Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(1), 2011, Pages 49–53
ISSN: 1223-6934

A note on totally umbilical proper slant submanifold of a Lorentzian $\beta-$Kenmotsu manifold

Khushwant Singh, Siraj Uddin, and Meraj Ali Khan

Abstract. In the present note, we study a slant submanifold of a Lorentzian β-Kenmotsu manifold which is totally umbilical. We prove that every totally umbilical proper slant submanifold M of a Lorentzian β-Kenmotsu manifold $\bar{M}$ is either minimal or if M is not minimal then we derive a formula for slant angle of M.

2010 Mathematics Subject Classification. Primary 53C25; Secondary 3C40, 53C42, 53D15.
Key words and phrases. totally umbilical, minimal, slant submanifold, Lorentzian β-Kenmotsu manifold.

1. Introduction

The idea of slant submanifolds was introduced by B.Y. Chen [2] in the setting of almost Hermitian manifolds. Since then many research articles have been appeared on the existence of these submanifolds in different known spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta [4].

Recently, M. A. Khan and others [3], studied these submanifolds in the setting of Lorentzian paracontact manifolds. In this paper, we study slant submanifolds of Lorentzian β-Kenmotsu manifolds. We consider, M as a totally umbilical proper slant submanifold of Lorentzian β-Kenmotsu manifold $\bar{M}$ and prove that M is either minimal or if it is not minimal then we get a formula of its slant angle $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$.

2. Preliminaries

Let $\bar{M}$ be a n-dimensional Lorentzian almost paracontact manifold with the almost paracontact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a contravariant vector field, η is a $1-$form and g is a Lorentzian metric with signature $(-,+,+,\cdots,+)$ on $\bar{M}$, satisfying [5]:

$$\phi^2 X = X + \eta(X)\xi, \ \eta(\xi) = -1, \ \phi(\xi) = 0, \ \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \ \eta(X) = g(X, \xi). \tag{2.2}$$

Also, if on $\bar{M}$ the following additional conditions hold:

$$(\bar{\nabla}_X \phi)Y = \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.3}$$

$$\bar{\nabla}_X \xi = \beta\{X - \eta(X)\xi\} \tag{2.4}$$

Received November 04, 2010. Revision received January 06, 2011.

for all $X, Y \in T\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection with respect to the Lorentzian metric g. Then $\bar{M}$ is said to be a *Lorentzian $\beta-$Kenmotsu*. Again if we put

$$\Phi(X, Y) = g(X, \phi Y),$$

then by (2.2), $\Phi(X, Y)$ is symmetric $(0, 2)$ tensor field [5], i.e., $\Phi(X, Y) = \Phi(Y, X)$.

Now, let M be a submanifold of $\bar{M}$. Let TM be the Lie algebra of vector fields in M and $T^{\perp}M$ the set of all vector fields normal to M. If ∇ be the Levi-Civita connection on M, then Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.6}$$

for any $X\ Y \in TM$ and any $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the induced connection in the normal bundle, h is the second fundamental form of M and A_V is the Weingarten endomorphism associated with V. The second fundamental form h and the shape operator A are related by

$$g(A_V X, Y) = g(h(X, Y), V) \tag{2.7}$$

where g denotes the metric on $\bar{M}$ as well as the induced metric on M.

For any $X \in TM$, we write

$$\phi X = TX + FX \tag{2.8}$$

where TX is the tangential component of ϕX and FX is the normal component of ϕX respectively. Similarly, for any $V \in T_x^{\perp}M$, we have

$$\phi V = tV + fV \tag{2.9}$$

where tV (resp. fV) is the tangential component (resp. normal component) of ϕV.

As we know metric g is Lorentzian it is easy to observe that for each $x \in M$ and $X,\ Y \in T_x M$

$$g(TX, Y) = g(X, TY). \tag{2.10}$$

The covariant derivative of the morphisms T and F are defined respectively as

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.11}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y \tag{2.12}$$

for any $X, Y \in TM$.

Throughout, the structure vector field ξ assumed to be tangential to M, otherwise M is simply anti-invariant. For any $X,\ Y \in TM$ on using (2.4) and (2.5) we may obtain

$$(a)\ \nabla_X \xi = \beta\{X - \eta(X)\xi\},\ (b)\ h(X, \xi) = 0. \tag{2.13}$$

On using (2.3), (2.5), (2.6), (2.7), (2.9), (2.11) and (2.12), we obtain

$$(\bar{\nabla}_X T)Y = \beta\{g(TX, Y)\xi - \eta(Y)TX\} + A_{FY}X + th(X, Y) \tag{2.14}$$

$$(\bar{\nabla}_X F)Y = fh(X, Y) - \beta\eta(Y)FX - h(X, TY). \tag{2.15}$$

A submanifold M is said to be *totally umbilical* if

$$h(X, Y) = g(X, Y)H, \tag{2.16}$$

where H is the mean curvature vector. Furthermore, if $h(X, Y) = 0$ for all $X, Y \in TM$, then M is said to be *totally geodesic* and if $H = 0$ then M is *minimal* in $\bar{M}$.

3. Slant submanifolds

Through out the section we consider M as an immersed submanifold of a Lorentzian manifold $\bar{M}$. Such submanifolds we always consider tangent to the structure vector field ξ.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a *slant submanifold* if for any $x \in M$ and $X \in T_xM - \langle\xi\rangle$, the angle between ϕX and T_xM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called *slant angle* of M in $\bar{M}$. The tangent bundle TM of M is decomposed as

$$TM = D \oplus \langle\xi\rangle \tag{3.1}$$

where the orthogonal complementary distribution D of $\langle\xi\rangle$ is known as the *slant distribution* on M. If μ is an invariant subspace of the normal bundle $T^\perp M$, then

$$T^\perp M = FTM \oplus \mu. \tag{3.2}$$

For a proper slant submanifold M of an Lorentzian paracontact manifold $\bar{M}$ with a slant angle θ, M. A. Khan et. al. [3] proved the following theorem.

Theorem 3.1. *Let M be a submanifold of an LP-contact manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{3.3}$$

Furthermore, if θ is slant angle of M, then $\lambda = \cos^2\theta$.

Thus, we have the following consequences of the formula (3.3),

$$g(TX, TX) = \cos^2\theta[g(X, Y) + \eta(X)\eta(Y)] \tag{3.4}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) + \eta(X)\eta(Y)] \tag{3.5}$$

for any $X, Y \in TM$.

In, the following theorems we consider M as a totally umbilical submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$.

Theorem 3.2. *Let M be a totally umbilical Riemannian submanifold of a Lorentzian β-Kenmotsu manifold $\bar{M}$, then atleast one of the following statements is true*
(i) $H \in \mu$
(ii) *M is trivial.*

Proof. For any $X, Y \in TM$ then from (2.14), we have

$$(\bar{\nabla}_X T)Y = A_{FY}X + th(X, Y) + \beta\{g(TX, Y)\xi - \eta(Y)TX\}.$$

Taking the product with ξ, we obtain

$$g(\nabla_X TY, \xi) = g(h(X, \xi), FY) + g(th(X, Y), \xi) - \beta(g(TX, Y)$$

As M is a totally umbilical slant submanifold of $\bar{M}$, then from (2.16), the above equation takes the form

$$-g(TY, \nabla_X\xi) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y).$$

Using (2.13), we get

$$-\beta g(TY, X) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y).$$

Then from (2.10), we obtain

$$-\beta g(TX, Y) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y). \tag{3.6}$$

The second term of right hand side in (3.6) is identically zero, then the above equation takes the form

$$g(H,FY)\eta(X)=0. \tag{3.7}$$

Thus from (3.7), it follows that either $H\in\mu$ or M is trivial. This proves the theorem completely. $\square$

Now, using the above theorem we have the following main result of this paper.

Theorem 3.3. *Let M be a non trivial totally umbilical proper slant submanifold of a Lorentzian $\beta-$Kenmotsu manifold $\bar{M}$. Then at least one of the following statements is true*

(i) *M is minimal*

(ii) *If M is not minimal, then the slant angle of M is $\theta=tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$ for any $X,Y\in TM$.*

Proof. For any $X,Y\in TM$, we have

$$\bar{\nabla}_X\phi Y-\phi\bar{\nabla}_XY=\beta\{g(\phi X,Y)\xi-\eta(Y)\phi X\}.$$

From (2.5) and (2.8), we obtain

$$\bar{\nabla}_XTY+\bar{\nabla}_XFY-\phi(\nabla_XY+h(X,Y))=\beta\{g(TX+FX,Y)\xi-\eta(Y)(TX+FX)\}.$$

Again using (2.5), (2.6) and (2.8), we get

$$\beta\{g(TX,Y)\xi-\eta(Y)TX\}-\beta\eta(Y)FX=\nabla_XTY+h(X,TY)$$
$$-A_{FY}X+\nabla^{\perp}_XFY-T\nabla_XY-F\nabla_XY-\phi h(X,Y).$$

As M is totally umbilical proper slant, then

$$\beta\{g(TX,Y)\xi-\eta(Y)TX\}-\beta\eta(Y)FX=\nabla_XTY+g(X,TY)H-A_{FY}X$$
$$+\nabla^{\perp}_XFY-T\nabla_XY-F\nabla_XY-g(X,Y)\phi H.$$

Taking product with ϕH and using the fact that $H\in\mu$ (Theorem 3.2), we obtain

$$g(X,TY)g(H,\phi H)+g(\nabla^{\perp}_XFY,\phi H)=g(F\nabla_XY,\phi H)+g(X,Y)g(\phi H,\phi H)$$
$$-\beta\eta(Y)g(FX,\phi H).$$

Using (2.2) and the fact that $H\in\mu$, then ϕH is also lies in μ, thus we have

$$g(\nabla^{\perp}_XFY,\phi H)=g(X,Y)\|H\|^2.$$

Then from (2.6), we derive

$$g(\bar{\nabla}_XFY,\phi H)=g(X,Y)\|H\|^2. \tag{3.8}$$

Now for any $X\in TM$, we have

$$(\bar{\nabla}_X\phi)H=\bar{\nabla}_X\phi H-\phi\bar{\nabla}_XH.$$

Using (2.3) and the fact that $H\in\mu$, we obtain

$$0=\bar{\nabla}_X\phi H-\phi\bar{\nabla}_XH.$$

Using (2.5), (2.6), (2.8) and (2.9), we obtain

$$-A_{\phi H}X+\nabla^{\perp}_X\phi H=-TA_HX-FA_HX+t\nabla^{\perp}_XH+f\nabla^{\perp}_XH. \tag{3.9}$$

Taking the product in (3.9) with FY for any $Y\in TM$ and using the fact $f\nabla^{\perp}_XH\in\mu$, the above equation gives

$$g(\nabla^{\perp}_X\phi H,FY)=-g(FA_HX,FY).$$

Using (3.4), we obtain that

$$g(\bar{\nabla}_X FY, \phi H) = \sin^2\theta[g(A_H X, Y) + \eta(A_H X)\eta(Y)],$$

that is,

$$g(\bar{\nabla}_X FY, \phi H) = \sin^2\theta[g(X, Y) + \eta(X)\eta(Y)]\|H\|^2. \tag{3.10}$$

Thus, from (3.8) and (3.10), we derive

$$[\cos^2\theta g(X, Y) - sin^2\theta\eta(X)\eta(Y)]\|H\|^2 = 0. \tag{3.11}$$

Thus, it follows from (3.11) that either $H = 0$, i.e., M is minimal or if M is not minimal then the slant angle of M is $\theta = tan^{-1}(\sqrt{\frac{g(X,Y)}{\eta(X)\eta(Y)}})$. This completes the proof. □

References

[1] N.S. Basavarajappa, C.S. Bagewadi and D.G. Prakasha, *Some results on Lorentzian β-Kenmotsu manifolds*, Ann. Math. Comp. Sci. Ser. 35 (2008), 7–14.

[2] B.Y. Chen, *Slant immersions*, Bull. Austral. Math. Soc. 41 (1990), 135–147.

[3] M.A. Khan, Khushwant Singh and V.A. Khan, *Slant submanifolds of LP-contact manifolds*, Diff. Geom. Dyn. Syst. 12 (2010), 102-108.

[4] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie 39 (1996), 183–198.

[5] K. Matsumoto, *On Lorentzian paracontact manifolds,* Bull. Yamagata Univ. Nat. Sci. 12 (1989), 151–156.

(Khushwant Singh, Meraj Ali Khan) School of Mathematics and Computer Applications, Thapar University, Patiala-147 004, India
E-mail address: `khushwantchahil@gmail.com, meraj79@gmail.com`

(Siraj Uddin) Institute of Mathematical Sciences, Faculty of Science,University of Malaya, 50603 Kuala Lumpur, MALAYSIA
E-mail address: `iraj.ch@gmail.com`