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About the precision in Jensen-Steffensen inequality

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Abstract. The main objective of the present paper is to estimate the precision of Jensen-Steffensen inequality. We obtain results that complement, generalize, unify and agree with some of the previously known results in this area.

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1. Introduction

In the paper [4] S. Furuichi estimates the difference between arithmetic and geometric means by the following formula to which we will refer as *Furuichi's inequality.*

Proposition 1.1. *We consider* $a_1, a_2, ..., a_n \geq 0$ *and* $p_1, p_2, ..., p_n \geq 0$ *with* $\sum_{i=1}^n p_i = 1$ *and* $\lambda = \min\{p_1, p_2, ..., p_n\}$. *Then we have*

$$\sum_{i=1}^n p_i a_i - \prod_{i=1}^n a_i^{p_i} \geq n\lambda \left(\frac{\sum_{i=1}^n a_i}{n} - \prod_{i=1}^n a_i^{1/n} \right).$$

If we assume that λ *is attained by only one weight* p_k*, then we have equality if and only if* $a_1 = a_2 = ... = a_n$.

The aim of this paper is to prove a much more general result concerning the precision in Jensen-Steffensen inequality.

2. Preliminary notions

For the convenience of the reader we briefly recall few definitions and theorems:

Definition 2.1. *Let* I, J *be two intervals and* $\varphi : I \to J$ *a continuous, increasing and bijective function. The* weighted quasi-arithmetic mean $M_{[\varphi]}$ *of a nonempty set of data* $\mathbf{x} = (x_1, x_2, ..., x_n) \in I^n$ *with weights* $\mathbf{p} = (p_1, p_2, ..., p_n)$, *where* $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$, *is defined by the formula*

$$M_{[\varphi]}(\mathbf{x}; \mathbf{p}) = \varphi^{-1}\left(\sum_{i=1}^n p_i \varphi(x_i)\right).$$

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Particularly the weighted arithmetic mean $A(\mathbf{x};\mathbf{p}) = \sum_{i=1}^{n} p_i x_i$ corresponds to $\varphi(x) = x$, and the weighted geometric mean $G(\mathbf{x};\mathbf{p}) = \prod_{i=1}^{n} x_i^{p_i}$ corresponds to $\varphi(x) = \log x$.

Definition 2.2. *We consider two continuous, bijective and increasing functions $\varphi : I \to I, \psi : J \to J$. A function $f : I \to J$ is called $(M_{[\varphi]}, M_{[\psi]})$ -* convex *if for every two points $a, b \in I$ and all $\lambda \in [0,1]$*

$$f\left(\varphi^{-1}\left((1-\lambda)\varphi(a) + \lambda\varphi(b)\right)\right) \leq \psi^{-1}\left((1-\lambda)\psi(f(a)) + \lambda\psi(f(b))\right).$$

The function is called strictly $(M_{[\varphi]}, M_{[\psi]})$ - convex *if the inequality is strict for all $a \neq b$ and $\lambda \in (0,1)$.*

According to the definition, we observe some particular cases, depending on which type of mean, arithmetic (A) or geometric (G), it is given on its domain and codomain (see also C.P. Niculescu [5]):

- (A, A)-convex functions (the usual convex functions)
- (A, G)-convex functions (the log-convex functions)
- (G, A)-convex functions
- (G, G)-convex functions (the multiplicatively convex functions).

If $f : I \subset (0,\infty) \to (0,\infty)$ is a $\left(M_{[\varphi]}, M_{[\psi]}\right)-$ convex function, then $g := \psi \circ f \circ \varphi^{-1}$ is convex.

A basic result concerning the convex functions is *Jensen's inequality.* Its formal statement is as follows:

Proposition 2.1 (Jensen's inequality)**.** *A real valued function f defined on I is convex if and only if for all $x_1, x_2, ..., x_n \in I$ and $p_1, p_2, ..., p_n \in (0,1)$ with $\sum_{i=1}^{n} p_i = 1$ we have*

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i).$$

If f is strictly convex then the equality holds if and only if $x_1 = x_2 = ... = x_n$.

It is well known that the assumption "$p_1, p_2, ..., p_n \in [0,1]$"can be slightly relaxed and the result is known as the *Jensen-Steffensen inequality:*

Proposition 2.2 (Jensen-Steffensen inequality)**.** *We consider $x_1, x_2, ..., x_n \in I$, $x_1 \geq x_2 \geq ... \geq x_n$ and the real numbers $w_1, w_2, ..., w_n$ such that the partial sums $W_k = \sum_{i=1}^{k} w_i$, $k \in \{1, 2, ..., n\}$, verify the relations:*

$$0 \leq W_k \leq W_n, \text{ for all } k \in \{1, 2, ..., n-1\}, \tag{JS}$$

$$W_n > 0.$$

Then every convex function f defined on I verifies the inequality:

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{1}$$

If f is strictly convex then the inequality is strict unless $x_1 = x_2 = ... = x_n$.

(See C.P. Niculescu and L.-E. Persson [6, Theorem 1.5.6] *for details.)*

The left term of the Jensen-Steffensen inequality is well defined. We can easily deduce this from the following lemma.

Lemma 2.1. *Let $\varphi : I \to J$ be a continuous, increasing and bijective function. We consider $\mathbf{x} = (x_1, x_2, ..., x_n) \in I^n$ such that $x_1 \geq x_2 \geq ... \geq x_n$ and $\mathbf{w} = (w_1, w_2, ..., w_n)$ an n -tuple of real numbers that satisfies the conditions (JS). Then*

$$\bar{x} = \varphi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \varphi(x_i)\right)$$

verifies that $x_1 \geq \bar{x} \geq x_n$.

Proof. Since we have,

$$\begin{aligned} W_n(\varphi(x_1) - \varphi(\bar{x})) &= \sum_{i=1}^{n} w_i [\varphi(x_1) - \varphi(x_i)] \\ &= \sum_{j=1}^{n-1} (\varphi(x_j) - \varphi(x_{j+1}))(W_n - W_j) \geq 0 \end{aligned}$$

and

$$\begin{aligned} W_n(\varphi(x_n) - \varphi(\bar{x})) &= \sum_{i=1}^{n} w_i [\varphi(x_i) - \varphi(x_n)] \\ &= \sum_{j=1}^{n-1} (\varphi(x_j) - \varphi(x_{j+1})) W_j \geq 0, \end{aligned}$$

we deduce $\varphi(x_1) \geq \varphi(\bar{x}) \geq \varphi(x_n)$. Because φ is increasing, we may say that $x_1 \geq \bar{x} \geq x_n$, which is the required result. □

According to Lemma 2.1, if $\mathbf{x} = (x_1, x_2, ..., x_n)$ denote a vector from I^n, then $\bar{x} \in I$.

Let $f : I \subset (0, \infty) \to (0, \infty)$ be a $\left(M_{[\varphi]}, A\right)$ −convex function. Then $f \circ \varphi^{-1}$ is convex and verifies the Jensen-Steffensen inequality; this means that:

$$\left(f \circ \varphi^{-1}\right)\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \varphi(x_i)\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \left(f \circ \varphi^{-1}\right)(x_i).$$

Let $\mathbf{u} = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$ denote the uniform distribution vector. We denote

$$\mathcal{T}(f, \mathbf{w}, \mathbf{x}) = \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i) - f\left(\varphi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \varphi(x_i)\right)\right).$$

Definition 2.3. *A* Steffensen-Popoviciu measure *on $[a, b]$ is any signed Borel measure μ such that*

$$\mu([a, b]) > 0 \text{ and } \int_a^b f^+(x)\, d\mu(x) \geq 0$$

for all convex functions f on $[a, b]$.

3. Main results

3.1. The discrete case. Now we can state and prove our main result:

Theorem 3.1. *We denote $\mathcal{P}_{\max} = \frac{\max_i\{p_i\}}{P_n}$, where $\mathbf{p} = (p_1, p_2, ..., p_n)$ is an n-tuple of real numbers that satisfies the conditions (JS) and $P_n = \sum_{i=1}^{n} p_i$. Under the assumptions of the Lemma 2.1, if f is $\left(M_{[\varphi]}, A\right)$ −convex on I, then*

$$(0 \leq)\mathcal{T}(f, \mathbf{p}, \mathbf{x}) \leq n\mathcal{P}_{\max}\mathcal{T}(f, \mathbf{u}, \mathbf{x}).$$

We assume that $\mathcal{P}_{\max}$ is attained by only one weight and that f is strictly $\left(M_{[\varphi]},A\right)-$convex. Then we have equality if and only if $x_1 = x_2 = ... = x_n$.

Proof. Let $j \in \{1,2,..,n\}$ be an index such that $\mathcal{P}_{\max} = \frac{p_j}{P_n} > 0$. Hence, after we divide the inequality by $n\mathcal{P}_{\max}$, we must prove that

$$\sum_{i=1}^{n}\frac{p_j-p_i}{np_j}f\left(x_i\right)+\frac{P_n}{np_j}f\left(\varphi^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\varphi\left(x_i\right)\right)\right)\geq f\left(M_{[\varphi]}\left(\mathbf{x};\mathbf{u}\right)\right).$$

For this purpose, we take $w_i := \frac{p_j-p_i}{np_j} \geq 0$ for $i = 1,...,n$ and $w_{n+1} := \frac{P_n}{np_j} > 0$. Observe that $W_{n+1} = \frac{P_n}{np_j} + \sum_{i=1}^{n}\frac{p_j-p_i}{np_j} = 1$.

The left side of this inequality can be developed as follows, using Jensen's inequality:

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{p_j-p_i}{np_j}f\left(x_i\right)+\frac{P_n}{np_j}f\left(\varphi^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\varphi\left(x_i\right)\right)\right)\\
=\;&\sum_{i=1}^{n}w_if\left(x_i\right)+w_{n+1}f\left(\varphi^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\varphi\left(x_i\right)\right)\right)\\
\geq\;& f\left(\varphi^{-1}\left(\sum_{i=1}^{n}w_i\varphi\left(x_i\right)+w_{n+1}\frac{1}{P_n}\sum_{i=1}^{n}p_i\varphi\left(x_i\right)\right)\right)\\
=\;& f\left(M_{[\varphi]}\left(\mathbf{x};\mathbf{u}\right)\right).
\end{aligned}$$

The equality case is a simple consequence of Jensen-Steffennsen inequality. □

We are in position to establish bounds for $\mathcal{T}\left(f,\mathbf{p},\mathbf{x}\right)$ under the Jensen-Steffensen conditions:

Theorem 3.2. *We consider $x_1 \geq x_2 \geq ... \geq x_n$ in I, $p_1,p_2,...,p_n$ and $q_1,q_2,...,q_n$ real numbers such that*

$$\begin{aligned}
0 &\leq P_k \leq 1,\ \text{for all } k\in\{1,2,..n-1\},\quad P_n=1, \qquad (2)\\
0 &< Q_k < 1,\ \text{for all } k\in\{1,2,..n-1\},\quad Q_n=1,
\end{aligned}$$

where $P_k = \sum_{i=1}^{k}p_i$ and $Q_k = \sum_{i=1}^{k}q_i$, $k = 1,...,n$. We denote

$$\tilde{m}=\min_{k=1..n-1}\left\{\frac{P_k}{Q_k};\frac{1-P_k}{1-Q_k}\right\}\ \text{and}\ \tilde{M}=\max_{k=1..n-1}\left\{\frac{P_k}{Q_k};\frac{1-P_k}{1-Q_k}\right\}.$$

If f is $\left(M_{[\varphi]},A\right)-$convex on I, then there holds the relation:

$$\tilde{m}\mathcal{T}\left(f,\mathbf{q},\mathbf{x}\right)\leq\mathcal{T}\left(f,\mathbf{p},\mathbf{x}\right)\leq\tilde{M}\mathcal{T}\left(f,\mathbf{q},\mathbf{x}\right). \qquad (3)$$

Proof. We prove only the first inequality (the second one has a similar approach).

Let m be a nonnegative real constant such that

$$m\geq 0,\qquad P_k-mQ_k\geq 0,\qquad \left(1-P_k\right)-m\left(1-Q_k\right)\geq 0,$$

for all $k = 1,...,n-1$. We have

$$\left(1-P_k\right)-m\left(1-Q_k\right)=1-m-\left(P_k-mQ_k\right)\geq 0.$$

Then, obviously,

$$1-m\geq P_k-mQ_k\geq 0.$$

It follows that it has to be $1-m=\sum_{i=1}^{n}(p_i-mq_i)\geq 0$. Clearly, under our assumptions we have

$$m\leq \min_{k=1..n-1}\left\{\frac{P_k}{Q_k};\frac{1-P_k}{1-Q_k}\right\}=\tilde{m}.$$

The inequality we intend to prove is

$$m\mathcal{T}(f,\mathbf{q},\mathbf{x})\leq \mathcal{T}(f,\mathbf{p},\mathbf{x}). \tag{4}$$

This means

$$f\left(\varphi^{-1}\left(\sum_{i=1}^{n}p_i\varphi(x_i)\right)\right)\leq\sum_{i=1}^{n}(p_i-mq_i)f(x_i)+mf\left(\varphi^{-1}\left(\sum_{i=1}^{n}q_i\varphi(x_i)\right)\right).$$

Via Lemma 2.1 we conclude that there is an integer k, $2\leq k\leq n$ such that

$$x_{k-1}\geq\varphi^{-1}\left(\sum_{i=1}^{n}q_i\varphi(x_i)\right)\geq x_k. \tag{5}$$

We apply the Jensen-Steffensen inequality for the monotonically decreasing $(n+1)$-tuple $\mathbf{y}=(y_1,...,y_{n+1})$

$$y_i=\begin{cases} x_i, \text{ for } i=1,..,k-1\\ \varphi^{-1}\left(\sum_{i=1}^{n}q_i\varphi(x_i)\right), \text{ for } i=k\\ x_{i-1}, \text{ for } i=k+1,..,n+1\end{cases}$$

and

$$w_i=\begin{cases} p_i-mq_i, \text{ for } i=1,..,k-1\\ m, \text{ for } i=k\\ p_{i-1}-mq_{i-1}, \text{ for } i=k+1,..,n+1\end{cases}.$$

It is clear that $\mathbf{w}$ satisfies the conditions JS:

$$\begin{aligned} W_j &= \begin{cases} P_j-mQ_j\geq 0 \text{ for } j=1,..,k-1\\ P_{j-1}-mQ_{j-1}+m\geq 0 \text{ for } j=k,..,n\end{cases}\\ W_{n+1} &= \sum_{i=1}^{n}(p_i-mq_i)+m=1,\\ W_{n+1}-W_j &= \begin{cases}(1-P_j)-m(1-Q_j)+m \text{ for } j=1,..,k-1\\ (1-P_{j-1})-m(1-Q_{j-1}) \text{ for } j=k,..,n\end{cases}\geq 0\end{aligned}$$

A simple computation leads us to the conclusion that the inequality (4) is true, as claimed:

$$\begin{aligned} &\sum_{i=1}^{n}(p_i-mq_i)f(x_i)+mf\left(\varphi^{-1}\left(\sum_{i=1}^{n}q_i\varphi(x_i)\right)\right)\\ \geq\ & f\left(\varphi^{-1}\left(\sum_{i=1}^{n}(p_i-mq_i)\varphi(x_i)+m\sum_{i=1}^{n}q_i\varphi(x_i)\right)\right)\\ =\ & f\left(\varphi^{-1}\left(\sum_{i=1}^{n}p_i\varphi(x_i)\right)\right).\end{aligned}$$

This completes the proof of the right side inequality. □

We may easily observe that $\mathcal{T}(f,\mathbf{p},\mathbf{x})$ does not change if we simultaneously permute the components of $\mathbf{p}$ and $\mathbf{x}$. We can put the condition $x_1\geq x_2\geq...\geq x_n$ to the hypothesis without loss of generality, since this will not restrict the values, only will put them in a different order. The second inequality of the following theorem is a particular case of the Theorem 3.1, namely for a probability vector $\mathbf{p} = (p_1, p_2, ..., p_n)$.

Theorem 3.3. *We consider* $\mathbf{p} = (p_1, p_2, ..., p_n)$ *such that* $p_i > 0$, $\sum_{i=1}^n p_i = 1$. *Then we have*

$$np_{\min}\mathcal{T}(f, \mathbf{u}, \mathbf{x}) \le \mathcal{T}(f, \mathbf{p}, \mathbf{x}) \le np_{\max}\mathcal{T}(f, \mathbf{u}, \mathbf{x}),$$

where

$$p_{\min} = \min\{p_1, p_2, ..., p_n\} \text{ and } p_{\max} = \max\{p_1, p_2, ..., p_n\}.$$

We assume that each value $p_{\min}$ *and* $p_{\max}$ *is attained by only one weight and that* f *is strictly* $\left(M_{[\varphi]}, A\right)$*−convex. Then we have equality if and only if* $x_1 = x_2 = ... = x_n$.

Proof. The first inequality:

Since $p_i > 0$, $i = 1, .., n$, we have $p_{\min} > 0$. Let $j \in \{1, 2, .., n\}$ be an index such that $p_{\min} = p_j$.

Then we may notice that $np_j + \sum_{i=1}^n (p_i - p_j) = 1$. Then

$$\sum_{i=1}^n p_i f(x_i) - np_j \mathcal{T}(f, \mathbf{u}, \mathbf{x}) = \sum_{i=1}^n (p_i - p_j) f(x_i) + np_j f\left(M_{[\varphi]}(\mathbf{x}; \mathbf{u})\right)$$

$$\ge f\left(\varphi^{-1}\left(p_j \sum_{i=1}^n \varphi(x_i) + \sum_{i=1}^n (p_i - p_j)\varphi(x_i)\right)\right) = f\left(M_{[\varphi]}(\mathbf{x}; \mathbf{p})\right).$$

The proof of *the second inequality* is completely similar to that of Theorem 3.1 and, hence, the details are omitted.

The equality case:

Since $p_{\min} \ne p_{\max}$, if f is strictly $\left(M_{[\varphi]}, A\right)$−convex then we have equality if and only if $\mathcal{T}(f, \mathbf{u}, \mathbf{x}) = 0$. This yields $x_1 = x_2 = ... = x_n$, thus the proof is completed. □

For $\varphi(x) = \log x$ and $f(x) = x$ the left side of the inequality of this theorem coincides with Furuichi's inequality and the right side of it has been proved recently by J.M. Aldaz [1].

The following corollary is useful in practice.

Corollary 3.1. *For* $i = 1, ..., n$, *we consider* $p_i > 0$ *with* $\sum_{i=1}^n p_i = 1$.

i) If f *is a convex function then*

$$np_{\min}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right)$$

$$\le \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i f(x_i)\right) \le np_{\max}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right)$$

ii) If f *is a* (GA) *-convex function then*

$$np_{\min}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\prod_{i=1}^n x_i^{1/n}\right)\right)$$

$$\le \sum_{i=1}^n p_i f(x_i) - f\left(\prod_{i=1}^n x_i^{p_i}\right) \le np_{\max}\left(\frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\prod_{i=1}^n x_i^{1/n}\right)\right).$$

iii) If f *is a log-convex function then*

$$\left(\frac{\prod_{i=1}^n f(x_i)^{\frac{1}{n}}}{f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)}\right)^{np_{\min}} \le \frac{\prod_{i=1}^n f(x_i)^{p_i}}{f\left(\sum_{i=1}^n p_i x_i\right)} \le \left(\frac{\prod_{i=1}^n f(x_i)^{\frac{1}{n}}}{f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)}\right)^{np_{\max}}.$$

Proof. Directly from the Theorem 3.3 we obtain i) and ii).

iii) For the convex mapping $\log f$ we can apply i). □

Remark 3.1. *We take another probability vector* $\mathbf{q} = (q_1, q_2, ..., q_n)$ *with* $q_i > 0$, $\sum_{i=1}^{n} q_i = 1$. *Applying one more time the Theorem 3.3 we see that the expression* $\mathcal{T}(f, \mathbf{p}, \mathbf{x})$ *can be estimated by lower and upper bounds as follows*

$$\begin{aligned}\frac{p_{\min}}{q_{\max}}\mathcal{T}(f,\mathbf{q},\mathbf{x}) &\leq np_{\min}\mathcal{T}(f,\mathbf{u},\mathbf{x}) \\ &\leq \mathcal{T}(f,\mathbf{p},\mathbf{x}) \\ &\leq np_{\max}\mathcal{T}(f,\mathbf{u},\mathbf{x}) \leq \frac{p_{\max}}{q_{\min}}\mathcal{T}(f,\mathbf{q},\mathbf{x}).\end{aligned}$$

The following theorem is improving the precision of the double inequality

$$\frac{p_{\min}}{q_{\max}}\mathcal{T}(f,\mathbf{q},\mathbf{x}) \leq \mathcal{T}(f,\mathbf{p},\mathbf{x}) \leq \frac{p_{\max}}{q_{\min}}\mathcal{T}(f,\mathbf{q},\mathbf{x}). \tag{6}$$

Theorem 3.4. *For* $i = 1, ..., n$, *we consider* $x_i \in I$, $p_i > 0$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $q_i > 0$ *with* $\sum_{i=1}^{n} q_i = 1$. *Then*

$$\min_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\mathcal{T}(f,\mathbf{q},\mathbf{x}) \leq \mathcal{T}(f,\mathbf{p},\mathbf{x}) \leq \max_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\mathcal{T}(f,\mathbf{q},\mathbf{x}). \tag{7}$$

If f *is strictly* $\left(M_{[\varphi]}, A\right)-$*convex then both sides of the inequality are equalities if and only if* $x_1 = x_2 = ... = x_n$ *or* $p_i = q_i$ *for all* $i = 1, ..., n$.

Proof. The first inequality:

Let m be a positive real constant such that $p_i - mq_i \geq 0$ for all $i = 1, ..., n$. Then $0 < m \leq \min_{i=1..n}\left\{\frac{p_i}{q_i}\right\} \leq 1$.

From $p_i - mq_i \geq 0$ for all $i = 1, ..., n$ follows that $1 - m = \sum_{i=1}^{n}(p_i - mq_i) \geq 0$.

If $\min_{i=1..n}\left\{\frac{p_i}{q_i}\right\} = 1$ then $p_i = q_i$ for all $i = 1, ..., n$ and the first inequality of (7) obviously is an equality.

Hence, it remains to consider the case when $\min_{i=1..n}\left\{\frac{p_i}{q_i}\right\} < 1$. Then $0 < m < 1$.

The inequality we intend to prove is

$$m\mathcal{T}(f,\mathbf{q},\mathbf{x}) \leq \mathcal{T}(f,\mathbf{p},\mathbf{x}). \tag{8}$$

Since $\sum_{i=1}^{n}(p_i - mq_i) + m = 1$, a simple computation leads us to the conclusion that (8) holds:

$$\begin{aligned}&\sum_{i=1}^{n}(p_i - mq_i) f(x_i) + mf\left(M_{[\varphi]}(\mathbf{x};\mathbf{q})\right) \\ \geq\ & f\left(\varphi^{-1}\left(\sum_{i=1}^{n}(p_i - mq_i)\varphi(x_i) + m\sum_{i=1}^{n} q_i\varphi(x_i)\right)\right) \\ =\ & f\left(M_{[\varphi]}(\mathbf{x};\mathbf{p})\right).\end{aligned}$$

The second inequality:

Let M be a positive real constant such that $Mq_i - p_i \geq 0$ for all $i = 1, ..., n$. Then $M \geq \max_{i=1..n}\left\{\frac{p_i}{q_i}\right\} \geq 1$.

From $Mq_i - p_i \geq 0$ for all $i = 1, ..., n$ follows that $M - 1 = \sum_{i=1}^{n}(Mq_i - p_i) \geq 0$.

If $\max_{i=1..n}\left\{\frac{p_i}{q_i}\right\} = 1$ then $p_i = q_i$ for all $i = 1, ..., n$ and the second inequality of (7) is an equality.

We consider the case when $\max_{i=1..n}\left\{\frac{p_i}{q_i}\right\}>1$. Then $M>1$.

We intend to prove that

$$\mathcal{T}(f,\mathbf{p},\mathbf{x})\leq M\mathcal{T}(f,\mathbf{q},\mathbf{x}). \tag{9}$$

Since $\sum_{i=1}^{n}\frac{Mq_i-p_i}{M}+\frac{1}{M}=1$, a simple computation bring us to the conclusion that (9) is true:

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{Mq_i-p_i}{M}f(x_i)+\frac{1}{M}f\left(M_{[\varphi]}(\mathbf{x};\mathbf{p})\right)\\
\geq\ & f\left(\varphi^{-1}\left(\sum_{i=1}^{n}\frac{Mq_i-p_i}{M}\varphi(x_i)+\frac{1}{M}\sum_{i=1}^{n}p_i\varphi(x_i)\right)\right)\\
=\ & f\left(M_{[\varphi]}(\mathbf{x};\mathbf{q})\right).
\end{aligned}$$

Equality case:

We consider that f is strictly $\left(M_{[\varphi]},A\right)$ −convex. Then all terms of the inequality are equal if and only if

$$\mathcal{T}(f,\mathbf{p},\mathbf{x})=\mathcal{T}(f,\mathbf{q},\mathbf{x})=0$$

(this yields $x_1=x_2=...=x_n$) or

$$\min_{i=1..n}\left\{\frac{p_i}{q_i}\right\}=\max_{i=1..n}\left\{\frac{p_i}{q_i}\right\},$$

(that is if $p_i=q_i$ for all $i=1,...,n$). □

The Theorem 3.4 is an improvement of a result due to S. S. Dragomir (see [3, Theorem 1]).

Remark 3.2. *We can see the first assertion of Theorem 3.3 as a corollary of the Theorem 3.4 if we consider the particular case* $\mathbf{q}=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)$.

When we combine the relations (6) and (7) we obtain

$$\begin{aligned}
\frac{p_{\min}}{q_{\max}}\mathcal{T}(f,\mathbf{q},\mathbf{x}) &\leq \min_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\mathcal{T}(f,\mathbf{q},\mathbf{x})\\
&\leq \mathcal{T}(f,\mathbf{p},\mathbf{x})\\
&\leq \max_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\mathcal{T}(f,\mathbf{q},\mathbf{x})\leq\frac{p_{\max}}{q_{\min}}\mathcal{T}(f,\mathbf{q},\mathbf{x}).
\end{aligned}$$

Moreover, every two probability vectors $\mathbf{p}$ and $\mathbf{q}$ are satisfying the conditions (2). Therefore, one has the relation 3 valid in this case. Since

$$\min_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\leq\tilde{m}\text{ and }\max_{i=1..n}\left\{\frac{p_i}{q_i}\right\}\geq\tilde{M}$$

(see J. Barić, A. Matković [2, Lemma 1] for proof), we conclude that the bounds we obtain via Theorem 3.2 are tighter than those obtained with Theorem 3.4.

3.2. The integral case. For μ a Steffensen-Popoviciu measure on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ a $\left(M_{[\varphi]},A\right)$ −convex function then we have

$$f\left(\varphi^{-1}\left(\frac{1}{\mu([a,b])}\int_a^b\varphi(x)\,d\mu(x)\right)\right)\leq\frac{1}{\mu([a,b])}\int_a^b f(x)\,d\mu(x).$$

(See C.P. Niculescu and L.-E. Persson [6, Chapter 4] for more results concerning the Steffensen-Popoviciu measures.) We consider $p:[a,b]\rightarrow\mathbb{R}$ such that $p(x)\,dx$ is an absolutely continuous measure and

$$0<\int_a^b p(x)\,dx,\qquad 0\leq\int_a^t p(x)\,dx\leq\int_a^b p(x)\,dx \tag{SP}$$

for all $t\in[a,b]$. Then $p(x)\,dx$ is a Steffensen-Popoviciu measure and f verifies

$$f\left(\varphi^{-1}\left(\frac{\int_a^b\varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right)\leq\frac{1}{\int_a^b p(x)\,dx}\int_a^b f(x)\,p(x)\,dx. \tag{10}$$

Let's define

$$\mathcal{T}(f,p):=\frac{\int_a^b f(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}-f\left(\varphi^{-1}\left(\frac{\int_a^b\varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right).$$

Theorem 3.5. *If $p(x)\,dx$ is an absolutely continuous Steffensen-Popoviciu measure that verifies the conditions (SP), then if $\mathcal{P}_{\sup}<\infty$ we have*

$$0\leq\mathcal{T}(f,p)\leq\mathcal{P}_{\sup}(b-a)\,\mathcal{T}(f,id),$$

for every $f:[a,b]\rightarrow\mathbb{R}$ a $\left(M_{[\varphi]},A\right)-$convex function, where

$$\mathcal{P}_{\sup}=\frac{1}{\int_a^b p(x)\,dx}\cdot\sup_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{t-s};s\neq t\right\}$$

(the integral analogue of Theorem 3.1).

Proof. The first inequality follows from (10).

We will prove *the second inequality.* We denote

$$M=\mathcal{P}_{\sup}(b-a)>0.$$

The inequality we intend to prove is

$$\mathcal{T}(f,p)\leq M\mathcal{T}(f,id).$$

If $M-1=0$ then we have an equality. We consider the case $M-1>0$ and define

$$q(x):=M\frac{1}{b-a}-\frac{p(x)}{\int_a^b p(x)\,dx}.$$

Then

$$\int_s^t q(x)\,dx=M\frac{t-s}{b-a}-\frac{\int_s^t p(x)\,dx}{\int_a^b p(x)\,dx}\geq 0.$$

It is obvious that

$$\begin{aligned}\int_a^b q(x)\,dx&=M-1>0,\\ \int_a^t q(x)\,dx&=M\frac{t-a}{b-a}-\frac{\int_a^t p(x)\,dx}{\int_a^b p(x)\,dx}\geq 0,\\ \int_a^b q(x)\,dx-\int_a^t q(x)\,dx&\geq 0.\end{aligned}$$

This means that $q(x)\,dx$ is a Steffensen-Popoviciu measure that verifies the conditions (SP). Since f is convex we have

$$\int_a^b q(x)\,dx \cdot f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,q(x)\,dx}{\int_a^b q(x)\,dx}\right)\right) \leq \int_a^b f(x)\,q(x)\,dx.$$

We must prove that

$$\begin{aligned} & \frac{\int_a^b f(x)}{b-a}dx - \frac{1}{M}\frac{\int_a^b f(x)\,p(x)\,dx}{\int_a^b p(x)\,dx} + \frac{1}{M}f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right) \\ \geq \; & f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,dx}{b-a}\right)\right). \end{aligned}$$

Indeed, using (10), we get

$$\begin{aligned} & \frac{1}{M}\int_a^b f(x)\,q(x)\,dx + \frac{1}{M}f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right) \\ \geq \; & \frac{M-1}{M}\cdot f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right) \\ & + \frac{1}{M}f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right) \\ \geq \; & f\left(\varphi^{-1}\left(\frac{1}{M}\int_a^b \varphi(x)\,q(x)\,dx + \frac{1}{M}\frac{\int_a^b \varphi(x)\,p(x)\,dx}{\int_a^b p(x)\,dx}\right)\right) \\ = \; & f\left(\varphi^{-1}\left(\frac{\int_a^b \varphi(x)\,dx}{b-a}\right)\right). \end{aligned}$$

□

Using this technique, the reader can easily prove the integral analogues of all theorems we have state in Section 3 for the discrete case. We succinctly state below the results but we are omitting their proofs.

Theorem 3.6. *Let $p(x)\,dx$ and $q(x)\,dx$ be two absolutely continuous, Popoviciu-Steffensen measures that verifies*

$$\begin{aligned} \int_a^b p(x)\,dx &= 1, \qquad 0 \leq \int_a^t p(x)\,dx \leq \int_a^b p(x)\,dx, \\ \int_a^b q(x)\,dx &= 1, \qquad 0 < \int_a^t q(x)\,dx < \int_a^b q(x)\,dx, \end{aligned}$$

for all $t \in (a,b)$. We denote

$$\begin{aligned} \tilde{m} &= \inf_{t\in(a,b)}\left\{\frac{\int_a^t p(x)\,dx}{\int_a^t q(x)\,dx}, \frac{\int_t^b p(x)\,dx}{\int_t^b q(x)\,dx}\right\}, \\ \tilde{M} &= \sup_{t\in(a,b)}\left\{\frac{\int_a^t p(x)\,dx}{\int_a^t q(x)\,dx}, \frac{\int_t^b p(x)\,dx}{\int_t^b q(x)\,dx}\right\}. \end{aligned}$$

Then

$$\tilde{m}\mathcal{T}(f,q) \leq \mathcal{T}(f,p) \leq \tilde{M}\mathcal{T}(f,q)$$

(the integral analogue of Theorem 3.2).

Theorem 3.7. *Let $p(x)\,dx$ be a absolutely continuous measure, where $p:[a,b]\to(0,\infty)$ is increasing, such that $\int_a^b p(x)\,dx=1$ and define*

$$\mathcal{T}(f,p):=\int_a^b f(x)\,p(x)\,dx-f\left(\varphi^{-1}\left(\int_a^b\varphi(x)\,p(x)\,dx\right)\right).$$

We denote

$$p_{\inf}=\inf_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{t-s};s\neq t\right\},\qquad p_{\sup}=\sup_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{t-s};s\neq t\right\}.$$

Then

$$(b-a)\,p_{\inf}\mathcal{T}\left(f,\frac{id}{b-a}\right)\leq\mathcal{T}(f,p)\leq(b-a)\,p_{\sup}\mathcal{T}\left(f,\frac{id}{b-a}\right)$$

for every $f:[a,b]\to\mathbb{R}$ a $\left(M_{[\varphi]},A\right)-$convex function (the integral analogue of Theorem 3.3).

Remark 3.3. *Let $p(x)\,dx$ and $q(x)\,dx$ be two absolutely continuous measures, where $p,q:[a,b]\to(0,\infty)$ are increasing such that $\int_a^b p(x)\,dx=1$ and $\int_a^b q(x)\,dx=1$. Applying one more time the Theorem 3.7 , we see that the expression $\mathcal{T}(f,p)$ can be estimated by lower and upper bounds as follows*

$$\begin{aligned}\frac{p_{\inf}}{q_{\sup}}\mathcal{T}(f,q)&\leq(b-a)\,p_{\inf}\mathcal{T}\left(f,\frac{id}{b-a}\right)\\&\leq\mathcal{T}(f,p)\\&\leq(b-a)\,p_{\sup}\mathcal{T}\left(f,\frac{id}{b-a}\right)\leq\frac{p_{\sup}}{q_{\inf}}\mathcal{T}(f,q).\end{aligned}$$

These bounds are improved by the following result:

Theorem 3.8. *Let $p(x)\,dx$ and $q(x)\,dx$ be two absolutely continuous measures, where $p,q:[a,b]\to(0,\infty)$ are increasing such that $\int_a^b p(x)\,dx=1$ and $\int_a^b q(x)\,dx=1$. Then the following inequalities hold*

$$\begin{aligned}&\inf_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx};s\neq t\right\}\mathcal{T}(f,q)\\\leq\;&\mathcal{T}(f,p)\leq\sup_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx};s\neq t\right\}\mathcal{T}(f,q),\end{aligned}$$

for every $f:[a,b]\to\mathbb{R}$ a $\left(M_{[\varphi]},A\right)-$convex function (the integral analogue of Theorem 3.4).

Remark 3.4. *Under the assumption of Theorem 3.8 we have*

$$\tilde{m}\geq\inf_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx};s\neq t\right\}\qquad\text{and}\qquad\tilde{M}\leq\sup_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx};s\neq t\right\}.$$

Indeed, these are true since

$$\sup_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx};s\neq t\right\}\geq\frac{\int_a^t p(x)\,dx}{\int_a^t q(x)\,dx}\geq\inf_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx};s\neq t\right\}$$

and

$$\sup_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx}; s\neq t\right\} \geq \frac{\int_s^b p(x)\,dx}{\int_s^b p(x)\,dx} \geq \inf_{t,s\in[a,b]}\left\{\frac{\int_s^t p(x)\,dx}{\int_s^t q(x)\,dx}; s\neq t\right\}$$

for all $t, s \in (a, b)$.

We conclude that the bounds we obtain via Theorem 3.6 are tighter than those obtained with Theorem 3.8.

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