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A note on *BL*-algebras with internal state

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Abstract. The scope of this paper is to put in evidence some properties of the BL-algebras with internal state. I introduce the concepts of prime and maximal state-filters, I prove a Prime state-filter theorem 4.7 and I characterize the set $\text{Rad}_\sigma(A)$, which represents the intersection of all maximal state-filters of a state BL-algebra (A,σ). Also, I introduce the concepts of simple, semisimple and local state BL-algebras relative to its state-filter set.

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1. Introduction

The concept of state $MV-$algebras was firstly introduced by Flaminio and Montagna in [4] and [5] as a $MV-$algebra endowed with a unary operation σ (called a state-operator), which preserves the usual properties of states. Di Nola and Dvurečenskij presented in [6] a stronger version of states $MV-$algebras namely state-morphism $MV-$algebras. Afterwards Ciungu, Dvurečenskij and Hyčko extended in [2] the concept of state (morphism) $MV-$algebra and in the case of $BL-$algebras and they extended the properties of a state-operator. The present article is structured into five sections.

In Section 2, basic properties regarding the concepts of $MV-$algebra, $BL-$algebra are being presented, as well as some basic properties of the operations defined on these algebras, which are to be used afterwards. The concept of state (morphism) $-$operator on a $BL-$algebra also belongs to this section, as well some of its properties.

In Section 3 some examples of state $BL-$algebras are presented. In Section 4 the concept of state-filter on a state $BL-$algebra is introduced. There are presented some examples of filters and state-filters, as well as the concepts of maximal state-filter, prime state-filter, some of their characteristics and, if the state-operator σ is a morphism, the set $Rad_\sigma(A)$ is characterised, in which $Rad_\sigma(A)$ represents the intersection of all maximal state-filters of a state $BL-$algebra (A,σ).

In Section 5, there are presented some classes of $BL-$algebras such as simple, semisimple and local as well as simple, semisimple and local state $BL-$algebras. There are introduced the concepts of simple, semisimple and local state $BL-$algebras relative to its state-filters set and there are establishised relations between these structures in certain conditions imposed to the state-operator σ.

2. Preliminaries

Definition 2.1. *An algebra* $(A,\wedge,\vee,\odot,\rightarrow,0,1)$ *of the type* $(2,2,2,2,0,0)$ *is called a* $BL-$*algebra if satisfies the following axioms:*

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(1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice;*
(2) $(A, \odot, 1)$ *is a commutative monoid;*
(3) $x \odot y \leq z$ *iff* $x \leq y \rightarrow z$;
(4) $x \wedge y = x \odot (x \rightarrow y)$;
(5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$;
for every $x, y, z \in A$.

We will denote $x^* = x \rightarrow 0$, $x \in A$. If $x \in A$, we define $x^0 = 1$ and for $n \geq 1$ we define $x^n = x^{n-1} \odot x$.

Definition 2.2. *Let* A *be a* $BL-$*algebra and* $x \in A$. *If there exists the least number* $n \in \mathbb{N}^*$ *such that* $x^n = 0$, *then we set* $ord\,(x) = n$. *If there is no such a number* $($*that is,* $x^n > 0$, *for every* $n \geq 0)$, *then we set* $ord\,(x) = \infty$.

We recall some results relative to $BL-$algebras:

Proposition 2.1. *Let* A *be a* $BL-$*algebra. Then:*
(1) *if* $a \leq b$ *and* $c \leq d$ *then* $a \odot c \leq b \odot d$;
(2) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$;
(3) $a \vee (b \odot c) \geq (a \vee b) \odot (a \vee c)$;
(4) $a^m \vee b^n \geq (a \vee b)^{mn}, m, n \in \mathbb{N}$;
(5) $(a \odot b)^* = a \rightarrow b^*$;
(6) $a \odot (a \rightarrow (a \odot b)) = a \odot b$;
for every $a, b, c \in A$.

Definition 2.3. *An algebra* $(A, \oplus, *, 0)$ *of the type* $(2, 1, 0)$ *is called a* $MV-$*algebra if satisfies the following axioms:*
(1) $(A, \oplus, 0)$ *is a commutative monoid;*
(2) $x^{**} = x$, *for every* $x \in A$;
(3) $x \oplus 0^* = 0^*$, *for every* $x \in A$;
(4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, *for every* $x, y \in A$.

On a $BL-$algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ we define the operation $\oplus$ on A by $x \oplus y = (x^* \odot y^*)^*, x, y \in A$. If $x^{**} = x$, for every $x \in A$, then $(A, \oplus, *, 0)$ it becomes a $MV-$algebra. We are now defining the concept of state-operator on a $BL-$algebra.

Definition 2.4. [2] *Let* A *be a* $BL-$*algebra. An application* $\sigma : A \rightarrow A$ *which verifies the properties:*
$(1)_{BL}$ $\sigma(0) = 0$;
$(2)_{BL}$ $\sigma(x \rightarrow y) = \sigma(x) \rightarrow \sigma(x \wedge y)$;
$(3)_{BL}$ $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \rightarrow x \odot y)$;
$(4)_{BL}$ $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$;
$(5)_{BL}$ $\sigma(\sigma(x) \rightarrow \sigma(y)) = \sigma(x) \rightarrow \sigma(y)$;
for every $x, y \in A$, *is called state-operator on* A, *and the pair* (A, σ) *is called a state* $BL-$*algebra or, more precisely, a* BL*-algebra with internal state.*

Some examples of state-operators will be presented in Section 3.

Proposition 2.2. [2] *In a state* $BL-$*algebra* (A, σ) *the following hold:*
(a) $\sigma(1) = 1$;
(b) $\sigma(x^*) = \sigma(x)^*$, *for every* $x \in A$;
(c) *if* $x, y \in A$ *and* $x \leq y$ *then* $\sigma(x) \leq \sigma(y)$;
(d) $\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y)$, *for every* $x, y \in A$;
(e) $\sigma(x \rightarrow y) \leq \sigma(x) \rightarrow \sigma(y)$, *for every* $x, y \in A$;

(*f*) $\sigma(\sigma(x)) = \sigma(x)$, *for every* $x \in A$;
(*g*) $\sigma(A)$ *is a* $BL-$*subalgebra of* A *and* $\sigma(A) = \{x \in A \mid \sigma(x) = x\}$.

Definition 2.5. [2] *A state-morphism operator on a* $BL-$*algebra* A *is an application* $\sigma : A \to A$ *which verifies* $(1)_{BL}, (2)_{BL}, (4)_{BL}, (5)_{BL}$ *and*
$(6)_{BL}$ $\sigma(x \odot y) = \sigma(x) \odot \sigma(y)$, *for every* $x, y \in A$.

Remark 2.1. *Any state-morphism operator* σ *on a* $BL-$*algebra* A *is a state-operator on* A. *Indeed, by using* $(6)_{BL}$ *we have:*
$\sigma(x) \odot \sigma(x \to x \odot y) = \sigma(x \odot (x \to x \odot y)) = \sigma(x \odot y)$, *according to Proposition 2.1.*
If σ *is a state-operator on* A, *we define* $\ker(\sigma) = \{x \in A \mid \sigma(x) = 1\}$.

Definition 2.6. *A state-operator* $\sigma : A \to A$ *is called faithful iff* $\ker(\sigma) = 1$.

3. Examples of state-operators on $BL-$algebras

Example 3.1. *If* A *is a* $BL-$*algebra, then* $\sigma : A \to A$, *defined by* $\sigma(x) = x$, *for every* $x \in A$, *is a state-operator on* A, *called the identity state-operator on* A. *Thus* (A, id_A) *is a state* $BL-$*algebra.*

Example 3.2. [2] *Let* $A = \{0, a, b, 1\}$ *be with* $0 < a < b < 1$.
Then $(A, \wedge, \vee, \odot, \to, 0, 1)$ *with the following operations:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

it becomes a $BL-$*algebra, but not a* $MV-$*algebra* (*since* $b^{**} = 1 \neq b$).
The fact that $\sigma : A \to A$, *given by* $\sigma(0) = 0, \sigma(a) = a, \sigma(b) = \sigma(1) = 1$, *is a state-operator on* A, *is verified. Moreover,* $(6)_{BL}$ *holds, so* σ *is a state-morphism operator on* A.

Example 3.3. [3] *Let* $A = \{0, a, b, c, d, 1\}$ *,with the operations* $\odot$ *and* $\to$ *given by the following tables:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then the $BL-$*algebra* $(A, \wedge, \vee, \odot, \to, 0, 1)$ *is a* $MV-$*algebra.*
We will determine the state-operators on A. *Let* $\sigma : A \to A$ *be a state-operator. From* $(1)_{BL}$ *we have* $\sigma(c \to a) = \sigma(c) \to \sigma(c \wedge a)$, *so*
$\sigma(c) = \sigma(c) \to \sigma(a)$. *From the table of the operation* $\to$ *we deduce that the equation* $x = x \to y$ *has only the solutions* $x = c, y = a$ *and* $x = y = 1$.
In the first case we have $\sigma(c) = c$ *and* $\sigma(a) = a$ *and then*
$\sigma(d) = \sigma(a^*) = \sigma(a)^*$ (*accordind to the Proposition 2.2,* (*b*)) $= a^* = d$, *and* $\sigma(b) = \sigma(c^*) = \sigma(c)^* = b$, *so* $\sigma = id_A$. *In the second case we have* $\sigma(c) = \sigma(a) = 1$, *and then* $\sigma(d) = \sigma(a)^* = 0, \sigma(b) = \sigma(c)^* = 0$, *thus*

$\sigma(0) = \sigma(b) = \sigma(d) = 0$ and $\sigma(c) = \sigma(a) = 1$, which verifies $(1)_{BL} - (6)_{BL}$, so this is also a state-morphism operator.

Example 3.4. [3] *Let $A = \{0, a, b, c, d, 1\}$, with the following tables of operations:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then it becomes a $BL-$algebra, which is a $MV-$algebra. Let $\sigma : A \rightarrow A$ be a state-operator. As in the Example 3.3 we have

$\sigma(d \rightarrow c) = \sigma(d) \rightarrow \sigma(d \wedge c) = \sigma(d) \rightarrow \sigma(c) = \sigma(d)$. Since the equation $x = x \rightarrow y$ has only the solutions $x = d, y = c$ and $x = y = 1$ we obtain $\sigma(d) = d, \sigma(c) = c$ or $\sigma(d) = \sigma(c) = 1$. In the first case we have $\sigma = id_A$, and in the second case we have $\sigma(a) = \sigma(b) = \sigma(0) = 0$ and $\sigma(c) = \sigma(d) = \sigma(1) = 1$, both operators being state-morphism operators.

Example 3.5. [3] *Let $A = \{0, c, a, b, 1\}$, in which $0 < c < a, b < 1$ and a, b are incomparable, with the following tables of operations :*

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	c
1	0	c	a	b	1

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

The application $\sigma : A \rightarrow A$, given by $\sigma(0) = 0$ and $\sigma(x) = 1$ otherwise, is a state-morphism operator.

We recall that a $t-$norm is a function $t : [0,1] \times [0,1] \rightarrow [0,1]$, which verifies the conditions:

(1) $t(x, y) = t(y, x)$, for every $x, y \in [0,1]$;

(2) $t(t(x, y), z) = t(x, t(y, z))$, for every $x, y, z \in [0,1]$;

(3) $t(x, 1) = x$, for every $x \in [0,1]$;

(4) if $x \leq y$ then $t(x, z) \leq t(y, z), x, y, z \in [0,1]$.

If t is continuous, we define $x \odot_t y = t(x, y)$ and

$x \rightarrow_t y = \sup\{z \in [0,1] \mid t(z, x) \leq y\}$, for $x, y \in [0,1]$. In these conditions $\mathbf{I}_t = ([0,1], \min, \max, \odot_t, \rightarrow_t, 0, 1)$ is a $BL-$algebra. Moreover, according to [1], the variety of $BL-$algebras is generated by all the $\mathbf{I}_t$ with a continuous norm t. There are three basic continuous $t-$norms on $[0,1]$:

(i) Łukasiewicz: $Ł(x, y) = \max\{x + y - 1, 0\}$, with

$x \rightarrow_Ł y = \min\{1 - x + y, 1\}$;

(ii) Gödel: $G(x, y) = \min\{x, y\}$, with $x \rightarrow_G y = 1$ if $x \leq y$ and $x \rightarrow_G y = y$ otherwise;

(iii) product: $P(x, y) = xy$, with $x \rightarrow_P y = 1$ if $x \leq y$ and $x \rightarrow_P y = \frac{y}{x}$ otherwise.

Then we have:

Proposition 3.1. [2]

(1)*If σ is a state-operator on $\mathbf{I}_L$, then $\sigma(x) = x$, for every $x \in [0,1]$.*

(2) *Let $a \in [0,1]$ and we define $\sigma_a(x) = x$ if $x \leq a$ and $\sigma_a(x) = 1$ otherwise. For $a \in (0,1]$ we define the application $\sigma^a(x) = x$ if $x < a$ and $\sigma_a(x) = 1$ otherwise.*

Then σ_a *şi* σ^a *are state-morphism operators on* $\mathbf{I}_G$ *and, if* σ *is a state-operator on* $\mathbf{I}_G$*, then* $\sigma = \sigma_a$ *or* $\sigma = \sigma^a$ *for a certain* $a \in [0,1]$.

(3) *If* σ *is a state-operator* $\mathbf{I}_P$*, then* $\sigma(x) = x$*, for every* $x \in [0,1]$ *or* $\sigma(x) = 1$*, for every* $x > 0$.

Proposition 3.2. [2] *Let* A *be a finite linear Gödel BL−algebra, that is,* $x^2 = x$*, for every* $x \in A$*. Then, with the notations from Proposition 3.1* σ^a *and* σ_a *are state-morphism operators, and any state-operator on* A *is of the form* σ^a *or* σ_a*, for a certain* $a \in [0,1]$.

Actually we have the following more general result:

Proposition 3.3. *Let* A *be a linear Gödel BL−algebra and* $B \subset A$ *such that* $0 \in B, 1 \notin B$ *and, if* $x \in B, y \in A \setminus B$*, then* $x < y$*. Then the application* $\sigma_B : [0,1] \to [0,1]$*, given by* $\sigma_B(x) = x$ *if* $x \in B$ *and* $\sigma_B(x) = 1$ *otherwise, is a state-morphism operator on* A*, and, conversely, any state-operator on* A *is of such a form.*

Proof. Firstly we observe that, if $x, y \in A$ then
$x \odot y \geq x \odot (x \wedge y) = x \odot (x \odot (x \to y)) = x^2 \odot (x \to y) = x \odot (x \to y)$
$= x \wedge y \geq x \odot y$, so $x \odot y = x \wedge y = \min\{x, y\}$, for every $x, y \in A$.
Then $x \to y = \sup\{z \in A \mid x \odot z \leq y\} = \sup\{z \in A \mid x \wedge z \leq y\}$.
If $x \leq y$, then $x \to y = 1$.
If $x > y$, then $\sup\{z \in A \mid x \wedge z \leq y\} = \sup\{z \in A \mid \min\{x, z\} \leq y\} = y$.

We will verify the $(1)_{BL} - (5)_{BL}$ axioms. Since $0 \in B$ we have that $\sigma_B(0) = 0$, so the $(1)_{BL}$ is proved.

If $x, y \in B$, then we have: $\sigma_B(x \to y) = 1 = \sigma_B(x) \to \sigma_B(x \wedge y)$, if $x \leq y$, and, if $x > y$ we have $\sigma_B(x \to y) = \sigma_B(y)$, and $\sigma_B(x) \to \sigma_B(x \wedge y) = x \to x \wedge y = x \to y = y = \sigma_B(y) = \sigma_B(x \to y)$.

If $x, y \in A \setminus B$, then, since $y \leq x \to y$, it follows that $x \to y \in A \setminus B$, so $\sigma_B(x \to y) = 1$, and $\sigma_B(x) \to \sigma_B(x \wedge y) = 1$ (since $x \wedge y \in A \setminus B$).

If $x \in B, y \in A \setminus B$, then $\sigma_B(x \to y) = \sigma_B(1) = 1 = \sigma_B(x) \to \sigma_B(x \wedge y)$.

If $y \in B, x \in A \setminus B$, then $\sigma_B(x \to y) = \sigma_B(y) = y$ and $\sigma_B(x) \to \sigma_B(x \wedge y) = 1 \to y = y$, so we have an equality again.

Thus $(2)_{BL}$ is proved.

We will now prove $(6)_{BL}$, which means that, according to the Remark 2.1

$(3)_{BL}$ is proved. Indeed, if $x, y \in B$, then $x \odot y = \min\{x, y\} \in B$, so $\sigma_B(x \odot y) = x \odot y = \sigma_B(x) \odot \sigma_B(y)$. If $x, y \in A \setminus B$, then $x \odot y = \min\{x, y\} \in A \setminus B$, so $\sigma_B(x \odot y) = 1 = \sigma_B(x) \odot \sigma_B(y)$.

If $x \in B, y \in A \setminus B$, then $x \odot y = \min\{x, y\} = x$, so $\sigma_B(x \odot y) = \sigma_B(x) = x = \sigma_B(x) \odot 1 = \sigma_B(x) \odot \sigma_B(y)$. Thus $(6)_{BL}$ is fulfilled.

If $x \in B$, then $\sigma_B(\sigma_B(x)) = \sigma_B(x)$, and if $x \in A \setminus B$, then we have $\sigma_B(\sigma_B(x)) = \sigma_B(1) = 1 = \sigma_B(x)$,so $\sigma_B(\sigma_B(x)) = \sigma_B(x), \forall x \in A$.

Then $\sigma_B(\sigma_B(x) \odot \sigma_B(y)) = \sigma_B(\sigma_B(x \odot y))$ (according to $(6)_{BL}$) $= \sigma_B(x \odot y) = \sigma_B(x) \odot \sigma_B(y), \forall x, y \in A$, so $(4)_{BL}$ is verified.

In order to complete the first part of the proof, we still have to verify $(5)_{BL}$. Indeed, if $x, y \in B$, then $\sigma_B(\sigma_B(x) \to \sigma_B(y)) = \sigma_B(x \to y)$, and $\sigma_B(x) \to \sigma_B(y) = x \to y$. If $x \leq y$, then $x \to y = 1$, so $\sigma_B(x \to y) = x \to y$. If $x > y$, $x \to y = y$, and $\sigma_B(x \to y) = \sigma_B(y) = y$, so equality once more. Let's now suppose that $x, y \in A \setminus B$. Then $\sigma_B(\sigma_B(x) \to \sigma_B(y)) = \sigma_B(1) = 1 = \sigma_B(x) \to \sigma_B(y)$.

If $x \in B, y \in A \setminus B$, then $\sigma_B(\sigma_B(x) \to \sigma_B(y)) = \sigma_B(x \to 1) = 1 = \sigma_B(x) \to \sigma_B(y)$.

Finally, if $y \in B, x \in A \setminus B$, then $\sigma_B(\sigma_B(x) \to \sigma_B(y)) = \sigma_B(1 \to y) = \sigma_B(y) = \sigma_B(x) \to \sigma B(y)$.

Conversely, let σ be a state-operator on A and let $a \in (0,1)$.

We are going to prove $\sigma(a) = a$ or $\sigma(a) = 1$. Let's suppose that $\sigma(a) < a$.

Then, according to $(2)_{BL}$, $\sigma(a \to \sigma(a)) = \sigma(a) \to \sigma(a \wedge \sigma(a)) = \sigma(a) \to \sigma(\sigma(a)) = \sigma(a) \to \sigma(a) = 1$.

But $a \to \sigma(a) = \sigma(a)$, so $\sigma(a \to \sigma(a)) = \sigma(a)$, so $\sigma(a) = 1$, a contradiction. If $a < \sigma(a)$, then , from $(2)_{BL}$ we have $\sigma(\sigma(a) \to a) = \sigma(\sigma(a)) \to \sigma(\sigma(a) \wedge a) = \sigma(a) \to \sigma(a) = 1$. Since $\sigma(a) \to a = a$, we obtain $\sigma(a) = 1$.

Let $B = \{a \in [0,1) \mid \sigma(a) = a\}$. Then $\sigma(x) = x$, if $x \in B$, and $\sigma(x) = 1$, if $x \in A \setminus B$. Obviously $0 \in B, 1 \notin B$. Let $x \in B, y \in A \setminus B$. If $y \leq x$, then $\sigma(y) \leq \sigma(x)$, that is $1 \leq x$, a contradiction. So $x < y$. Thus $\sigma = \sigma_B$, in which B fulfills the conditions from the enounciation. $\square$

Example 3.6. [2] *Let A be a $BL-$algebra. Then $(A \times A, \wedge, \vee, \odot, \to, 0, 1)$ it becomes a $BL-$algebra, where $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \leq d$, and the operations are defined on the components. Let $\sigma : A \times A \to A \times A$ be, defined by $\sigma(a,b) = (a,a)$, for every $(a,b) \in A \times A$. It is easily to prove that σ is a state-morphism operator on $A \times A$.*

4. Filters and state-filters

Definition 4.1. *Let A be a $BL-$algebra. A nonvoid subset $F \subseteq A$ is called filter if the following conditions are verified:*

(1) *$x, y \in F$ implies $x \odot y \in F$;*

(2) *$x \in F$ and $x \leq y$ implies $y \in F$.*

A proper filter of A is called a maximal filter if it doesn't belong to any other proper filter of A. The intersection all the maximal filters of A is denoted by $Rad(A)$.

Definition 4.2. [2] *Let (A, σ) be a state(morphism) $BL-$algebra. A nonvoid subset $F \subseteq A$ is called a state(morphism)$-$filter of (A, σ), if F is a filter of A with the property that if $x \in F$, then $\sigma(x) \in F$. A proper state-filter of (A, σ) is called a maximal state-filter if it doesn't belong to any other proper state-filter of (A, σ). The intersection all the maximal state-filters of (A, σ) is denoted by $Rad_\sigma(A)$.*

Example 4.1. *If we consider the Example 3.1 then the filters of A and the state-filters of (A, σ) are the same.*

For A the $BL-$algebra from Example 3.2 the filters are $\{1\}, \{b,1\}, A$, and the state-filters of (A, σ) are $\{1\}, \{b,1\}, A$. The (state)filter $\{b,1\}$ is a maximal (state)filter. In this case $Rad(A) = Rad_\sigma(A) = \{b,1\}$.

Let's now consider A the BL-algebra from Example 3.3 and the state-operator $\sigma : A \to A$, defined by $\sigma(0) = \sigma(b) = \sigma(d) = 0, \sigma(a) = \sigma(c) = 1$. The filters of A are $\{1\}, \{d,1\}, \{a,c,1\}, A$, and the state-filters of (A, σ) are $\{1\}, \{a,c,1\}, A$. The BL-algebra A has two maximal filters: $\{d,1\}$ şi $\{a,c,1\}$. There exists only an maximal state-filter of (A, σ), namely $\{a,c,1\}$. In this case we have $Rad(A) = \{1\}$, and $Rad_\sigma(A) = \{a,c,1\}$.

Let's now the $BL-$algebra from Example 3.4 and the state-operator $\sigma : A \to A$, defined by $\sigma(d) = \sigma(c) = \sigma(1) = 1, \sigma(a) = \sigma(b) = \sigma(0) = 0$. The filters of A are $\{1\}, \{b,1\}, \{c,d,1\}, A$, and the state-filters of (A, σ) are $\{1\}, \{c,d,1\}, A$. There are two maximal filters, namely $\{b,1\}$ and $\{c,d,1\}$, so $Rad(A) = \{1\}$, and a single maximal state-filter, $\{c,d,1\}$, so $Rad_\sigma(A) = \{c,d,1\}$.

For A the $BL-$algebra from Example 3.5 and the state-operator $\sigma : A \to A$, defined by $\sigma(0) = 0$ and $\sigma(x) = 1$ otherwise, the filters and the state-filters are the same: $\{1\}, \{a,1\}, \{b,1\}, \{c,a,b,1\}, A$. We have $Rad(A) = Rad_\sigma(A) = \{c,a,b,1\}$. For the algebra $\mathbf{I}_L$, since $ord(x) < \infty$, for every $x \neq 1$, the only filters are $\{1\}$ and $[0,1]$. Since the single state-operator on $\mathbf{I}_L$ is $id_{\mathbf{I}_L}$, these are also the only state-filters.

In the case of the algebra $\mathbf{I}_G$, the filters are the sets of the form $[x,1]$ or $(x,1]$, where $x \in [0,1]$. $\mathbf{I}_G$ has an only maximal filter, namely $(0,1]$. According to Proposition 3.1(2) if σ is a state-operator on $\mathbf{I}_G$, then $\sigma = \sigma^a$ or $\sigma = \sigma_a$ (with thoses notations). For any of these state-operators, the state-filters and the filters of $\mathbf{I}_G$ are the same. In the case of the algebra $\mathbf{I}_P$, since $ord(x) < \infty$, for every $x \neq 1$, the only filters are $\{1\}$ şi $[0,1]$, which are therefore the only state-filters.

Proposition 4.1. *Let A and B be two $BL-$algebras and let us consider $A \times B$ the $BL-$algebra product of A and B. If F_1, F_2 are filters of A, respectively B, then $F_1 \times F_2$ is a filter of $A \times B$ and, conversely, any filter of $A \times B$ is of the form $F_1 \times F_2$, where F_1, F_2 are filters of A, respectively B.*

Proof. If F_1, F_2 are filters of A, respectively B, then it is imediate that $F_1 \times F_2$ is a filter of $A \times B$. Conversely, let F be a filter of $A \times B$. Since F is nonvoid, then the sets $F_1 := \{x \in A \mid \text{there exists } y \in B \text{ such that } (x,y) \in F\} \subseteq A$ and $F_2 := \{y \in B \mid \text{there exists } x \in A \text{ such that } (x,y) \in F\} \subseteq B$ will be too.

We are going to prove that F_1, F_2 are filters and $F = F_1 \times F_2$. Indeed, if $a, b \in F_1$, then there exists $c, d \in B$ such that $(a,c), (b,d) \in F$, so $(a \odot b, c \odot d) \in F$, so $a \odot b \in F_1$. If $a \in F_1$ and $a \le c$, then, since there exists $b \in B$ such that $(a,b) \in F$ and since $(a,b) \le (c,b)$, it follows that $(c,b) \in F$, therefore $c \in F_1$. Thus F_1 is a filter and analogously it shows that F_2 is a filter. Let $(a,b) \in F$. Then $a \in F_1, b \in F_2$, so $(a,b) \in F_1 \times F_2$, so $F \subseteq F_1 \times F_2$. Let's now $(a,b) \in F_1 \times F_2$. Since $a \in F_1, b \in F_2$, there exist $x \in A, y \in B$ such that $(a,y), (x,b) \in F$. Then $(a,1), (1,b) \in F$ and so $(a \odot 1, 1 \odot b) \in F$, that is, $(a,b) \in F$, therefore $F_1 \times F_2 \subseteq F$. Thus $F = F_1 \times F_2$. □

Let's now consider an $BL-$algebra A which contains proper filters and the state-operator $\sigma : A \times A \to A \times A$, $\sigma(a,b) = (a,a)$, for every $(a,b) \in A \times A$, from the Example 3.6. According to Proposition 4.1 any filter of $A \times A$ is of the form $F_1 \times F_2$, with F_1, F_2 filters of A. If $F_1 \times F_2$ is a state-filter of $(A \times A, \sigma)$, then $F_1 \subseteq F_2$. Indeed, let $a \in F_1$. Then $(a,1) \in F_1 \times F_2$, so $\sigma(a,1) = (a,a) \in F_1 \times F_2$, that is, $a \in F_2$.

Conversely, if $F_1 \times F_2$ is a filter of $A \times A$ such that $F_1 \subseteq F_2$, and $(a,b) \in F_1 \times F_2$, then $\sigma(a,b) = (a,a) \in F_1 \times F_2$, so the state-filters of $(A \times A, \sigma)$ are the sets of the form $F_1 \times F_2$, in which F_1, F_2 are filters of A with $F_1 \subseteq F_2$.

Remark 4.1. [2] *Let A be a $BL-$algebra and σ a state-operator on A. Then $\ker(\sigma)$ is a state-filter of (A, σ).*

Proposition 4.2. [2] *Let A be a $BL-$algebra. A proper filter F of A is a maximal filter iff for any $a \notin F$, there exists $n \in \mathbb{N}^*$ such that $(a^n)^* \in F$.*

Proposition 4.3. [7] *Let A be a $BL-$algebra.*
Then $Rad(A) = \left\{x \in A \mid (x^n)^* \le x, \text{ for every } n \in \mathbb{N}\right\}$.

Proposition 4.4. [2] *Let (A, σ) be a state $BL-$algebra and $X \subseteq A$. Then the state-filter $F_\sigma(X)$ generated by X is the set*
$\{x \in A \mid x \ge (x_1 \odot \sigma(x_1))^{n_1} \odot ... \odot (x_k \odot \sigma(x_k))^{n_k}, x_i \in X, n_i \ge 1, k \ge 1\}$.

If F is a state-filter of (A,σ) and $a \notin F$, then the state-filter generated by F and a is the set $F_\sigma(F,a) = \{x \in A \mid x \geq i \odot (a \odot \sigma(a))^n, i \in F, n \geq 1\}$. A proper state-filter F is a maximal state-filter iff for any $a \notin F$ there exists $n \in \mathbb{N}^*$ such that $(\sigma(a)^n)^* \in F$.

In watt follow we will introduce the concept of a prime state-filter, we will establish some results related to this concept on the basis of which we are going to characterise the set $Rad_\sigma(A)$, in the case of a state-morphism $BL-$algebra (A,σ).

Proposition 4.5. *Let (A,σ) be a state $BL-$algebra and P a proper state-filter of (A,σ). Then the following statements are equivalent:*

(i) *If P_1, P_2 are two state-filters of (A,σ) such that $P = P_1 \cap P_2$, then $P = P_1$ or $P = P_2$;*

(ii) *If $(a \odot \sigma(a)) \vee (b \odot \sigma(b)) \in P$, $a, b \in A$, then $a \in P$ or $b \in P$.*

Proof. $(i) \Rightarrow (ii)$. Let $a, b \in A$ such that $(a \odot \sigma(a)) \vee (b \odot \sigma(b)) \in P$. We consider the sets $F_\sigma(P,a) = \{x \in A \mid x \geq i \odot (a \odot \sigma(a))^n, i \in P, n \geq 1\}$ and $F_\sigma(P,b) = \{x \in A \mid x \geq i \odot (b \odot \sigma(b))^n, i \in P, n \geq 1\}$, which represent state-filters generated by P and a, respectively P and b (according to Proposition 4.4).

Obviously, $P \subseteq F_\sigma(P,a) \cap F_\sigma(P,b)$. If $x \in F_\sigma(P,a) \cap F_\sigma(P,b)$, then there exist $i_1, i_2 \in P$ and $m, n \in \mathbb{N}^*$ such that $x \geq i_1 \odot (a \odot \sigma(a))^m$ and $x \geq i_2 \odot (b \odot \sigma(b))^n$, so $x \geq (i_1 \odot (a \odot \sigma(a))^m) \vee (i_2 \odot (b \odot \sigma(b))^n) \geq (i_1 \vee i_2) \odot (i_1 \vee (b \odot \sigma(b))^n) \odot (i_2 \vee (a \odot \sigma(a))^m) \odot ((a \odot \sigma(a))^m \vee (b \odot \sigma(b))^n)$ (according to Proposition 2.1, (3))

$\geq (i_1 \vee i_2) \odot (i_1 \vee (b \odot \sigma(b))^n) \odot (i_2 \vee (a \odot \sigma(a))^m) \odot ((a \odot \sigma(a)) \vee (b \odot \sigma(b)))^{mn}$ (according to Proposition 2.1, (4)).

But $i_1 \vee i_2, i_1 \vee (b \odot \sigma(b))^n, i_2 \vee (a \odot \sigma(a))^m$ and $((a \odot \sigma(a)) \vee (b \odot \sigma(b)))^{mn}$ belong to P, and then it follows that $x \in P$. Thus $P = F_\sigma(P,a) \cap F_\sigma(P,b)$, and, from the hypothesis, we obtain that $P = F_\sigma(P,a)$ or $P = F_\sigma(P,b)$, that is, $a \in P$ or $b \in P$.

$(ii) \Rightarrow (i)$. Let P_1, P_2 be two state-filters of (A,σ) such that $P = P_1 \cap P_2$. Let's suppose that $P \neq P_1$ and $P \neq P_2$. Then there exist $a \in P_1 \setminus P$ and $b \in P_2 \setminus P$. Then $a \odot \sigma(a) \in P_1, b \odot \sigma(b) \in P_2$, so $(a \odot \sigma(a)) \vee (b \odot \sigma(b)) \in P_1 \cap P_2 = P$, hence $a \in P$ or $b \in P$, a contradiction. Therefore $P = P_1$ or $P = P_2$.

□

Definition 4.3. *Let (A,σ) be a state $BL-$algebra. A proper state-filter P of (A,σ) is called a prime state-filter if it verify one of the equivalent conditions from the Proposition 4.5.*

Proposition 4.6. *Let (A,σ) be a state $BL-$algebra. Then any maximal state-filter of (A,σ) is a prime state-filter.*

Proof. Let F be a maximal state-filter of (A,σ) and P_1, P_2 two state-filters such that $F = P_1 \cap P_2$. If $F \neq P_1$, then F is strictly contained in P_1, and, since F is a maximal state-filter, it follows that $P_1 = A$. Then $F = A \cap P_2 = P_2$. Therefore F is a prime state-filter.

□

Definition 4.4. *Let (A,σ) be a state $BL-$algebra. A nonovoid subset I of A is called state-ideal if the following conditions are verified:*

(1) *$a, b \in I$ implies $a \oplus b \in I$;*

(2) *$a \in I, b \leq a$ implies $b \in I$;*

(3) *$a \in I$ implies $\sigma(a) \in I$.*

Proposition 4.7. (***Prime state-filter theorem***) *Let I be a state-ideal and F a state-filter on a state $BL-$algebra (A, σ) such that $F \cap I = \varnothing$. Then there is a prime state-filter P such that $F \subseteq P$ and $P \cap I = \varnothing$.*

Proof. Consider the set

$\mathbf{F}(F) = \{F' \mid F'$ is a state-filter such that $F \subseteq F'$ and $F' \cap I = \varnothing\}$.

Since $F \in \mathbf{F}(F)$, it follows that $\mathbf{F}(F)$ is nonvoid. It is easily to prove that the set $\mathbf{F}(F)$ is inductively ordered, so, by Zorn's Lemma in $\mathbf{F}(F)$ then is P a maximal element. I want to prove that P is a prime state-filter. Since $P \in \mathbf{F}(F)$, it follows that P is a proper state-filter and $P \cap I = \varnothing$.

Let $a, b \in A$ such that $(a \odot \sigma(a)) \vee (b \odot \sigma(b)) \in P$. Let's suppose that $a \notin P$ and $b \notin P$. Consider the sets $F_\sigma(P, a)$ şi $F_\sigma(P, b)$, which represent state-filters generated by P and a, respectively P and b. Then P is strictly contained in $F_\sigma(P, a)$ and $F_\sigma(P, b)$ and, by the maximality of P, we deduce that $F_\sigma(P, a) \notin \mathbf{F}(F)$ and $F_\sigma(P, b) \notin \mathbf{F}(F)$. Thus $F_\sigma(P, a) \cap I \neq \varnothing$ and $F_\sigma(P, b) \cap I \neq \varnothing$. Let $x \in F_\sigma(P, a) \cap I$ and $y \in F_\sigma(P, b) \cap I$. Then there exist $i_1, i_2 \in P$ and $m, n \in \mathbb{N}$ such that $x \geq i_1 \odot (a \odot \sigma(a))^m$ and $y \geq i_2 \odot (b \odot \sigma(b))^n$, so $x \vee y \geq (i_1 \odot (a \odot \sigma(a))^m) \vee (i_2 \odot (b \odot \sigma(b))^n) \geq (i_1 \vee i_2) \odot (i_1 \vee (b \odot \sigma(b))^n) \odot (i_2 \vee (a \odot \sigma(a))^m) \odot ((a \odot \sigma(a)) \vee (b \odot \sigma(b)))^{mn} \in P$, that is, $x \vee y \in P$. But $x, y \in I$, so $x \vee y \in I$, hence $P \cap I \neq \varnothing$, a contradiction.

Thus P is a prime state-filter.

□

Proposition 4.8. *Let (A, σ) be a state $BL-$algebra and $a \in A, a < 1$. Then there exists a prime state-filter P of (A, σ) such that $a \notin P$.*

Proof. Like in the Proposition 4.7 we consider the set

$\mathbf{F}(a) = \{F \mid F$ is a state-filter and $a \notin F\}$. Since $\{1\} \in \mathbf{F}(a)$, it follows that $\mathbf{F}(a)$ is nonvoid.

We can easily prove that the set $\mathbf{F}(a)$ is inductively ordered, so by Zorn's Lemma then is P a maximal element of $\mathbf{F}(a)$. I want to prove that P is a prime state-filter. Let $x, y \in A$ such that $(x \odot \sigma(x)) \vee (y \odot \sigma(y)) \in P$. Let's suppose that $x \notin P$ and $y \notin P$. Considering the sets $F_\sigma(P, x)$ and $F_\sigma(P, y)$, which represent state-filters generated by P and x, respectively P and y, it follows that P is strictly contained in $F_\sigma(P, x)$ and $F_\sigma(P, y)$ and, by the maximality of P, we deduce that $a \in F_\sigma(P, x) \cap F_\sigma(P, y)$. Then there exist $i_1, i_2 \in P$ and $m, n \in \mathbb{N}$ such that $a \geq i_1 \odot (x \odot \sigma(x))^m$ and $a \geq i_2 \odot (y \odot \sigma(y))^n$, so $a \geq (i_1 \odot (x \odot \sigma(x))^m) \vee (i_2 \odot (y \odot \sigma(y))^n) \geq (i_1 \vee i_2) \odot (i_1 \vee (y \odot \sigma(y))^n) \odot (i_2 \vee (x \odot \sigma(x))^m) \odot ((x \odot \sigma(x)) \vee (y \odot \sigma(y)))^{mn} \in P$, so $a \in P$, a contradiction. Thus P is a prime state-filter and $a \notin P$.

□

Corollary 4.1. *Let (A, σ) be a state $BL-$algebra and P a proper state-filter of (A, σ). Then there exists a maximal state-filter F_0 of (A, σ) such that $P \subseteq F_0$.*

Proof. The Proposition 4.7 is applied for $I = \{0\}$ and $F = P$. Let F_0 be a maximal element of the set $\mathbf{F}(P) = \{F' \mid F'$ is a proper state-filter and $P \subseteq F'\}$. I want to prove that F_0 is a maximal state-filter of (A, σ). Indeed, if F_1 is a state-filter of (A, σ) such that $F_0 \subseteq F_1$ then, the maximality of F_0, it follows that $F_1 \notin \mathbf{F}(P)$, so F_1 is not a proper state-filter, so $F_1 = A$.

□

On the basis of the previous results, we will be able to characterize the set $Rad_\sigma(A)$, of the intersection of all maximal state-filters of a state-morphism $BL-$algebra (A, σ). Firstly, we will establish the following result:

Proposition 4.9. *Let (A,σ) be a state $BL-$algebra. Then* $\left\{x \in A \mid (\sigma(x)^n)^* \leq \sigma(x),\ \textit{for every } n \in \mathbb{N}\right\} \subseteq Rad_\sigma(A)$.

Proof. Consider $B = \left\{x \in A \mid (\sigma(x)^n)^* \leq \sigma(x),\ \text{for every } n \in \mathbb{N}\right\}$ and let $x \in B$. Let's suppose that $x \notin Rad_\sigma(A)$, therefore there exists a maximal state-filter F of (A,σ) such that $x \notin F$. According to Proposition 4.8, there exists $n \in \mathbb{N}$ such that $(\sigma(x)^n)^* \in F$. Since $(\sigma(x)^n)^* \leq \sigma(x)$, we deduce that $\sigma(x) \in F$. But then $\sigma(x)^n \in F$ and, since $(\sigma(x)^n)^* \in F$, we obtain that $F = A$, a contradiction. Therefore $B \subseteq Rad_\sigma(A)$.

□

Proposition 4.10. *Let (A,σ) be a state-morphism $BL-$algebra. Then* $Rad_\sigma(A) \subseteq \left\{x \in A \mid (\sigma(x)^n)^* \leq \sigma(x),\ \textit{for every } n \in \mathbb{N}\right\}$.

Proof. Consider $B = \left\{x \in A \mid (\sigma(x)^n)^* \leq \sigma(x),\ \text{for every } n \in \mathbb{N}\right\}$ and let $x \in Rad_\sigma(A)$. Let's suppose that $x \notin B$, so there exists $n \in \mathbb{N}$ such that $(\sigma(x)^n)^* \nleq \sigma(x)$, that is, $(\sigma(x)^n)^* \to \sigma(x) < 1$. According to Proposition 4.8 there exists a prime state-filter P of (A,σ) such that $(\sigma(x)^n)^* \to \sigma(x) \notin P$. On the other hand $\sigma\left((\sigma(x)^n)^* \to \sigma(x)\right) = \sigma\left(\sigma\left((x^n)^*\right) \to \sigma(x)\right)$ (since σ is a morphism) $=$ $\sigma\left((x^n)^*\right) \to \sigma(x)$ (from the $(4)_{BL}$) $= (\sigma(x)^n)^* \to \sigma(x)$ and, analogously,
$\sigma\left(\sigma(x) \to (\sigma(x)^n)^*\right) = \sigma(x) \to (\sigma(x)^n)^*$.

Then $\left(\left((\sigma(x)^n)^* \to \sigma(x)\right) \odot \sigma\left((\sigma(x)^n)^* \to \sigma(x)\right)\right)$
$\vee \left(\left(\sigma(x) \to (\sigma(x)^n)^*\right) \odot \sigma\left(\sigma(x) \to (\sigma(x)^n)^*\right)\right)$
$= \left((\sigma(x)^n)^* \to \sigma(x)\right)^2 \vee \left(\sigma(x) \to (\sigma(x)^n)^*\right)^2$
$\geq \left(\left((\sigma(x)^n)^* \to \sigma(x)\right) \vee \left(\sigma(x) \to (\sigma(x)^n)^*\right)\right)^4$
(according to Proposition 2.1, (4)) $= 1 \in P$, and, since P is prime and $(\sigma(x)^n)^* \to \sigma(x) \notin P$, we deduce that $\sigma(x) \to (\sigma(x)^n)^* \in P$.

But $\sigma(x) \to (\sigma(x)^n)^* = (\sigma(x) \odot \sigma(x)^n)^*$ (from Proposition 2.1, (5)),

thus $\left(\sigma(x)^{n+1}\right)^* \in P$. According to Corrollary 4.1, there exists a maximal state-filter F_0 of (A,σ) such that $P \subseteq F_0$, so $\left(\sigma(x)^{n+1}\right)^* \in F_0$, that is, $\sigma(x)^{n+1} \notin F_0$. Then $\sigma(x) \notin F_0$ and so $x \notin F_0$, namely $x \notin Rad_\sigma(A)$, a contradiction. Therefore $Rad_\sigma(A) \subseteq B$.

□

From Propositions 4.9 and 4.10 we obtain:

Theorem 4.1. *Let (A,σ) be a state-morphism $BL-$algebra. Then* $Rad_\sigma(A) = \left\{x \in A \mid (\sigma(x)^n)^* \leq \sigma(x),\ \textit{for every } n \in \mathbb{N}\right\}$.
Moreover, $Rad(A) \subseteq Rad_\sigma(A)$.

Proof. The first part result from Propositions 4.9 and 4.10. For the second part, let $x \in Rad(A)$, so $(x^n)^* \leq x$, for every $n \in \mathbb{N}$.

Then $\sigma\left((x^n)^*\right) \leq \sigma(x)$, for every $n \in \mathbb{N}$, so $(\sigma(x)^n)^* \leq \sigma(x)$, for every $n \in \mathbb{N}$, that is, $x \in Rad_\sigma(A)$.

□

5. Classes of $BL-$ algebras

Within this section, we are going to present some classes of $BL-$algebras, such as simple, semisimple and local $BL-$algebras, we will then define the concepts of simple, semisimple and local state $BL-$algebras (A,σ), next we will introduce the concepts of simple, semisimple and local state $BL-$algebras (A,σ) relative to its state-filters set, and we will finally establish relations between these concepts, which occur in some conditions imposed to the state-operator σ.

Definition 5.1. *A $BL-$algebra A is called simple if its only filters are $\{1\}$ and A. A state $BL-$algebra (A,σ) is called simple if $\sigma(A)$ is simple.*

We will now define a new concept:

Definition 5.2. *A state $BL-$algebra (A,σ) is called simple relative to its state-filters set if it has only two state-filters: $\{1\}$ and A.*

Example 5.1. *Let's consider a state $BL-$algebra (A,σ). If $\sigma = id_A$, then the three concepts from Definition 5.1 are the same. Let's consider the state $BL-$algebra (A,σ) from Example 3.2. We have $\sigma(A) = \{0,a,1\}$.*

If $I \subseteq \sigma(A)$ is a filter, $I \neq \{1\}$, and if $a \in I$, then $a \odot a = 0 \in I$, so $I = \sigma(A)$. Thus $\sigma(A)$ is simple, so (A,σ) is simple. By the contrary, according to Example 4.1 A is not simple and (A,σ) is not simple relative to its state-filters set. For each state $BL-$algebras (A,σ) from Examples 3.3, 3.4, 3.5 we have $\sigma(A) = \{0,1\}$, so (A,σ) is simple, but A is notsimple and (A,σ) is not simple relative to its state-filters set.

Remark 5.1. *According to* [2]*, if (A,σ) is a state $BL-$algebra such that A is simple, then $\sigma(A)$ is simple, so (A,σ) is simple.*

Theorem 5.1. [2] *Let (A,σ) be a state-morphism $BL-$algebra. Then the following conditions are equivalent:*

(1) *(A,σ) is simple;*
(2) $\ker(\sigma)$ *is a maximal filter of A.*

Proposition 5.1. *Let (A,σ) be a state $BL-$algebra. If (A,σ) is simple relative to its state-filters set, then (A,σ) is simple.*

Proof. Let J be a filter of $\sigma(A)$, $J \neq \{1\}$. We will prove that $J = \sigma(A)$. Consider $\mathbf{F}_J = \{z \in A \mid z \geq j, \text{for a certain } j \in J\}$. If $x, y \in \mathbf{F}_J$, then there exist $j_1, j_2 \in J$ such that $x \geq j_1, y \geq j_2$, so $x \odot y \geq j_1 \odot j_2 \in J$, hence $x \odot y \in \mathbf{F}_J$. If $x \in \mathbf{F}_J$ and $x \leq y$, then obviously $y \in \mathbf{F}_J$.

If $x \in \mathbf{F}_J$, then $x \geq j, j \in J$, so $\sigma(x) \geq \sigma(j) = j$ (since $j \in \sigma(A)$), hence $\sigma(x) \in \mathbf{F}_J$. Therefore $\mathbf{F}_J$ is a state-filter of (A,σ). Since (A,σ) is simple relative to its state-filters set, and $\mathbf{F}_J \neq \{1\}$ (since $J \subseteq \mathbf{F}_J$), it follows that $\mathbf{F}_J = A$, so $0 \in \mathbf{F}_J$, hence $0 \in J$, that is, $J = \sigma(A)$.

□

Remark 5.2. *If (A,σ) is a simple state $BL-$algebra relative to its state-filters set, then, since* $\ker(\sigma)$ *is a state filter and* $\ker(\sigma) \neq A$, *it follows that* $\ker(\sigma) = \{1\}$, *thus σ is a faithful operator.*

Remark 5.3. *If (A,σ) is a simple state $BL-$algebra, then it doesn't necessarly follow that σ is faithful. For instance, for the simple state BL-algebra (A,σ) from the Example 3.2 we have* $\ker(\sigma) = \{b,1\} \neq \{1\}$.

Theorem 5.2. *Let (A,σ) be a state $BL-$ algebra. Then the following conditions are equivalent:*

(*i*) *(A,σ) is simple relative to its state-filters set;*
(*ii*) *(A,σ) is simple and σ is faithful.*

Proof. $(i) \Rightarrow (ii)$ Results from the Proposition 5.1 and the Remark 5.2.

$(ii) \Rightarrow (i)$ Let I be a state-filter of (A, σ). Then $I \cap \sigma(A)$ is a filter of $\sigma(A)$, and so $I \cap \sigma(A) = \{1\}$ or $I \cap \sigma(A) = \sigma(A)$. If $I \cap \sigma(A) = \sigma(A)$, then $\sigma(A) \subseteq I$ and, since $0 \in \sigma(A)$, we deduce that $I = A$. If $I \cap \sigma(A) = \{1\}$, let $x \in I$. Then $\sigma(x) \in I \cap \sigma(A)$, so $\sigma(x) = 1$, that is, $x = 1$ (since σ is faithful), so $I = \{1\}$. Therefore the only state-filters of (A, σ) are $\{1\}$ and A. □

Theorem 5.3. *Let (A, σ) be a state-morphism $BL-$algebra. Then the following conditions are equivalent:*

(i) *(A, σ) is simple relative to its state-filters set;*

(ii) *A is simple.*

Proof. $(i) \Rightarrow (ii)$ According to Theorem 5.2 it follows that (A, σ) is simple and σ is faithful. According to Theorem 5.1 $\ker(\sigma)$ is a maximal state-filter of A. Let now F be a filter of A, $F \neq \{1\}$. Since $\ker(\sigma) = \{1\} \subseteq F$ and $\ker(\sigma)$ is maximal, we deduce that $F = A$, so A is simple.

$(ii) \Rightarrow (i)$ Clearly. □

From of the Theorems 5.3 and 5.3 it follows:

Theorem 5.4. *Let (A, σ) be a state-morphism $BL-$algebra and σ is faithful. Then the following conditions are equivalent:*

(i) *A is simple;*

(ii) *(A, σ) is simple.*

Proof. $(i) \Rightarrow (ii)$ Results from the Remark 5.1.

$(ii) \Rightarrow (i)$ If (A, σ) is simple, since σ is faithful, then from the Theorem 5.2 it follows that (A, σ) is simple relative to its state-filters set and then, from the Theorem 5.3 we deduce that A is simple. □

Definition 5.3. *A $BL-$algebra A is called local if it has only a maximal filter. A state $BL-$algebra (A, σ) is called local if $\sigma(A)$ is local.*

Next we define a new concept:

Definition 5.4. *A state $BL-$algebra (A, σ) is local relative to its state-filters set if it has only a maximal state-filter.*

Example 5.2. *Let's consider the $BL-$algebra A and the state-operator $\sigma : A \to A$ from Example 3.2. Then A is local, (A, σ) is local and (A, σ) is local relative to its state-filters set. In Example 3.3 the $BL-$algebra A is not local, but (A, σ) is local relative to its state-filters set.*

Theorem 5.5. *Let (A, σ) be a state $BL-$algebra. Then the following conditions are equivalent:*

(i) *(A, σ) is local relative to its state-filters set;*

(ii) *(A, σ) is local.*

Proof. $(i) \Rightarrow (ii)$ Let F be the only maximal state-filter of (A, σ). Then $F \cap \sigma(A)$ is a filter of $\sigma(A)$. We will prove that $F \cap \sigma(A)$ is the only maximal filter of $\sigma(A)$. If $F \cap \sigma(A) = \sigma(A)$, then $\sigma(A) \subseteq F$, so $0 \in F$, a contradiction. Let I be an arbitrary proper filter of $\sigma(A)$. We consider the set $F_\sigma(I) = \{z \in A \mid z \geq i, i \in I\}$,which represents the state–filter generated by I in (A, σ). If $F_\sigma(I) = A$, then $0 \in F_\sigma(I)$, so $0 \in I$, false. Then $F_\sigma(I)$ is a proper state-filter, so $F_\sigma(I) \subseteq F$, that is,

$I = I \cap \sigma(A) \subseteq F_\sigma(I) \cap \sigma(A) \subseteq F \cap \sigma(A)$.

Then $F \cap \sigma(A)$ is a proper filter which contains any proper filter I of $\sigma(A)$, thus it is the only maximal filter of $\sigma(A)$, so (A, σ) is local.

$(ii) \Rightarrow (i)$ Let I be the only maximal filter of $\sigma(A)$ and the set

$F_\sigma(I) = \{z \in A \mid z \geq i, i \in I\}$, which represents the state–filter generated by I in (A, σ). Let $\mathbf{F}(I) = \{F \mid F$ is a proper state-filter of (A, σ) and $I \subseteq F\}$.

If $F_\sigma(I)$ is not proper, then $0 \in F_\sigma(I)$, so $0 \in I$, false. Thus $F_\sigma(I) \in \mathbf{F}(I)$, so $\mathbf{F}(I)$ is nonvoid. It is easily to verify that $\mathbf{F}(I)$ is inductively ordered, so by Zorn's Lemma then is F a maximal element of $\mathbf{F}(I)$. We will prove that F is the only maximal state-filter of (A, σ). Indeed, let F_1 be an arbitrary proper state-filter of (A, σ). Let's suppose that there exists an element $x \in F_1 \setminus F$. Then $\sigma(x) \in F_1 \cap \sigma(A)$. If $F_1 \cap \sigma(A) = \sigma(A)$ it follows that $\sigma(A) \subseteq F_1$, so $0 \in F_1$, a contradiction. Thus $F_1 \cap \sigma(A) \neq \sigma(A)$, $F_1 \cap \sigma(A)$ is a filter of $\sigma(A)$ and, since I is a maximal filter of $\sigma(A)$, it follows that $F_1 \cap \sigma(A) \subseteq I$, so $\sigma(x) \in I$.

Then $\sigma(x) \in F_\sigma(I)$, so $\sigma(x) \in F$. Since $x \notin F$ and F is a maximal state-filter, then, according to Proposition 4.8, it follows that there exists $n \in \mathbb{N}^*$ such that $(\sigma(x)^n)^* \in F$.

But $\sigma(x)^n \in F$, a contradiction. Thus $F_1 \subseteq F$, so F is the only maximal state-filter of (A, σ), so (A, σ) is local relative to its state-filters set. □

Definition 5.5. *A $BL-$algebra A is called semisimple if $Rad(A) = \{1\}$. Let (A, σ) be a state $BL-$algebra. (A, σ) is called semisimple if $Rad(\sigma(A)) = \{1\}$.*

Concerning all this, we are now going to define a new concept:

Definition 5.6. *A state $BL-$algebra (A, σ) is called semisimple relative to its state-filters set if $Rad_\sigma(A) = \{1\}$.*

Example 5.3. *Let's consider the state $BL-$algebra (A, σ) from Example 3.2. The A algebra is not semisimple, but (A, σ) is semisimple because $Rad(\sigma(A)) = \{1\}$. It is not semisimple relative to its state-filters set.*

The A algebras from Examples 3.3, 3.4 are semisimple, (A, σ) is not semisimple, but they are semisimple relative to its state-filters set.

The A algebra from Example 3.5 is not semisimple, (A, σ) is not semisimple relative to its state-filters set, but (A, σ) is semisimple.

The $\mathbf{I}_L$ algebra from Proposition 3.1 is semisimple, and, since $\sigma = id_L$, $(\mathbf{I}_L, \sigma)$ is semisimple and semisimple relative to its state-filters set.

Proposition 5.2. ([2]) *Let (A, σ) be a state $BL-$algebra. Then*
$\sigma(Rad(A)) \supseteq Rad(\sigma(A)) = \sigma(Rad_\sigma(A))$.

Theorem 5.6. *Let (A, σ) be a state $BL-$algebra. Then the following conditions are equivalent:*

(i) (A, σ) is semisimple and σ is faithful;

(ii) (A, σ) is semisimple relative to its state-filters set.

Proof. $(i) \Rightarrow (ii)$ According to Proposition 5.2 we have $\sigma(Rad_\sigma(A)) = Rad(\sigma(A)) = \{1\}$, so $Rad_\sigma(A) \subseteq \ker(\sigma) = \{1\}$, that is, $Rad_\sigma(A) = \{1\}$.

$(ii) \Rightarrow (i)$ $Rad(\sigma(A)) = \sigma(Rad_\sigma(A)) = \sigma(\{1\}) = \{1\}$, so (A, σ) is semisimple. We will prove that σ is faithful. Let $x \in \ker(\sigma)$, that is, $\sigma(x) = 1$. Let's suppose that $x \notin Rad_\sigma(A)$.Then there exists a maximal state-filter F such that $x \notin F$. According to Proposition 4.4 there exists $n \in \mathbb{N}^*$ such that $(\sigma(x)^n)^* \in F$, so $0 \in F$, a contradiction. Thus $x \in Rad_\sigma(A) = \{1\}$, so σ is faithful.

□

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