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Projection Algorithm for Split Feasibility Problem

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Abstract. The split feasibility problem has many applications in various fields of science and technology (for example solving systems of linear equalities and/or inequalities). The class of methods, generally called projection methods, has witnessed great progress in recent years and the algorithms have been applied with success in different areas. The paper reviews algorithmic structures and specific algorithms for the split feasibility problem.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space and $C \subset \mathcal{H}$ be a closed and convex subset. In general the optimization problems have the form:

find an element $x^* \in C$ which satisfies some additional condition leading to a solution set being a closed and convex subset.

Examples for this optimization problem are: the convex feasibility problem (studied by Bauschke and Borwein in 1996, [1]), the split feasibility problem (introduced by Censor and Elfving in 1994, [4]). Iterative methods, which try to solve such problems consist very often in adaptive construction of the operator $T : C \to \mathcal{H}$ with $FixT$ the solution of the problem.

Classical iterative method used to solve optimization problem is Mann iteration process:

$$x_{k+1} = (1 - t_k)x_k + t_k T(x_k)$$

where t_k is a sequence of real numbers satisfying some properties ($t_k \in (0, 1)$), usually called the control sequence (or the weight factor).

Starting from the iterative methods we have projection algorithm use to solve this type of optimization problem. Projection algorithms employ projections onto convex sets in various ways. This class of algorithms has a great progress in recent years and its algorithms have been applied with success in different problems. Projections onto sets are used in a wide variety of methods in optimization problems. Projection methods are iterative algorithms that use projections onto sets.

A projection algorithm reaches its goal, related to the whole family of sets, by performing projections onto the individual sets. Projection algorithms employ projections onto convex sets in various ways. They may use different kinds of projections and, sometimes, even use different projections within the same algorithm.

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2. Projection algorithm for the split feasibility problem

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

The **split feasibility problem** is formulated as finding a point x satisfying the property:

$$x \in C, \; Ax \in Q,$$

where $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator.

The split feasibility problem was introduced by Censor and Elfving (1994) ([4]) and was studied by Byrne (2002, 2004) ([3]). It attracts many authors attention due to its application in signal processing. The split feasibility problem has many practical applications and that is way various algorithms have been invented to solve it ([2]). One of thees algorithms was given by Byrne [3] who introduced the CQ algorithm.

Let's consider the initial point $x_0 \in \mathcal{H}_1$ arbitrarily, and define $(x_n)_{n\geq 0}$ recursively as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n \tag{1}$$

where $0 < \gamma < 2/\rho(A * A)$ and where P_C denotes the projector onto C and $\rho(A * A)$ is the spectral radius of the self-adjoint operator $A * A$.

Then the sequence $(x_n)_{n\geq 0}$ generated by the **CQ algorithm** converges strongly to a solution of the split feasibility problem whenever $\mathcal{H}_1$ is finite-dimensional and whenever there exists a solution to the split feasibility problem.

It can be observed that the CQ algorithm is a particular case of the Mann iteration for approximating fixed points of nonexpansive mappings.

Take an initial point $x_0 \in C$ arbitrarily, and construct $(x_n)_{n\geq 0}$ recursively as

$$x_{n+1}{=}t_n x_n + (1 - t_n)Tx_n,$$

where $t_n \in [0, 1]$ satisfying $\sum\limits_{n=0}^{\infty} t_n(1 - t_n) = \infty$. If T is nonexpansive with a nonempty fixed point set, then the sequence $(x_n)_{n\geq 0}$ generated by the Mann iteration converges weakly to a fixed point of T. It is known that Mann algorithm is in general not strongly convergent ([6]) and neither is the CQ algorithm.

So, the CQ algorithm need not necessarily converge strongly in the case when $\mathcal{H}_1$ is infinite dimensional.

Definition 2.1. *The mapping T is said to be **nonexpansive** if*

$$\|Tx - Ty\| \leq \|x - y\|, \; \forall x, y \in \mathcal{H}_1.$$

Definition 2.2. *The mapping T is said to be **contractive** if there exist 0< $\alpha < 1$ such that*

$$\|Tx - Ty\| \leq\alpha\|x - y\|, \; \forall x, y \in \mathcal{H}_1.$$

Remark 2.1. *Obviously, contractions are nonexpansive, and if T is nonexpansive, then I - T is monotone.*

Lemma 2.1. *Let C be a nonempty closed convex subset of $\mathcal{H}_1$ and $T : C \to C$ a nonexpansive mapping with nonempty fix set ($Fix(T) \neq \emptyset$). If $(x_n)_{n\geq 1}$ is a sequence in C weakly converging to x and if the nsequence $((I - T)x_n$ converges strongly to y, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

Let P_C be the projection from $\mathcal{H}_1$ onto a nonempty closed convex subset C of $\mathcal{H}_1$; usually defined as:

$$P_C x = \min_{y \in C} \|x - y\|,\ x \in \mathcal{H}_1$$

For every $c \in C$ we have

$$\langle x - P_C x, c - P_C x \rangle \leq 0,$$

so P_C is nonexpansive.

In order to use nonexpasive mappings we can approximate them by using contractions. We know that the mapping $I - \gamma A^*(I - P_Q)A$ is nonexpansive.

Consider x_0 arbitrary chosen

Consider a sequence $\{\alpha_n\}_{n \geq 0} \in (0, 1)$

$$x_{n+1} = P_C[(1 - \alpha_n)(I - \gamma A^*(I - P_Q)A)]x_n$$

The sequence $\{\alpha_n\}_{n \geq 0}$ must be considered such that

(1) $\lim_{n \to \infty} \alpha_n$=0;

(2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \to \infty} \frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}$=0

Remark 2.2. *Usually the sequence*$\{\alpha_n\}$ *is considered* $\alpha_n = \frac{1}{(1+n)}$ *for all* $n \geq 0$.

Theorem 2.1. [7] *The sequence* $\{x_n\}_{n \geq 0}$ *generated by algorithm CQ converges strongly to the minimum-norm solution* $\widetilde{x}$ *of the split feasibility problem.*

3. Experimental test

Let's consider the following split feasibility problem:

Let $C \subset \mathcal{R}^n$, $Q \subset \mathcal{R}^m$ *be nonempty, closed convex sets, and* A *a real matrix,* $A \in \mathcal{R}^{m \times n}$. *Find* $x \in C$ *so that* $Ax \in Q$ *if such an element exists.*

In order to use the CQ algorithm we must find a fix point of the following operator $P_C(I - \gamma A^*(I - P_Q)A)x_n$, where $\gamma > 0$.

In general, the split feasibility problem considered in real space has the form:

$$minimize \frac{1}{2} ||P_Q(Ax) - Ax||^2.$$

The convexly constrained linear problem requires to solve the constrained linear system:

$$Ax = b$$
$$x \in C,$$

where $b \in Q$.

This problem can be solved by finding a point that satisfies the relation:

$$\min_{x \in C} \|Ax - b\|^2 + \alpha \|x\|^2,$$

where $\alpha > 0$ is a parameter that influence the number of iteration calculated until a solution of the split feasibility problem is found.

To solve the split feasibility problem we consider the next algorithm:

So the split feasibility problem is equivalent with the following minimization problem

$$\min_{x \in C} \|(I - P_Q)Ax\|.$$

From the last relations we can observe that the split feasibility problem for a real space is equivalent with finding a point that minimize the relation:

Algorithm 1 Modified CQ algorithm for the split feasibility problem

Consider the set C nonempty, closed and convex, $C \subset \mathcal{R}^n$
Consider the set Q nonempty, closed and convex, $Q \subset \mathcal{R}^m$
Consider $b \in Q$
Consider the matrix A, $A \in \mathcal{R}^{m \times n}$
Consider a real number $\alpha \in (0, 1)$
Find a point from C that minimize the relation:
$$\|Ax - b\|^2 + \alpha\|x\|^2$$

$$\|(I - P_Q)Ax\|^2 + \alpha\|x\|^2,$$

where $\alpha > 0$ is the regularization parameter.

If we note the point that satisfies the previous relation by x^*, then we have:

$$x^* := \min_{x \in C}\{\|(I - P_Q Ax)\|^2 + \alpha\|x\|^2\}$$

Theorem 3.1. [7] *Let x^* be given by the relation $x^* := \min_{x \in C}\{\|(I - P_Q Ax)\|^2 + \alpha\|x\|^2\}$. Then x^* converges strongly as $\alpha \to 0$ to the minimum-norm solution $\widetilde{x}$ of the split feasibility problem.*

In order to solve the convex feasibility problem we have the following algorithm:

Algorithm 2 Modified CQ algorithm

Consider the set C nonempty, closed and convex, $C \subset \mathcal{R}^n$
Consider the set Q nonempty, closed and convex, $Q \subset \mathcal{R}^m$
Consider $b \in Q$
Consider the matrix A, $A \in \mathcal{R}^{m \times n}$
Consider a real number $\alpha \in (0, 1)$
Determine the point from C so that the relation is minimized:
$$\|(I - P_Q Ax)\|^2 + \alpha\|x\|^2$$

The general CQ algorithm for the split feasibility problem, and the one that show us how the regularization factor influence the number of iterations calculated until a solution of the problem is found, is:

This algorithm permits us to analyze how the regularization factor α influence the number of points calculated until the solution of the convex feasibility problem is determinate.

Example 3.1. *Let's consider the following system:*

$$\begin{cases} x + 2y + 3 = 0 \\ 2x + 3y + 6 = 0 \\ -4x + y + 6 = 0 \\ -2x - 7y + 1 = 0 \\ x - 3y + 2 = 0 \\ -5x + y + 7 = 0 \end{cases}$$

The results obtained for this example are presented in the table 1

Algorithm 3 CQ algorithm

Consider the set C nonempty, closed and convex, $C \subset \mathcal{R}^n$
Consider the set Q nonempty, closed and convex, $Q \subset \mathcal{R}^m$
Consider $b \in Q$
Consider the matrix A, $A \in \mathcal{R}^{m \times n}$
Let $x_0 \in C$ be an initial point
Consider a real number $\alpha \in (0, 1)$
Calculate $\rho(A^*A)$
Take $\gamma \in (0, \frac{2}{\rho(A^*A)})$
Determine the operator $I - \gamma A^*(I - P_Q)A$
$n = 0$
while x_n is not a solution for the split feasibility problem **do**
  Calculate $(I - \gamma A^*(I - P_Q)A)x_n$
  $x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n$
  $n = n + 1$
end while

TABLE 1. The number of iterations calculated for the starting point (1,4)

initial approximation	*regularization factor*	*number of iterations*
(1,1)	*0.1*	*2*
(1,1)	*0.2*	*3*
(1,1)	*0.3*	*19*
(1,1)	*0.4*	*25*
(1,1)	*0.5*	*67*
(1,1)	*0.6*	*83*
(1,1)	*0.7*	*106*
(1,1)	*0.8*	*143*
(1,1)	*0.9*	*168*

Example 3.2. *Let's consider a linear system with $m = 8$ equations and $n = 2$ unknowns.*

$$\begin{cases} 2x + 4y + 1 = 0 \\ 2x - 3y + 3 = 0 \\ 4x - y + 2 = 0 \\ -2x + 7y + 1 = 0 \\ -2x - 3y + 2 = 0 \\ 2x - 7y + 1 = 0 \\ 3x - 3y + 2 = 0 \\ 5x + y + 2 = 0 \end{cases}$$

For different starting points, we observe that the regularization factor α influence the number of iterations calculated. For small regularization factor (closer to 0) the number of iterations is very small, but when the regularization factor is increasing the number of iterations is also getting bigger.

Applying this algorithm for a lot of numerical data we can analyze the influence of the regularization factor. For different n and m all the numerical results give us the

TABLE 2. The number of iterations calculated for the starting point (1,4)

initial approximation	*regularization factor*	*number of iterations*
(1,4)	*0.1*	*5*
(1,4)	*0.2*	*7*
(1,4)	*0.3*	*16*
(1,4)	*0.4*	*31*
(1,4)	*0.5*	*75*
(1,4)	*0.6*	*153*
(1,4)	*0.7*	*182*
(1,4)	*0.8*	*218*
(1,4)	*0.9*	*234*

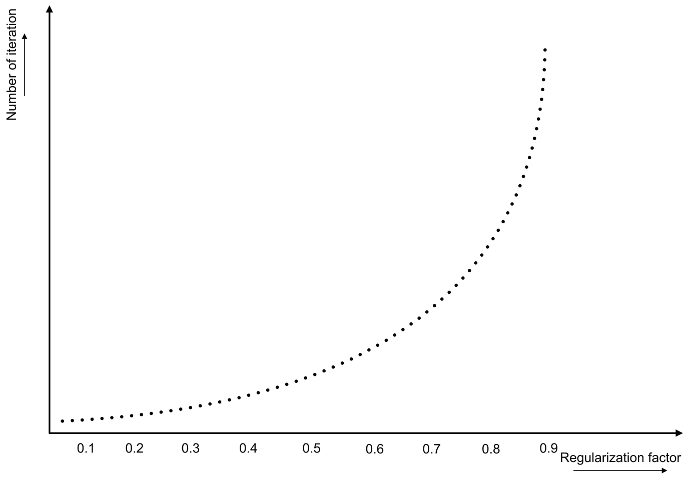

FIGURE 1. The relation between the regularization factor and the iterations number

same conclusion: an optimal regularization factor is the one that is very small (closer to 0).

4. Conclusion

The projection algorithm used in this paper to solve the split feasibility problem is the CQ algorithm. The paper present the algorithm and analyze the influence of the regularization factor on the number of iterations calculated until the solution of the split feasibility problem is computed. For the same algorithm is interesting to find how the parameter $\gamma \in (0, \frac{2}{\rho(A^*A)})$ influence the number of iterations and if there is a relation between the regularization factor and the parameter γ.

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