A note on totally umbilical proper slant submanifold of a Lorentzian $\beta$–Kenmotsu manifold

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Abstract. In the present note, we study a slant submanifold of a Lorentzian $\beta$-Kenmotsu manifold which is totally umbilical. We prove that every totally umbilical proper slant submanifold $M$ of a Lorentzian $\beta$-Kenmotsu manifold $\bar{M}$ is either minimal or if $M$ is not minimal then we derive a formula for slant angle of $M$.

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1. Introduction

The idea of slant submanifolds was introduced by B.Y. Chen [2] in the setting of almost Hermitian manifolds. Since then many research articles have been appeared on the existence of these submanifolds in different known spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta [4].

Recently, M. A. Khan and others [3], studied these submanifolds in the setting of Lorentzian paracontact manifolds. In this paper, we study slant submanifolds of Lorentzian $\beta$-Kenmotsu manifolds. We consider, $M$ as a totally umbilical proper slant submanifold of Lorentzian $\beta$-Kenmotsu manifold $\bar{M}$ and prove that $M$ is either minimal or if it is not minimal then we get a formula of its slant angle $\theta = \tan^{-1}(\sqrt{g(Y, Y)/\eta(Y, Y)})$.

2. Preliminaries

Let $\bar{M}$ be a $n$-dimensional Lorentzian almost paracontact manifold with the almost paracontact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a contravariant vector field, $\eta$ is a $1$–form and $g$ is a Lorentzian metric with signature $(-, +, +, \cdots, +)$ on $M$, satisfying [5]:

\begin{align*}
\phi^2X &= X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad (2.1) \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi). \quad (2.2)
\end{align*}

Also, if on $\bar{M}$ the following additional conditions hold:

\begin{align*}
(\bar{\nabla}_X \phi)Y &= \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \quad (2.3) \\
\bar{\nabla}_X \xi &= \beta\{X - \eta(X)\xi\}. \quad (2.4)
\end{align*}

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for all $X, Y \in TM$, where $\nabla$ is the Levi-Civita connection with respect to the Lorentzian metric $g$. Then $M$ is said to be a Lorentzian $\beta-$Kenmotsu. Again if we put

$$\Phi(X, Y) = g(X, \phi Y),$$

then by (2.2), $\Phi(X, Y)$ is symmetric $(0, 2)$ tensor field [5], i.e., $\Phi(X, Y) = \Phi(Y, X)$.

Now, let $M$ be a submanifold of $\bar{M}$. Let $TM$ be the Lie algebra of vector fields in $M$ and $T^\perp M$ the set of all vector fields normal to $M$. If $\nabla$ be the Levi-Civita connection on $M$, then Gauss-Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

(2.5)

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V$$

(2.6)

for any $X, Y \in TM$ and any $V \in T^\perp M$, where $\nabla^\perp$ is the induced connection in the normal bundle, $h$ is the second fundamental form of $M$ and $A_V$ is the Weingarten endomorphism associated with $V$. The second fundamental form $h$ and the shape operator $A$ are related by

$$g(A_V X, Y) = g(h(X, Y), V)$$

(2.7)

where $g$ denotes the metric on $\bar{M}$ as well as the induced metric on $M$.

For any $X \in TM$, we write

$$\phi X = TX + FX$$

(2.8)

where $TX$ is the tangential component of $\phi X$ and $FX$ is the normal component of $\phi X$ respectively. Similarly, for any $V \in T^\perp_M$, we have

$$\phi V = tV + fV$$

(2.9)

where $tV$ (resp. $fV$) is the tangential component (resp. normal component) of $\phi V$.

As we know metric $g$ is Lorentzian it is easy to observe that for each $x \in M$ and $X, Y \in T_x \bar{M}$

$$g(TX, Y) = g(X, TY).$$

(2.10)

The covariant derivative of the morphisms $T$ and $F$ are defined respectively as

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y$$

(2.11)

$$(\bar{\nabla}_X F)Y = \nabla^\perp_X FY - F\nabla_X Y$$

(2.12)

for any $X, Y \in TM$.

Throughout, the structure vector field $\xi$ assumed to be tangential to $M$, otherwise $M$ is simply anti-invariant. For any $X, Y \in TM$ on using (2.4) and (2.5) we may obtain

$$(a) \nabla_X \xi = \beta [X - \eta(X)\xi], \ (b) h(X, \xi) = 0.$$ 

(2.13)

On using (2.3), (2.5), (2.6), (2.7), (2.9), (2.11) and (2.12), we obtain

$$(\nabla_X T)Y = \beta [g(TX, Y)\xi - \eta(Y)TX] + A_F Y + th(X, Y)$$

(2.14)

$$(\nabla_X F)Y = fh(X, Y) - \beta \eta(Y)FX - h(X, TY).$$

(2.15)

A submanifold $M$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H,$$

(2.16)

where $H$ is the mean curvature vector. Furthermore, if $h(X, Y) = 0$ for all $X, Y \in TM$, then $M$ is said to be totally geodesic and if $H = 0$ then $M$ is minimal in $\bar{M}$.
3. Slant submanifolds

Throughout the section we consider $M$ as an immersed submanifold of a Lorentzian manifold $\tilde{M}$. Such submanifolds we always consider tangent to the structure vector field $\xi$.

A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ is said to be a slant submanifold if for any $x \in M$ and $X \in T_xM - \langle \xi \rangle$, the angle between $\phi X$ and $T_xM$ is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of $M$ in $\tilde{M}$. The tangent bundle $TM$ of $M$ is decomposed as

$$TM = D \oplus \langle \xi \rangle$$

where the orthogonal complementary distribution $D$ of $\langle \xi \rangle$ is known as the slant distribution on $M$. If $\mu$ is an invariant subspace of the normal bundle $T^\perp M$, then

$$T^\perp M = FTM \oplus \mu.$$  

For a proper slant submanifold $M$ of an Lorentzian paracontact manifold $\tilde{M}$ with a slant angle $\theta$, M. A. Khan et. al. [3] proved the following theorem.

**Theorem 3.1.** Let $M$ be a submanifold of an LP-contact manifold $\tilde{M}$ such that $\xi \in TM$. Then, $M$ is slant submanifold if and only if there exist a constant $\lambda \in [0, 1]$ such that

$$T^2 = \lambda(I + \eta \otimes \xi).$$

Furthermore, if $\theta$ is slant angle of $M$, then $\lambda = \cos^2 \theta$.

Thus, we have the following consequences of the formula (3.3),

$$g(TX, TX) = \cos^2 \theta[g(X, Y) + \eta(X)\eta(Y)]$$

$$g(FX, FY) = \sin^2 \theta[g(X, Y) + \eta(X)\eta(Y)]$$

for any $X, Y \in TM$.

In, the following theorems we consider $M$ as a totally umbilical slant submanifold of a Lorentzian $\beta$-Kenmotsu manifold $\tilde{M}$.

**Theorem 3.2.** Let $M$ be a totally umbilical Riemannian submanifold of a Lorentzian $\beta$-Kenmotsu manifold $\tilde{M}$, then at least one of the following statements is true

(i) $H \in \mu$

(ii) $M$ is trivial.

**Proof.** For any $X, Y \in TM$ then from (2.14), we have

$$(\nabla_X Y) = A_{FY}X + th(X, Y) + \beta(g(TX, Y)\xi - \eta(Y)TX).$$

Taking the product with $\xi$, we obtain

$$g(\nabla_X Y, \xi) = g(h(X, \xi), FY) + g(th(X, Y), \xi) - \beta(g(TX, Y)$$

As $M$ is a totally umbilical slant submanifold of $\tilde{M}$, then from (2.16), the above equation takes the form

$$-g(TY, \nabla_X \xi) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y).$$

Using (2.13), we get

$$-\beta g(TY, X) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y).$$

Then from (2.10), we obtain

$$-\beta g(TX, Y) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y).$$

$$-\beta g(TX, Y) = g(H, FY)\eta(X) + g(X, Y)g(tH, \xi) - \beta g(TX, Y).$$
The second term of right hand side in (3.6) is identically zero, then the above equation takes the form
\[ g(H, FY)\eta(X) = 0. \] (3.7)
Thus from (3.7), it follows that either \( H \in \mu \) or \( M \) is trivial. This proves the theorem completely.

Now, using the above theorem we have the following main result of this paper.

**Theorem 3.3.** Let \( M \) be a non trivial totally umbilical proper slant submanifold of a Lorentzian \( \beta-Kenmotsu \) manifold \( \bar{M} \). Then at least one of the following statements is true

\begin{enumerate}[(i)]
  \item \( M \) is minimal
  \item If \( M \) is not minimal, then the slant angle of \( M \) is \( \theta = \tan^{-1}\left(\frac{\sqrt{g(X,Y)}}{\eta(X)\eta(Y)}\right) \) for any \( X, Y \in T\bar{M} \).
\end{enumerate}

**Proof.** For any \( X, Y \in T\bar{M} \), we have
\[ \nabla_X \phi Y - \phi \nabla_X Y = \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}. \]
From (2.5) and (2.8), we obtain
\[ \nabla_X TY + \nabla_X FY - \phi(\nabla_X Y + h(X,Y)) = \beta\{g(TX + FX, Y)\xi - \eta(Y)(TX + FX)\}. \]
Again using (2.5), (2.6) and (2.8), we get
\[ \beta\{g(TX, Y)\xi - \eta(Y)TX\} - \beta\eta(Y)FX = \nabla_X TY + h(X, TY) \]
\[ -A_{FY}X + \nabla_X FY - T\nabla_X Y - F\nabla_X Y - \phi h(X, Y). \]
As \( M \) is totally umbilical proper slant, then
\[ \beta\{g(TX, Y)\xi - \eta(Y)TX\} - \beta\eta(Y)FX = \nabla_X TY + g(X, TY)H - A_{FY}X \]
\[ + \nabla_X FY - T\nabla_X Y - F\nabla_X Y - g(X, Y)\phi H. \]
Taking product with \( \phi H \) and using the fact that \( H \in \mu \) (Theorem 3.2), we obtain
\[ g(X, TY)g(H, \phi H) + g(\nabla_X FY, \phi H) = g(F\nabla_X Y, \phi H) + g(X, Y)g(\phi H, \phi H) \]
\[ - \beta\eta(Y)g(FX, \phi H). \]
Using (2.2) and the fact that \( H \in \mu, \) then \( \phi H \) is also lies in \( \mu, \) thus we have
\[ g(\nabla_X FY, \phi H) = g(X, Y)||H||^2. \]
Then from (2.6), we derive
\[ g(\nabla_X FY, \phi H) = g(X, Y)||H||^2. \] (3.8)
Now for any \( X \in TM \), we have
\[ (\nabla_X \phi)H = \nabla_X \phi H - \phi \nabla_X H. \]
Using (2.3) and the fact that \( H \in \mu, \) we obtain
\[ 0 = \nabla_X \phi H - \phi \nabla_X H. \]
Using (2.5), (2.6), (2.8) and (2.9), we obtain
\[ -A_{\phi H}X + \nabla_X \phi H = -TA_HX - FA_HX + t\nabla_X H + f\nabla_X H. \] (3.9)
Taking the product in (3.9) with \( FY \) for any \( Y \in TM \) and using the fact \( f\nabla_X H \in \mu, \) the above equation gives
\[ g(\nabla_X \phi H, FY) = -g(FA_HX, FY). \]
Using (3.4), we obtain that
\[ g(\nabla_X F Y, \phi H) = \sin^2 \theta [g(A_H X, Y) + \eta(A_H X)\eta(Y)], \]
that is,
\[ g(\nabla_X F Y, \phi H) = \sin^2 \theta [g(X, Y) + \eta(X)\eta(Y)]\|H\|^2. \quad (3.10) \]
Thus, from (3.8) and (3.10), we derive
\[ [\cos^2 \theta g(X, Y) - \sin^2 \theta \eta(X)\eta(Y)]\|H\|^2 = 0. \quad (3.11) \]
Thus, it follows from (3.11) that either \( H = 0 \), i.e., \( M \) is minimal or if \( M \) is not minimal then the slant angle of \( M \) is \( \theta = \tan^{-1}(\sqrt{\frac{g(X, Y)}{\eta(X)\eta(Y)}}) \). This completes the proof. \( \square \)

References


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