the $(1)_{BL} - (5)_{BL}$ axioms. Since $0 \in B$ we have that $\sigma_B(0) = 0$, so the $(1)_{BL}$ is proved.

If $x, y \in B$, then we have: $\sigma_B(x \lor y) = 1 = \sigma_B(x) \lor \sigma_B(y),$ if $x < y$, and, if $x > y$ we have $\sigma_B(x \lor y) = \sigma_B(y),$ and $\sigma_B(x) \lor \sigma_B(y) = x \lor y = x \land y = x \lor y.$

If $x, y \in A \setminus B$, then, since $y \leq x \lor y$, it follows that $x \lor y \in A \setminus B,$ so $\sigma_B(x \lor y) = 1$, and $\sigma_B(x) \lor \sigma_B(y) = 1$ (since $x \land y \in A \setminus B$).

If $x \in B, y \in A \setminus B$, then $\sigma_B(x \lor y) = \sigma_B(1) = 1 = \sigma_B(x) \lor \sigma_B(y).$

If $y \in B, x \in A \setminus B$, then $\sigma_B(x \lor y) = \sigma_B(y) = y$ and $\sigma_B(x) \lor \sigma_B(y) = 1 \lor y = y,$ so we have an equality again.

Thus $(2)_{BL}$ is proved.

We will now prove $(6)_{BL},$ which means that, according to the Remark 2.1

$(3)_{BL}$ is proved. Indeed, if $x, y \in B$, then $x \lor y = \min \{x, y\} \in B,$ so $\sigma_B(x \lor y) = x \lor y = \sigma_B(x) \lor \sigma_B(y).$ If $x, y \in A \setminus B$, then $x \lor y = \min \{x, y\} \in A \setminus B$, so $\sigma_B(x \lor y) = 1 = \sigma_B(x) \lor \sigma_B(y).$ Thus $(6)_{BL}$ is fulfilled.

If $x \in B$, then $\sigma_B(\sigma_B(x)) = \sigma_B(x),$ and if $x \in A \setminus B$, then we have $\sigma_B(\sigma_B(x)) = \sigma_B(1) = 1 = \sigma_B(x) \lor \sigma_B(\sigma_B(x)) = \sigma_B(x),$ for all $x \in A.

Then $\sigma_B(\sigma_B(x) \lor \sigma_B(y)) = \sigma_B(\sigma_B(x \lor y))$ (according to $(6)_{BL}) = \sigma_B(x \lor y) = \sigma_B(x) \lor \sigma_B(y),$ for all $x, y \in A,$ so $(4)_{BL}$ is verified.

In order to complete the first part of the proof, we still have to verify $(5)_{BL},$ Indeed, if $x, y \in B$, then $\sigma_B(\sigma_B(x) \lor \sigma_B(y)) = \sigma_B(x \lor y),$ and $\sigma_B(x \lor \sigma_B(y)) = x \lor y.$

If $x \leq y$, then $x \lor y = 1,$ so $\sigma_B(x \lor y) = x \lor y.$ If $x > y$, $x \lor y = y,$ and $\sigma_B(x \lor y) = \sigma_B(y) = y$, so equality once more. Let’s now suppose that $x, y \in A \setminus B.$ Then $\sigma_B(\sigma_B(x) \lor \sigma_B(y)) = \sigma_B(1) = 1 \lor \sigma_B(x) \lor \sigma_B(y).$

If $x \in B, y \in A \setminus B$, then $\sigma_B(\sigma_B(x) \lor \sigma_B(y)) = \sigma_B(x) \lor 1 = \sigma_B(x) \lor \sigma_B(y).$
Finally, if \( y \in B, x \in A \setminus B \), then \( \sigma_B (\sigma_B (x) \rightarrow \sigma_B (y)) = \sigma_B (1 \rightarrow y) = \sigma_B (y) = \sigma_B (x) \rightarrow \sigma_B (y) \).

Conversely, let \( \sigma \) be a state-operator on \( A \) and let \( a \in (0,1) \).

We are going to prove \( \sigma (a) = a \) or \( \sigma (a) = 1 \). Let’s suppose that \( \sigma (a) < a \).

Then, according to (2)\(_{BL} \), \( \sigma (a \rightarrow \sigma (a)) = \sigma (a) \rightarrow (\sigma (a) \land \sigma (a)) = \sigma (a) \rightarrow \sigma (a) = a \).

But \( a \rightarrow \sigma (a) = \sigma (a) \), so \( \sigma (a \rightarrow \sigma (a)) = \sigma (a) \), so \( \sigma (a) = 1 \), a contradiction. If \( a < \sigma (a) \), then, from (2)\(_{BL} \), we have \( \sigma (\sigma (a) \rightarrow a) = (\sigma (\sigma (a)) \rightarrow \sigma (\sigma (a) \land a)) = \sigma (a) \rightarrow \sigma (a) = a \).

Let \( B = \{a \in [0,1) \mid \sigma (a) = a\} \). Then \( \sigma (x) = x \), if \( x \in B \), and \( \sigma (x) = 1 \), if \( x \in A \setminus B \). Obviously \( 0 \in B, 1 \notin B \). Let \( x \in B, y \in A \setminus B \). If \( y \preceq x \), then \( \sigma (y) \preceq \sigma (x) \), that is \( 1 \preceq x \), a contradiction. So \( x < y \). Thus \( \sigma = \sigma_B \), in which \( B \) fulfills the conditions from the enunciation. \( \square \)

Example 3.6. [2] Let \( A \) be a \( BL \)-algebra. Then \((A \times A, \land, \lor, \& , \rightarrow, 0, 1)\) becomes a \( BL \)-algebra, where \((a, b) \leq (c, d)\) if \( a \leq c \) and \( b \leq d \), and the operations are defined on the components. Let \( \sigma : A \times A \rightarrow A \times A \) be, defined by \( \sigma (a, b) = (a, a) \), for every \((a, b) \in A \times A \). It is easily to prove that \( \sigma \) is a state-morphism operator on \( A \times A \).

4. Filters and state-filters

**Definition 4.1.** Let \( A \) be a \( BL \)-algebra. A nonvoid subset \( F \subseteq A \) is called a filter if the following conditions are verified:

1. \( x, y \in F \) implies \( x \circ y \in F \);
2. \( x \in F \) and \( x \preceq y \) implies \( y \in F \).

A proper filter of \( A \) is called a maximal filter if it doesn’t belong to any other proper filter of \( A \). The intersection all the maximal filters of \( A \) is denoted by \( \text{Rad} (A) \).

**Definition 4.2.** [2] Let \((A, \sigma)\) be a state (morphism) \( BL \)-algebra. A nonvoid subset \( F \subseteq A \) is called a state (morphism) – filter of \((A, \sigma)\), if \( F \) is a filter of \( A \) with the property that if \( x \in F \), then \( \sigma (x) \in F \). A proper state-filter of \((A, \sigma)\) is called a maximal state-filter if it doesn’t belong to any other proper state-filter of \((A, \sigma)\). The intersection all the maximal state-filters of \((A, \sigma)\) is denoted by \( \text{Rad}_\sigma (A) \).

**Example 4.1.** If we consider the Example 3.1 then the filters of \( A \) and the state-filters of \((A, \sigma)\) are the same.

For \( A \) the \( BL \)-algebra from Example 3.2 the filters are \( \{1\}, \{b, 1\}, A \), and the state-filters of \((A, \sigma)\) are \( \{1\}, \{b, 1\}, A \). The (state)filter \( \{b, 1\} \) is a maximal (state)filter.

In this case \( \text{Rad} (A) = \text{Rad}_\sigma (A) = \{b, 1\} \).

Let’s now consider \( A \) the \( BL \)-algebra from Example 3.3 and the state-operator \( \sigma : A \rightarrow A \), defined by \( \sigma (0) = \sigma (b) = \sigma (d) = 0, \sigma (a) = \sigma (c) = 1 \). The filters of \( A \) are \( \{1\}, \{d, 1\}, \{a, c, 1\}, A \), and the state-filters of \((A, \sigma)\) are \( \{1\}, \{a, c, 1\}, A \). The \( BL \)-algebra \( A \) has two maximal filters: \( \{d, 1\} \) and \( \{a, c, 1\} \). There exists only an maximal state-filter of \((A, \sigma)\), namely \( \{a, c, 1\} \). In this case we have \( \text{Rad} (A) = \{1\}, \) and \( \text{Rad}_\sigma (A) = \{a, c, 1\} \).

Let’s now the \( BL \)-algebra from Example 3.4 and the state-operator \( \sigma : A \rightarrow A \), defined by \( \sigma (d) = \sigma (c) = \sigma (1) = 1, \sigma (a) = \sigma (b) = \sigma (0) = 0 \). The filters of \( A \) are \( \{1\}, \{b, 1\}, \{c, d, 1\}, A \), and the state-filters of \((A, \sigma)\) are \( \{1\}, \{c, d, 1\}, A \). There are two maximal filters, namely \( \{b, 1\} \) and \( \{c, d, 1\} \), so \( \text{Rad} (A) = \{1\}, \) and a single maximal state-filter, \( \{c, d, 1\} \), so \( \text{Rad}_\sigma (A) = \{c, d, 1\} \).
For a the BL-algebra from Example 3.5 and the state-operator $\sigma : A \to A$, defined
by $\sigma (0) = 0$ and $\sigma (x) = 1$ otherwise, the filters and the state-filters are the same:
$\{1\}, \{a, 1\}, \{b, 1\}, \{c, a, b, 1\}, A$. We have $Rad (A) = Rad \sigma (A) = \{c, a, b, 1\}$. For the
algebra $I_L$, since $ord (x) < \infty$, for every $x \neq 1$, the only filters are $\{1\}$ and $\{0, 1\}$. Since
the single state-operator on $I_L$ is $id_L$, these are also the only state-filters.

In the case of the algebra $I_G$, the filters are the sets of the form $\{x, 1\}$ or $\{x, 1\}$, where
$x \in \{0, 1\}, I_G$ has an only maximal filter, namely $\{0, 1\}$. According to Proposition
3.1(2) if $\sigma$ is a state-operator on $I_G$, then $\sigma = \sigma^a$ or $\sigma = \sigma_a$ (with those notations).
For any of these state-operators, the state-filters and the filters of $I_G$ are the same.

In the case of the algebra $I_P$, since $ord (x) < \infty$, for every $x \neq 1$, the only filters are
$\{1\}$ and $\{0, 1\}$, which are therefore the only state-filters.

**Proposition 4.1.** Let $A$ and $B$ be two BL-algebras and let us consider $A \times B$ the
BL-algebra product of $A$ and $B$. If $F_1, F_2$ are filters of $A$, respectively $B$, then $F_1 \times F_2$
is a filter of $A \times B$ and, conversely, any filter of $A \times B$ is of the form $F_1 \times F_2$, where
$F_1, F_2$ are filters of $A$, respectively $B$.

Proof. If $F_1, F_2$ are filters of $A$, respectively $B$, then it is immediate that $F_1 \times F_2$
is a filter of $A \times B$. Conversely, let $F$ be a filter of $A \times B$. Since $F$ is nonvoid,
then the sets $F_1 := \{x \in A \mid \text{there exists } y \in B \text{ such that } (x, y) \in F\} \subseteq A$ and $F_2 :=
\{y \in B \mid \text{there exists } x \in A \text{ such that } (x, y) \in F\} \subseteq B$ will be too.

We are going to prove that $F_1, F_2$ are filters and $F = F_1 \times F_2$. Indeed, if $a, b \in F_1$,
then there exists $c, d \in B$ such that $(a, c), (b, d) \in F$, so $(a \circ b, c \circ d) \in F$, so
$a \circ b \in F_1$. If $a \in F_1$ and $a \leq c$, then, since there exists $b \in B$ such that $(a, b) \in F$
and since $(a, b) \leq (c, b)$, it follows that $(c, b) \in F$, therefore $c \in F_1$. Thus $F_1$ is a filter
and analogously it shows that $F_2$ is a filter. Let $(a, b) \in F$. Then $a \in F_1, b \in F_2$, so
$(a, b) \in F_1 \times F_2$, so $F \subseteq F_1 \times F_2$. Let's now $(a, b) \in F_1 \times F_2$. Since $a \in F_1, b \in F_2$,
there exist $x \in A, y \in B$ such that $(a, y), (x, b) \in F$. Then $(a, 1), (1, b) \in F$ and so
$(a \circ 1, 1 \circ b) \in F$, that is, $(a, b) \in F$, therefore $F_1 \times F_2 \subseteq F$. Thus $F = F_1 \times F_2$.

Let's now consider an BL-algebra $A$ which contains proper filters and the state-operator $\sigma : A \times A \to A \times A, \sigma (a, b) = (a, a)$, for every $(a, b) \in A \times A$, from the
Example 3.6. According to Proposition 4.1 any filter of $A \times A$ is of the form $F_1 \times F_2$, with
$F_1, F_2$ filters of $A$. If $F_1 \times F_2$ is a state-filter of $(A \times A, \sigma)$, then $F_1 \subseteq F_2$. Indeed,
let $a \in F_1$. Then $(a, 1) \in F_1 \times F_2$, so $\sigma (a, 1) = (a, a) \in F_1 \times F_2$ that is, $a \in F_2$.

Conversely, if $F_1 \times F_2$ is a filter of $A \times A$ such that $F_1 \subseteq F_2$, and $(a, b) \in F_1 \times F_2$,
then $\sigma (a, b) = (a, a) \in F_1 \times F_2$, so the state-filters of $(A \times A, \sigma)$ are the sets of the form
$F_1 \times F_2$, in which $F_1, F_2$ are filters of $A$ with $F_1 \subseteq F_2$.

**Remark 4.1.** [2] Let $A$ be a BL-algebra and $\sigma$ a state-operator on $A$. Then $\ker (\sigma)$
is a state-filter of $(A, \sigma)$.

**Proposition 4.2.** [2] Let $A$ be a BL-algebra. A proper filter $F$ of $A$ is a maximal
filter iff for any $a \notin F$, there exists $n \in \mathbb{N}^*$ such that $(a^n)^* \in F$.

**Proposition 4.3.** [7] Let $A$ be a BL-algebra. Then $Rad (A) = \{x \in A \mid (x^n)^* \leq x, \text{ for every } n \in \mathbb{N}\}$.

**Proposition 4.4.** [2] Let $(A, \sigma)$ be a state BL-algebra and $X \subseteq A$. Then the state-
filter $F_\sigma (X)$ generated by $X$ is the set
$\{x \in A \mid x \geq (x_1 \circ \sigma (x_1))^{n_1} \circ ... \circ (x_k \circ \sigma (x_k))^{n_k}, x_i \in X, n_i \geq 1, k \geq 1\}$.
If $F$ is a state-filter of $(A, \sigma)$ and $a \notin F$, then the state-filter generated by $F$ and $a$ is the set $F_a(F, a) = \{x \in A \mid x \geq i \circ (a \circ \sigma(a))^n, i \in F, n \geq 1\}$. A proper state-filter $F$ is a maximal state-filter iff for any $a \notin F$ there exists $n \in \mathbb{N}^\ast$ such that $(\sigma(a))^n \in F$.

In what follows we will introduce the concept of a prime state-filter, we will establish some results related to this concept on the basis of which we are going to characterise the set $Rad_\sigma(A)$, in the case of a state-morphism $BL$–algebra $(A, \sigma)$.

**Proposition 4.5.** Let $(A, \sigma)$ be a state $BL$–algebra and $P$ a proper state-filter of $(A, \sigma)$. Then the following statements are equivalent:

(i) If $P_1, P_2$ are two state-filters of $(A, \sigma)$ such that $P = P_1 \cap P_2$, then $P = P_1$ or $P = P_2$;

(ii) If $(a \circ \sigma(a)) \lor (b \circ \sigma(b)) \in P$, $a, b \in A$, then $a \in P$ or $b \in P$.

**Proof.** (i) $\Rightarrow$ (ii). Let $a, b \in A$ such that $(a \circ \sigma(a)) \lor (b \circ \sigma(b)) \in P$. We consider the sets $F_n(P, a) = \{x \in A \mid x \geq i \circ (a \circ \sigma(a))^n, i \in F, n \geq 1\}$ and $F_n(P, b) = \{x \in A \mid x \geq i \circ (b \circ \sigma(b))^n, i \in F, n \geq 1\}$, which represent state-filters generated by $P$ and $a$, respectively $P$ and $b$ (according to Proposition 4.4).

Obviously, $P \subseteq F_n(P, a) \cap F_n(P, b)$. If $x \in F_n(P, a) \cap F_n(P, b)$, then there exist $i_1, i_2 \in P$ and $m, n \in \mathbb{N}^\ast$ such that $x \geq i_1 \circ (a \circ \sigma(a))^m$ and $x \geq i_2 \circ (b \circ \sigma(b))^n$, so $x \geq (i_1 \circ (a \circ \sigma(a))^m) \lor (i_2 \circ (b \circ \sigma(b))^n) \geq (i_1 \lor i_2) \circ ((a \circ \sigma(a))^m \lor (b \circ \sigma(b))^n)$ (according to Proposition 2.1, (3))

$\geq (i_1 \lor i_2) \circ (i_1 \lor i_2) \circ ((a \circ \sigma(a))^m \lor (b \circ \sigma(b))^n)$ (according to Proposition 2.1, (4)).

But $i_1 \lor i_2, i_1 \lor (b \circ \sigma(b))^n, i_2 \lor (a \circ \sigma(a))^m$ and $(a \circ \sigma(a))^m \lor (b \circ \sigma(b))^n$ belong to $P$, and then it follows that $x \in P$. Thus $P = F_n(P, a) \cap F_n(P, b)$, and, from the hypothesis, we obtain that $P = F_n(P, a)$ or $P = F_n(P, b)$, that is, $a \in P$ or $b \in P$.

(ii) $\Rightarrow$ (i). Let $P_1, P_2$ be two state-filters of $(A, \sigma)$ such that $P = P_1 \cap P_2$. Let’s suppose that $P \neq P_1$ and $P \neq P_2$. Then there exist $a \in P \setminus P_1$ and $b \in P \setminus P_2$. Then $a \circ \sigma(a) \in P_1, b \circ \sigma(b) \in P_2$, so $(a \circ \sigma(a)) \lor (b \circ \sigma(b)) \in P_1 \cap P_2 = P$, hence $a \in P$ or $b \in P$, a contradiction. Therefore $P = P_1$ or $P = P_2$.

**Definition 4.3.** Let $(A, \sigma)$ be a state $BL$–algebra. A proper state-filter $P$ of $(A, \sigma)$ is called a prime state-filter if it verify one of the equivalent conditions from the Proposition 4.5.

**Proposition 4.6.** Let $(A, \sigma)$ be a state $BL$–algebra. Then any maximal state-filter of $(A, \sigma)$ is a prime state-filter.

**Proof.** Let $F$ be a maximal state-filter of $(A, \sigma)$ and $P_1, P_2$ two state-filters such that $F = P_1 \cap P_2$. If $F \neq P_1$, then $F$ is strictly contained in $P_1$, and, since $F$ is a maximal state-filter, it follows that $P_1 = A$. Then $F = A \cap P_2 = P_2$. Therefore $F$ is a prime state-filter.

**Definition 4.4.** Let $(A, \sigma)$ be a state $BL$–algebra. A nonvoid subset $I$ of $A$ is called state-ideal if the following conditions are verified:

(1) $a, b \in I$ implies $a \circ b \in I$;

(2) $a \in I, b \leq a$ implies $b \in I$;

(3) $a \in I$ implies $\sigma(a) \in I$. 
**Proposition 4.7.** (Prime state-filter theorem) Let $I$ be a state-ideal and $F$ a state-filter on a state BL-algebra $(A, \sigma)$ such that $F \cap I = \varnothing$. Then there is a prime state-filter $P$ such that $F \subseteq P$ and $P \cap I = \varnothing$.

**Proof.** Consider the set

$$F(F) = \{ F' \mid F' \text{ is a state-filter such that } F \subseteq F' \text{ and } F' \cap I = \varnothing \}.$$ 

Since $F \in F(F)$, it follows that $F(F)$ is nonvoid. It is easily to prove that the set $F(F)$ is inductively ordered, so, by Zorn's Lemma in $F(F)$ then is $P$ a maximal element. I want to prove that $P$ is a prime state-filter. Since $P \in F(F)$, it follows that $P$ is a proper state-filter and $P \cap I = \varnothing$.

Let $a, b \in A$ such that $(a \circ \sigma(a)) \cup (b \circ \sigma(b)) \in P$. Let's suppose that $a \notin P$ and $b \notin P$. Consider the sets $F_{\sigma}(P, a) \cup F_{\sigma}(P, b)$, which represent state-filters generated by $P$ and $a$, respectively $P$ and $b$. Then $P$ is strictly contained in $F_{\sigma}(P, a)$ and $F_{\sigma}(P, b)$ and, by the maximality of $P$, we deduce that $F_{\sigma}(P, a) \notin F(F)$ and $F_{\sigma}(P, b) \notin F(F)$. Thus $F_{\sigma}(P, a \cup b) \cap I \notin \varnothing$ and $F_{\sigma}(P, b \cup a) \cap I \notin \varnothing$. Let $x \in F_{\sigma}(P, a) \cap I$ and $y \in F_{\sigma}(P, b) \cap I$. Then there exist $i_1, i_2 \in P$ and $m, n \in \mathbb{N}$ such that $x \geq i_1 \circ (a \circ \sigma(a))^m$ and $y \geq i_2 \circ (b \circ \sigma(b))^n$, so $x \lor y \geq (i_1 \circ (a \circ \sigma(a))^m) \lor (i_2 \circ (b \circ \sigma(b))^n) \geq (i_1 \land i_2) \circ (i_1 \lor (b \circ \sigma(b))^n) \lor (i_2 \lor (a \circ \sigma(a))^m) \lor (x \circ \sigma(a)) \lor (b \circ \sigma(b))^n \in P$, that is, $x \lor y \in P$. But $x, y \notin I$, so $x \lor y \notin I$, hence $P \cap I \notin \varnothing$, a contradiction.

Thus $P$ is a prime state-filter.

□

**Proposition 4.8.** Let $(A, \sigma)$ be a state BL-algebra and $a \in A, a < 1$. Then there exists a prime state-filter $P$ of $(A, \sigma)$ such that $a \notin P$.

**Proof.** Like in the Proposition 4.7 we consider the set

$$F(a) = \{ F \mid F \text{ is a state-filter and } a \notin F \}.$$ 

Since $\{ 1 \} \notin F(a)$, it follows that $F(a)$ is nonvoid.

We can easily prove that the set $F(a)$ is inductively ordered, so by Zorn's Lemma then is $P$ a maximal element of $F(a)$. I want to prove that $P$ is a prime state-filter. Let $x, y \in A$ such that $(x \circ \sigma(x)) \cup (y \circ \sigma(y)) \in P$. Let's suppose that $x \notin P$ and $y \notin P$. Considering the sets $F_{\sigma}(P, x)$ and $F_{\sigma}(P, y)$, which represent state-filters generated by $P$ and $x$, respectively $P$ and $y$, it follows that $P$ is strictly contained in $F_{\sigma}(P, x)$ and $F_{\sigma}(P, y)$ and, by the maximality of $P$, we deduce that $a \in F_{\sigma}(P, x) \cap F_{\sigma}(P, y)$. Then there exist $i_1, i_2 \in P$ and $m, n \in \mathbb{N}$ such that $a \geq i_1 \circ (x \circ \sigma(x))^m$ and $a \geq i_2 \circ (y \circ \sigma(y))^n$, so $a \geq (i_1 \circ (x \circ \sigma(x))^m) \lor (i_2 \circ (y \circ \sigma(y))^n) \geq (i_1 \land i_2) \circ (i_1 \lor (x \circ \sigma(x))^m) \lor (i_2 \lor (y \circ \sigma(y))^n) \in P$, that is, $a \notin P$, a contradiction. Thus $P$ is a prime state-filter and $a \notin P$.

□

**Corollary 4.1.** Let $(A, \sigma)$ be a state BL-algebra and $P$ a proper state-filter of $(A, \sigma)$. Then there exists a maximal state-filter $F_0$ of $(A, \sigma)$ such that $P \subseteq F_0$.

**Proof.** The Proposition 4.7 is applied for $I = \{ 0 \}$ and $F = P$. Let $F_0$ be a maximal element of the set $F(P) = \{ F' \mid F' \text{ is a proper state-filter and } P \subseteq F' \}$. I want to prove that $F_0$ is a maximal state-filter of $(A, \sigma)$. Indeed, if $F_1$ is a state-filter of $(A, \sigma)$ such that $F_0 \subseteq F_1$, then, the maximality of $F_0$, it follows that $F_1 \notin F(P)$, so $F_1$ is not a proper state-filter, so $F_1 = A$.

□

On the basis of the previous results, we will be able to characterize the set $Rad_{\sigma}(A)$, of the intersection of all maximal state-filters of a state-morphism BL-algebra $(A, \sigma)$. Firstly, we will establish the following result:
Proposition 4.9. Let \((A, \sigma)\) be a state BL-algebra. Then
\[
\{ x \in A \mid (\sigma (x^n))^* \leq \sigma (x), \text{ for every } n \in \mathbb{N} \} \subseteq \text{Rad}_\sigma (A).
\]

Proof. Consider \(B = \{ x \in A \mid (\sigma (x^n))^* \leq \sigma (x), \text{ for every } n \in \mathbb{N} \}\) and let \(x \in B\).
Let’s suppose that \(x \notin \text{Rad}_\sigma (A)\), therefore there exists a maximal state-filter \(F\) of \((A, \sigma)\) such that \(x \notin F\). According to Proposition 4.8, there exists \(n \in \mathbb{N}\) such that \((\sigma (x^n))^* \in F\). Since \((\sigma (x^n))^* \leq \sigma (x)\), we deduce that \(\sigma (x) \in F\). But then \(\sigma (x^n) \in F\) and, since \((\sigma (x^n))^* \in F\), we obtain that \(F = A\), a contradiction. Therefore \(B \subseteq \text{Rad}_\sigma (A)\).

PROOF

Proposition 4.10. Let \((A, \sigma)\) be a state-morphism BL-algebra. Then
\[
\text{Rad}_\sigma (A) \subseteq \{ x \in A \mid (\sigma (x^n))^* \leq \sigma (x), \text{ for every } n \in \mathbb{N} \}.
\]

Proof. Consider \(B = \{ x \in A \mid (\sigma (x^n))^* \leq \sigma (x), \text{ for every } n \in \mathbb{N} \}\) and let \(x \in \text{Rad}_\sigma (A)\). Let’s suppose that \(x \notin B\), so there exists \(n \in \mathbb{N}\) such that \((\sigma (x^n))^* \notin (\sigma (x))\), that is, \((\sigma (x^n))^* \rightarrow \sigma (x) < 1\). According to Proposition 4.8 there exists a prime state-filter \(P\) of \((A, \sigma)\) such that \(\sigma (x^n) \rightarrow \sigma (x) \notin P\). On the other hand \(\sigma ((\sigma (x^n))^* \rightarrow \sigma (x)) = \sigma (\sigma (x^n))^* \rightarrow \sigma (x)\) (since \(\sigma\) is a morphism) = \((\sigma (x^n))^* \rightarrow (\sigma (x))\) (from Proposition 2.1, (4)) = \((\sigma (x^n))^* \rightarrow \sigma (x)\) and, analogously,
\[
\sigma (\sigma (x) \rightarrow (\sigma (x^n))^*) = \sigma (x) \rightarrow (\sigma (x^n))^*.
\]

Then
\[
((\sigma (x^n))^* \rightarrow (\sigma (x)) \circ (\sigma (x^n))^* \rightarrow (\sigma (x))) \vee ((\sigma (x) \rightarrow (\sigma (x^n))^*) \circ (\sigma (x) \rightarrow (\sigma (x^n))^*))
\]
\[
= (\sigma (x^n))^* \rightarrow \sigma (x))^2 \vee (\sigma (x) \rightarrow (\sigma (x^n))^*)^2
\]
\[
\geq ((\sigma (x^n))^* \rightarrow \sigma (x)) \vee (\sigma (x) \rightarrow (\sigma (x^n))^*)^4
\]

(according to Proposition 2.1, (4)) = 1 \in P, and, since \(P\) is prime and \((\sigma (x^n))^* \rightarrow (\sigma (x)) \notin P\), we deduce that \(\sigma (x) \rightarrow (\sigma (x^n))^* \in P\). But \(\sigma (x) \rightarrow (\sigma (x^n))^* = (\sigma (x) \circ (\sigma (x^n))^*)^4\) (from Proposition 2.1, (5)),

thus \((\sigma (x^n))^* \in P\). According to Corollary 4.1, there exists a maximal state-filter \(F_0\) of \((A, \sigma)\) such that \(P \subseteq F_0\), so \((\sigma (x^n+1))^* \in F_0\), that is, \((\sigma (x^n))^* \notin F_0\). Then \(\sigma (x) \notin F_0\) and so \(x \notin F_0\), namely \(x \notin \text{Rad}_\sigma (A)\), a contradiction. Therefore \(\text{Rad}_\sigma (A) \subseteq B\).

PROOF

From Propositions 4.9 and 4.10 we obtain:

Theorem 4.1. Let \((A, \sigma)\) be a state-morphism BL-algebra. Then
\[
\text{Rad}_\sigma (A) = \{ x \in A \mid (\sigma (x^n))^* \leq \sigma (x), \text{ for every } n \in \mathbb{N} \}.
\]
Moreover, \(\text{Rad} (A) \subseteq \text{Rad}_\sigma (A)\).

Proof. The first part result from Propositions 4.9 and 4.10. For the second part, let \(x \in \text{Rad} (A)\), so \((x^n)^* \leq x\), for every \(n \in \mathbb{N}\).

Then \(\sigma ((x^n)^*) \leq \sigma (x)\), for every \(n \in \mathbb{N}\), so \((\sigma (x^n))^* \leq \sigma (x)\), for every \(n \in \mathbb{N}\), that is, \(x \in \text{Rad}_\sigma (A)\).

PROOF

5. Classes of BL-algebras

Within this section, we are going to present some classes of BL-algebras, such as simple, semisimple and local BL-algebras, we will then define the concepts of simple,
semisimple and local state $BL-$algebras $(A, \sigma)$, next we will introduce the concepts of simple, semisimple and local state $BL-$algebras $(A, \sigma)$ relative to its state-filters set, and we will finally establish relations between these concepts, which occur in some conditions imposed to the state-operator $\sigma$.

**Definition 5.1.** A $BL-$algebra $A$ is called simple if its only filters are $\{1\}$ and $A$. A state $BL-$algebra $(A, \sigma)$ is called simple if $\sigma(A)$ is simple.

We will now define a new concept:

**Definition 5.2.** A state $BL-$algebra $(A, \sigma)$ is called simple relative to its state-filters set if it has only two state-filters: $\{1\}$ and $A$.

**Example 5.1.** Let’s consider a state $BL-$algebra $(A, \sigma)$. If $\sigma = id_A$, then the three concepts from Definition 5.1 are the same. Let’s consider the state $BL-$algebra $(A, \sigma)$ from Example 3.2. We have $\sigma(A) = \{0, a, 1\}$.

If $I \subseteq \sigma(A)$ is a filter, $I \neq \{1\}$, and if $a \in I$, then $a \otimes a = 0 \in I$, so $I = \sigma(A)$. Thus $\sigma(A)$ is simple, so $(A, \sigma)$ is simple. By the contrary, according to Example 4.1 $A$ is not simple and $(A, \sigma)$ is not simple relative to its state-filters set. For each state $BL-$algebras $(A, \sigma)$ from Examples 3.3, 3.4, 3.5 we have $\sigma(A) = \{0, 1\}$, so $(A, \sigma)$ is simple, but $A$ is not simple and $(A, \sigma)$ is not simple relative to its state-filters set.

**Remark 5.1.** According to [2], if $(A, \sigma)$ is a state $BL-$algebra such that $A$ is simple, then $\sigma(A)$ is simple, so $(A, \sigma)$ is simple.

**Theorem 5.1.** [2] Let $(A, \sigma)$ be a state-morphism $BL-$algebra. Then the following conditions are equivalent:

1. $(A, \sigma)$ is simple;
2. $\ker(\sigma)$ is a maximal filter of $A$.

**Proposition 5.1.** Let $(A, \sigma)$ be a state $BL-$algebra. If $(A, \sigma)$ is simple relative to its state-filters set, then $(A, \sigma)$ is simple.

**Proof.** Let $J$ be a filter of $\sigma(A), J \neq \{1\}$. We will prove that $J = \sigma(A)$. Consider $F_J = \{z \in A \mid z \geq j, \text{ for a certain } j \in J\}$. If $x, y \in F_J$, then there exist $j_1, j_2 \in J$ such that $x \geq j_1, y \geq j_2$, so $x \otimes y \geq j_1 \otimes j_2 \in J$, hence $x \otimes y \in F_J$. If $x \in F_J$ and $x \leq y$, then obviously $y \in F_J$.

If $x \in F_J$, then $x \geq j, j \in J$, so $\sigma(x) \geq \sigma(j) = j$ (since $j \in \sigma(A)$), hence $\sigma(x) \in F_J$. Therefore $F_J$ is a state-filter of $(A, \sigma)$. Since $(A, \sigma)$ is simple relative to its state-filters set, and $F_J \neq \{1\}$ (since $J \subseteq F_J$), it follows that $F_J = A$, so $0 \in F_J$, hence $0 \in J$, that is, $J = \sigma(A)$. \(\square\)

**Remark 5.2.** If $(A, \sigma)$ is a simple state $BL-$algebra relative to its state-filters set, then, since $\ker(\sigma)$ is a state filter and $\ker(\sigma) \neq A$, it follows that $\ker(\sigma) = \{1\}$, thus $\sigma$ is a faithful operator.

**Remark 5.3.** If $(A, \sigma)$ is a simple state $BL-$algebra, then it doesn’t necessarily follow that $\sigma$ is faithful. For instance, for the simple state $BL$-algebra $(A, \sigma)$ from the Example 3.2 we have $\ker(\sigma) = \{b, 1\} \neq \{1\}$.

**Theorem 5.2.** Let $(A, \sigma)$ be a state $BL-$algebra. Then the following conditions are equivalent:

1. $(A, \sigma)$ is simple relative to its state-filters set;
2. $(A, \sigma)$ is simple and $\sigma$ is faithful.
Let’s consider the $A$ state-

\[ F \text{ false. Then conditions are equivalent:} \]

$A \subseteq F \text{ and, since } 0 \in \sigma (A), \text{ we deduce that } I = A. \text{ If } I \cap \sigma (A) = \{ 1 \}, \text{ let } x \in I. \text{ Then } \sigma (x) \in I \cap \sigma (A), \text{ so } \sigma (x) = 1, \text{ that is, } x = 1 (\text{since } \sigma \text{ is faithful}), \text{ so } I = \{ 1 \}. \text{ Therefore the only state-filters of } (A, \sigma) \text{ are } \{ 1 \} \text{ and } A.

\[ \square \]

**Theorem 5.3.** Let $(A, \sigma)$ be a state-morphism $BL-$ algebra. Then the following conditions are equivalent:

(i) $(A, \sigma)$ is simple relative to its state-filters set;

(ii) $A$ is simple.

**Proof.** (i) ⇒ (ii) Results from the Proposition 5.1 and the Remark 5.2.

(ii) ⇒ (i) Let $I$ be a state-filter of $(A, \sigma)$. Then $I \cap \sigma (A)$ is a filter of $\sigma (A)$, and so $I \cap \sigma (A) = \{ 1 \}$ or $I \cap \sigma (A) = \sigma (A)$. If $I \cap \sigma (A) = \sigma (A)$, then $\sigma (A) \subseteq I$ and, since $0 \in \sigma (A)$, we deduce that $I = A$. If $I \cap \sigma (A) = \{ 1 \}$, let $x \in I$. Then $\sigma (x) \in I \cap \sigma (A), \text{ so } \sigma (x) = 1, \text{ that is, } x = 1 (\text{since } \sigma \text{ is faithful}), \text{ so } I = \{ 1 \}$. Therefore the only state-filters of $(A, \sigma)$ are $\{ 1 \}$ and $A$.

From of the Theorems 5.3 and 5.3 it follows:

**Theorem 5.4.** Let $(A, \sigma)$ be a state-morphism $BL-$ algebra and $\sigma$ is faithful. Then the following conditions are equivalent:

(i) $A$ is simple;

(ii) $(A, \sigma)$ is simple.

**Proof.** (i) ⇒ (ii) Results from the Remark 5.1.

(ii) ⇒ (i) If $(A, \sigma)$ is simple, since $\sigma$ is faithful, then from the Theorem 5.2 it follows that $(A, \sigma)$ is simple relative to its state-filters set and then, from Theorem 5.3 we deduce that $A$ is simple.

\[ \square \]

**Definition 5.3.** A $BL-$ algebra $A$ is called local if it has only a maximal filter. A state $BL-$ algebra $(A, \sigma)$ is called local if $\sigma (A)$ is local.

Next we define a new concept:

**Definition 5.4.** A state $BL-$ algebra $(A, \sigma)$ is local relative to its state-filters set if it has only a maximal state-filter.

**Example 5.2.** Let’s consider the $BL-$ algebra $A$ and the state-operator

$\sigma : A \rightarrow A$ from Example 3.2. Then $A$ is local, $(A, \sigma)$ is local and $(A, \sigma)$ is local relative to its state-filters set. In Example 3.3 the $BL-$ algebra $A$ is not local, but $(A, \sigma)$ is local relative to its state-filters set.

**Theorem 5.5.** Let $(A, \sigma)$ be a state $BL-$ algebra. Then the following conditions are equivalent:

(i) $(A, \sigma)$ is local relative to its state-filters set;

(ii) $(A, \sigma)$ is local.

**Proof.** (i) ⇒ (ii) Let $F$ be the only maximal state-filter of $(A, \sigma)$. Then $F \cap \sigma (A)$ is a filter of $\sigma (A)$. We will prove that $F \cap \sigma (A)$ is the only maximal filter of $\sigma (A)$. If $F \cap \sigma (A) = \sigma (A)$, then $\sigma (A) \subseteq F$, so $0 \in F$, a contradiction. Let $I$ be an arbitrary proper filter of $\sigma (A)$. We consider the set $F_{\sigma} (I) = \{ z \in A \mid z \geq i, i \in I \}$, which represents the state–filter generated by $I$ in $(A, \sigma)$. If $F_{\sigma} (I) = A$, then $0 \in F_{\sigma} (I)$, so $0 \in I$, false. Then $F_{\sigma} (I)$ is a proper state-filter, so $F_{\sigma} (I) \subseteq F$, that is,
\[ I = I \cap \sigma(A) \subseteq F_{\sigma}(I) \cap \sigma(A) \subseteq F \cap \sigma(A). \]

Then \( F \cap \sigma(A) \) is a proper filter which contains any proper filter \( I \) of \( \sigma(A) \), thus it is the only maximal filter of \( \sigma(A) \), so \( (A, \sigma) \) is local.

\((ii) \Rightarrow (i)\) Let \( I \) be the only maximal filter of \( \sigma(A) \) and the set \( F_{\sigma}(I) = \{ z \in A \mid z \geq i, i \in I \} \), which represents the state-filter generated by \( I \) in 
\((A, \sigma)\). Let \( F(I) = \{ F \mid F \text{ is a proper state-filter of } (A, \sigma) \text{ and } I \subseteq F \} \).

If \( F_{\sigma}(I) \) is not proper, then \( 0 \in F_{\sigma}(I) \), so \( 0 \in I \), false. Thus \( F_{\sigma}(I) \in F(I) \), so \( F(I) \) is nonvoid. It is easily to verify that \( F(I) \) is inductively ordered, so by Zorn’s Lemma then is \( F \) a maximal element of \( F(I) \). We will prove that \( F \) is the only maximal state-filter of \((A, \sigma)\). Indeed, let \( F_1 \) be an arbitrary proper state-filter of 
\((A, \sigma)\). Let’s suppose that there exists an element \( x \in F_1 \setminus F \). Then \( \sigma (x) \in F_1 \cap \sigma(A) \).

If \( F_1 \cap \sigma(A) = \sigma(A) \) it follows that \( \sigma(A) \subseteq F_1 \), so \( 0 \in F_1 \), a contradiction. Thus \( F_1 \cap \sigma(A) \neq \sigma(A) \), \( F_1 \cap \sigma(A) \) is a filter of \( \sigma(A) \) and, since \( I \) is a maximal filter of 
\( \sigma(A) \), it follows that \( F_1 \cap \sigma(A) \subseteq I \), so \( \sigma(x) \in I \).

Then \( \sigma(x) \in F_\sigma(I) \), so \( \sigma(x) \in F \). Since \( x \notin F \) and \( F \) is a maximal state-filter, then, according to Proposition 4.8, it follows that there exists \( n \in \mathbb{N}^* \) such that 
\((\sigma(x)^n)^* \in F \).

But \( (x)^n \notin F \), a contradiction. Thus \( F_1 \subseteq F \), so \( F \) is the only maximal state-filter of 
\((A, \sigma)\), so \( (A, \sigma) \) is local relative to its state-filters set.

\[ \square \]

**Definition 5.5.** A BL-algebra \( A \) is called semisimple if \( \text{Rad}(A) = \{1\} \). Let \( (A, \sigma) \) be a state BL-algebra. \((A, \sigma)\) is called semisimple if \( \text{Rad}(\sigma(A)) = \{1\} \).

Concerning all this, we are now going to define a new concept:

**Definition 5.6.** A state BL-algebra \((A, \sigma)\) is called semisimple relative to its state-filters set if \( \text{Rad}_\sigma(A) = \{1\} \).

**Example 5.3.** Let’s consider the state BL-algebra \((A, \sigma)\) from Example 3.2. The \( A \) algebra is not semisimple, but \((A, \sigma)\) is semisimple because \( \text{Rad}(\sigma(A)) = \{1\} \). It is not semisimple relative to its state-filters set.

The \( A \) algebras from Examples 3.3, 3.4 are semisimple, \((A, \sigma)\) is not semisimple, but they are semisimple relative to its state-filters set.

The \( A \) algebra from Example 3.5 is not semisimple, \((A, \sigma)\) is not semisimple relative to its state-filters set, but \((A, \sigma)\) is semisimple.

The \( \text{IL}_\sigma \) algebra from Proposition 3.1 is semisimple, and, since \( \sigma = \text{id}_{\text{IL}}, (\text{IL}_\sigma, \sigma) \) is semisimple and semisimple relative to its state-filters set.

**Proposition 5.2.** ([2]) Let \((A, \sigma)\) be a state BL-algebra. Then 
\[ \sigma(\text{Rad}(A)) \supseteq \text{Rad}(\sigma(A)) = \sigma(\text{Rad}_\sigma(A)). \]

**Theorem 5.6.** Let \((A, \sigma)\) be a state BL-algebra. Then the following conditions are equivalent:

(i) \((A, \sigma)\) is semisimple and \( \sigma \) is faithful;

(ii) \((A, \sigma)\) is semisimple relative to its state-filters set.

**Proof.** (i) \( \Rightarrow \) (ii) According to Proposition 5.2 we have \( \sigma(\text{Rad}_\sigma(A)) = \text{Rad}(\sigma(A)) = \{1\} \), so \( \text{Rad}_\sigma(A) \subseteq \ker(\sigma) = \{1\} \), that is, \( \text{Rad}_\sigma(A) = \{1\} \).

(ii) \( \Rightarrow \) (i) \( \text{Rad}(\sigma(A)) = \sigma(\text{Rad}_\sigma(A)) = \sigma(\{1\}) = \{1\} \), so \((A, \sigma)\) is semisimple.

We will prove that \( \sigma \) is faithful. Let \( x \in \ker(\sigma) \), that is, \( \sigma(x) = 1 \). Let’s suppose that \( x \notin \text{Rad}_\sigma(A) \). Then there exists a maximal state-filter \( F \) such that \( x \notin F \). According to Proposition 4.4 there exists \( n \in \mathbb{N}^* \) such that 
\( (\sigma(x)^n)^* \in F \), so \( 0 \in F \), a contradiction. Thus \( x \notin \text{Rad}_\sigma(A) = \{1\} \), so \( \sigma \) is faithful.

\[ \square \]
References


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