Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 37(3), 2010, Pages 110–115
ISSN: 1223-6934

Pressure field of acoustic sensor of order q

Ana-Maria Chiselev and Luminita Moraru

Abstract. The order of sensors has been defined related to the order of the Taylor series approximation of the pressure field at that point. Following this definition, the pressure, vector, and sensor is of order zero, one, and two, respectively. For this theoretical study, a multi-channel three-dimensional spatial filter is derived for a directional acoustic sensor of arbitrary order. There are found formulas for pressure field and beam pattern of the directional acoustic sensor.

2010 Mathematics Subject Classification. Primary 30G30; Secondary 40B05.
Key words and phrases. pressure field, array gain, vector-sensor array, acoustic sensor.

1. Introduction

There is a qualitative relationship between the Taylor series expansion of a pressure field and a vector sensor. Cray and Nuttall [2] utilized the four physical quantities measured by a vector sensor as inputs to a multi-channel filter and demonstrated that a single vector sensor can perform spatial filtering or beamforming. There are studies about the factors that lead to the improved direction-of arrival estimation performance of a vector sensor-array as compared to the pressure sensor array [4].

The Taylor series and the multi-channel filtering concept introduce and investigate the acoustic dyadic sensor [6]. This sensor arises from the second-order approximation of the Taylor series expansion of the pressure field. It was shown that a dyadic sensor can produce a beamwidth and a maximum array gain of 65° and 9.5 dB as compared to 105° and 6 dB for a vector sensor.

Cray [2] presented theory for acoustic receivers of order greater than two. It was pointed out that, although further improvement in directionality is achieved by high-order directional sensors, these sensors can be significantly more sensitive to non-acoustic noise sources. In spite of practical limitations, it was demonstrated that a super-directive array can provide substantial improvement over a conventional array in particular instances.

2. Acoustic Sensor of Order q

It can be define an acoustic sensor located at a point (x, y, z) in three dimensional space. To determine the pressure field at any point (x, y, z) we use the expansion of the Taylor series of the pressure field about the point (x, y, z), we assume the sensor is located at $(0, 0, 0)$ and we have

Received July 05, 2010. Revision received September 03, 2010.
The work of Ana-Maria Chiselev was partly supported by Project SOP HRD-SIMBAD 6853, 1.5/S/15-01.10.2008.

$$P(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} P_{m,n,p}(t)\frac{x^m y^n z^p}{m!n!p!} \tag{1}$$

where

$$P_{m,n,p} = \frac{\partial^{m+n+p}P(x,y,z,t)}{\partial x^m \partial y^n \partial z^p} \mid x=0, y=0, z=0 \tag{2}$$

are the various spatial partial derivatives of the pressure evaluated at the point $(0,0,0)$. The above equations show that the sensor must measure at the origin the pressure and all of its spatial partial derivatives. Clearly, this is not possible in practice. More practical sensors can be defined by truncating the three-dimensional Taylor series. In this order, it is useful to express the sum (1) in the form

$$P(x,y,z,t) = \sum_{d=0}^{\infty} \frac{P_d(x,y,z,t)}{d!} \tag{3}$$

where

$$P(x,y,z,t) = \sum_{m+n+p+d=0} P_{m,n,p}(t) \binom{d}{m,n,p} x^m y^n z^p \tag{4}$$

is an homogeneous polynomial of degree d in the variables x, y, z and $\binom{d}{m,n,p} = \frac{d!}{m!n!p!}$ is a multinomial coefficient [4]. A generalized acoustic sensor of order q is defined as

$$\hat{P}(x,y,z,t) = \sum_{d=0}^{q} \frac{P_d(x,y,z,t)}{d!}. \tag{5}$$

The approximation to the pressure field given by Eq. (5) is a multivariable polynomial in x, y, z of degree q. When $q = 0$, Eq. (5) defines a generalized acoustic sensor of order 0, which is the ordinary acoustic pressure sensor. If $q = 1$, Eq. (5) defines a generalized acoustic sensor of order one [3]. This sensor is the pressure gradient or vector sensor. The vector sensor measures the pressure and the gradient of the pressure. When $q = 2$, Eq. (5) defines a generalized acoustic sensor of order two. This sensor is the acoustic dyadic sensor. The dyadic sensor measures the pressure, the pressure gradient, and the Hessian of the pressure.

To study the directionality of acoustic sensor, we consider a plane wave (Fig. 1) propagating towards the origin of a rectangular coordinate system.

Located at the origin is an acoustic sensor of order q. The pressure of the planar wave front can be written

$$P(x,y,z,t) = P(t + \frac{ar}{c}) = P(t + \frac{a_x x + a_y y + a_z z}{c}) \tag{6}$$

$$a_x = \sin\phi\cos\theta,\ a_y = \sin\phi\sin\theta,\ a_z = \cos\phi. \tag{7}$$

Equation (2) becomes

$$P_{m,n,p}(t) = \frac{a_x^m a_y^n a_z^p}{c^{m+n+p}} \cdot \frac{\partial^{m+n+p}P(t)}{\partial t^{m+n+p}}. \tag{8}$$

The temporal Fourier transform of Eq. (4) is

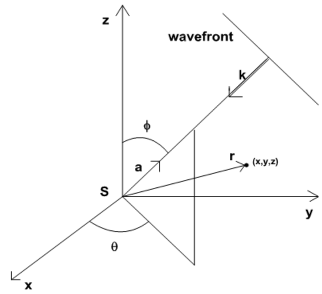

FIGURE 1. Plane wave propagating towards the origin of a rectangular coordinate system.

$$P_{m,n,p}(\omega) = \frac{a_x^m a_y^n a_z^p}{c^{m+n+p}}(j\omega)^{m+n+p}P(\omega). \tag{9}$$

Taking the temporal Fourier transform of Eq. (4) results in

$$P_d(x,y,z,\omega) = \sum_{m+n+p=d} P_{m,n,p}(\omega)\binom{d}{m,n,p}x^m y^n z^p \tag{10}$$

$$P_d(x,y,z,\omega) = (\frac{j\omega}{c})^d P(\omega) \times \sum_{m+n+p=d}\binom{d}{m,n,p}(a_x x)^m(a_y y)^n(a_z z)^p. \tag{11}$$

Applying the multinomial formula to the summation in Eq. (11) leads to

$$P_d(x,y,z,\omega) = (\frac{j\omega}{c})^d(a_x x + a_y y + a_z z)^d P(\omega) \tag{12}$$

$$P_d(a,r) = \left(\frac{j\omega}{c})^d(a\cdot r)^d P(\omega) \tag{13}$$

And we have

$$\hat{P}(r,\omega) = \sum_{d=0}^{q}\frac{P_d(r,\omega)}{d!}. \tag{14}$$

Equation (14) represents the pressure-field extrapolation in the frequency domain made by a generalized acoustic sensor of order q. As the value of q increases, the accuracy of the extrapolation increases. Consider now a more general form of Eq. (14)

$$\hat{P}_b(r,\omega) = \sum_{d=0}^{q} B_d(\omega)P_d(r,\omega) \tag{15}$$

where the various $B_d(\omega)$ are to be interpreted as weights. Eq. (15) may be interpreted as the temporal Fourier transform of the output of a multi-channel filter when the inputs are the derivatives $P_{m,n,p}(t)$ defined by Eq. (2). When each $B_d(\omega)$ is equal to $\frac{1}{d!}$, the multi-channel filter is an extrapolator. The vector r specifies the point at which the extrapolation is made. Consider the following values for the weights:

$$B_d(\omega) = b_d(\frac{c}{j\omega})^d, d = 0, 1, 2, \cdots \tag{16}$$

For Eq.(15) we have

$$\hat{P}_b(r, \omega) = P(\omega)g(a \cdot r) \tag{17}$$

$$g(a \cdot r) = \sum_{d=0}^{q} b_d(a \cdot r)^d. \tag{18}$$

Since Eq. (17) now corresponds to a multi-channel spatial filter, the vector r will be replaced by

$$a_s = \sin\phi_s \cos\theta_s \hat{x} + \sin\phi_s \sin\theta_s \hat{y} + \cos\phi_s \hat{z} \tag{19}$$

$$g(a \cdot a_s) = g(\phi, \theta) = \sum_{d=0}^{q} b_d(a \cdot a_s)^d \tag{20}$$

where

$$a \cdot a_s = \sin\phi \sin\phi_s \cos(\theta - \theta_s) + \cos\phi \cos\phi_s). \tag{21}$$

The function given by Eq. (20) is the beam pattern of the multi-channel filter, or equivalently, the beam pattern of the associated directional acoustic sensor of order q. The basic structural equations for the multi-channel filter are

$$\hat{P}_b(\phi, \theta, \omega) = \sum_{d=0}^{q} B_d(\omega)P_d(\phi, \theta, \omega) \tag{22}$$

$$P_d(\phi, \theta, \omega) = \sum_{m+n+p=d} f_{m,n,p}P_{m,n,p}(\omega) \tag{23}$$

$$f_{m,n,p} = \binom{d}{m, n, p} \sin^{m+n}(\phi_s)\cos^p(\phi_s)\cos^m(\theta_s)\sin^n(\theta_s). \tag{24}$$

Equation (22) is the same as Eq. (15) with the exception that the dependence on the angles ϕ and θ is made explicit. There are two sets of coefficients associated with the multi-channel filter, namely, $f_{m,n,p}$ and b_d. The $f_{m,n,p}$ coefficients, which are computed from Eq. (24), determine the angles ϕ_s and b_d at which the magnitude of the beam pattern $g(\phi, \theta)$ attains its maximum value. The coefficients b_d are to be chosen so as to achieve some optimality condition such as maximum array gain.

The input $P(t)$ is the temporal variation of the pressure at the acoustic sensor. The dependence of the output on ϕ and θ is explicitly determined by the beam pattern function $g(\phi, \theta)$. The beam pattern for the acoustic sensor of order q is given by Eq. (20). It can be express the beam pattern in the alternative forms:

$$g_u(u) = \sum_{d=0}^{q} b_d \cdot u^d \tag{25}$$

$$g_\Psi(\Psi) = \sum_{d=0}^{q} b_d \cos^d \Psi \tag{26}$$

where

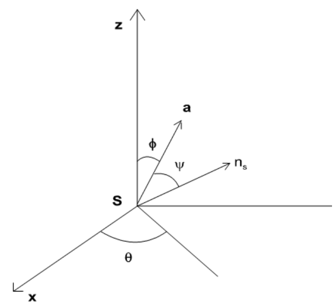

FIGURE 2. The angle ψ between the vector n_s and the unit vector a

$$u = a \cdot n_s. \tag{27}$$

Equations (25) and (26) represent the beam pattern in u space and Ψ space, respectively. The variable Ψ, which is illustrated in Fig. 2, is the angle between the unit vectors a and n_s. The directional acoustic sensor is has the direction indicated by the steering vector n_s.

The unit vector a is collinear with the wave number vector k (Fig. 1). The ranges of the variables u and Ψ are $-1 \leq \ u \leq 1$ and $0 \leq \Psi \leq \pi$.

The beam pattern of the directional acoustic sensor has been expressed in the three forms defined by Eqs. (20), (25), and (26).

Correspondingly, the beam power pattern (the square of the beam pattern), has the three forms [4, 5]:

$$B'(\phi, \theta) = g^2(\phi, \theta) = \sum_{n=0}^{2q} c_n (a \cdot n_s)^n \tag{28}$$

$$B'_u(u) = g_u^2(u) = \sum_{n=0}^{2q} c_n u^n \tag{29}$$

$$B'_\Psi(\Psi) = g_\Psi^2(\Psi) = \sum_{n=0}^{2q} c_n \cos^n \Psi \tag{30}$$

with

$$c_n = \sum_{m=0}^{p} b_{p-m} b_m, n = 0, 1, 2 \cdots, 2q. \tag{31}$$

If $u = 1$ in Eq.(25)

$$\sum_{d=1}^{q} b_d = 1 \tag{32}$$

then the value of the beam power pattern at $u = 1$ is unity, $B'_u(1) = 1$. In the design of the multi-channel filter, which gives the acoustic sensor its directionality, it will be assumed that the filter coefficients b_d satisfy the constraint specified by Eq. (32). The filter coefficients must be selected such that the beam power pattern $B'_u(u)$ have values generally less than unity when $u \neq 1$. One design criterion that achieves this objective is the maximization of the array gain.

The array gain of a spatial filter is defined as the ratio of the noise power out of an omni-directional device to the noise power out of the spatial filter (directional sensor).

Burdic [1] defines the array gain (expressed in decibel) as

$$AG = 10\log\frac{\int_0^2 i\int_0^\pi |N(\phi,\theta)|^2\sin\phi d\phi d\theta}{\int_0^2 i\int_0^\pi |N(\phi,\theta)|^2 B'(\phi,\theta)\sin\phi d\phi d\theta} \tag{33}$$

$|N(\phi,\theta)|^2$ is the noise field angular intensity distribution. If the noise field is isotropic, then ϕ and θ are constant. The array gain is

$$AG = -\log I \tag{34}$$

where

$$I = \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi B'(\phi,\theta)\sin\phi d\phi d\theta \tag{35}$$

The quantity I represents the normalized noise power out of the spatial filter. The power is normalized with respect to the noise power out of an omni-directional sensor.

3. Conclusion

In this paper we presented formula for the pressure-field extrapolation in the frequency domain made by a generalized acoustic sensor of order q. With this formula it can be express the value of beam power pattern of an acoustic sensor of order q. Studying sensors of order 0, 1, 2 and their factors, this study can improve directional of arrival performance of a vector-sensor array as compared to a pressure sensor array. For future work, we propose finding explicit formulas for first, second and third order directional acoustic sensor.

References

[1] W.S. Burdic, *Underwater Acoustic System Analysis*, Prentice-Hall, 1991.

[2] B.A Cray and A.H. Nuttall, A comparison of vector sensing and scalar-sensing linear arrays, Technical Report No. 10632 (1997).

[3] A.-M. Chiselev, L. Moraru and A. Gogu, Localization of an object using a bat model, inspired from biology, *Romanian Journal of Biophysics* **19** (2009), 251–258.

[4] M. Hawkes and A. Nehorai, Acoustic vector-sensor beamforming and capon direction estimation, *IEEE Trans. Signal Process.* **46** (1998), 2291–2304.

[5] D.J. Schmidlin, Directionality of generalized acoustic sensor of arbitrary order, *J.Acoust.Soc.Am.* **121** (2007), 3569–3578.

[6] M.T. Silvia, A theoretical and experimental investigation of acoustic dyadic sensors, SITTEL Technical Report No. TP-4 (2001).

(Ana-Maria Chiselev, Luminita Moraru) Department of Science, University of Galati, 47 Domneasca Street, Galati, 800008, Romania

E-mail address: anachiselev@yahoo.com, luminita.moraru@ugal.ro