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Weak convergence theorem for Lipschitzian pseudocontraction semigroups in Banach spaces

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Abstract. In this work, theorems of weak convergence of an implicit iterative algorithm with errors for treating a nonexpansive semigroup and a Lipschitzian pseudocontractive semigroup are estabilshed in the framework of real Banach spaces.

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1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) : \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \forall x \in E$, where E^* denotes the dual space of E and $\langle ., .\rangle$ denotes the generalized duality pairing. In the following, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $\omega_w(x_n)$ denote the weak ω-limit set.

Definition 1.1. *One-parameter family $\{T(t) : t \geq 0\}$ of mappings from K into itself is said to be a pseudo-contraction semigroup on K, if the following conditions are satisfied:*

(1) $T(0)x = x$ *for each* $x \in K$*;*
(2) $T(t + s)x = T(t)T(s)x$ *for any* $t, s \in \mathbb{R}_+$ *and* $x \in K$*;*
(3) *for each* $x \in E$*, the mapping* $T(.)x$ *from* $\mathbb{R}_+$ *into* K *is continuous;*
(4) *for any* $x, y \in C$*, there exists* $j(x - y) \in J(x - y)$ *such that*

$$\langle T(t)x - T(t)y, j(x - y)\rangle \leq \|x - y\|^2, \text{ for each } t > 0.$$

A pseudocontraction semigroup $\{T(t) : t \geq 0\}$ is said to be Lipschitzian [4], if the conditions (1)-(4) and the following condition (5) are satisfied:

(5) *There exists a bounded measurable function* $L : (0, \infty) \to [0, \infty)$ *such that, for any* $x, y \in K$ *then*

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\|, \text{ for each } t > 0. \quad (1)$$

In the sequel, we denote

$$M := \sup_{t\geq 0} L(t) < \infty \quad \text{and} \quad F := \bigcap_{t\geq 0} Fix(T(t)).$$

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Recently, the problems of convergence of an implicit iterative algorithm to a common fixed point for a family of nonexpansive mappings and its extensions to Hilbert spaces or Banach spaces have been considered by several authors; see [1-10] for more details.

In 2008, Hao [3] considered the implicit iterative algorithm for treating a family of Lipschitz pseudocontractions $\{T_1, T_2, ..., T_N\}$ in a Banach space:

$$x_0 \in K, \quad x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n, \quad \forall n \geq 1, \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and u_n is a bounded sequence in K. Theorems of weak convergence to common fixed points are estabilshed in a uniformly convex Banach space.

In 2010, X. Qin and S. Y. Cho [7] considered the implicit iterative algorithm (2) for treating strongly continuous semigrous of Lipschit pseudocontractions in a reflexive Banach sapce.

In this work, motivated by recent work going in this direction, we consider the weak convergence of a implicit iterative algorithm for a nonexpansive semigroup and a Lipschitzian pseudocontractive semigroup $\{T(t) : t \geq 0\}$ on K as follows:

$$x_0 \in K, \quad x_n = f(x_n) + \beta_n f(x_n) + \gamma_n u_n.$$

In the sequel, we will need the following definition and results.

Definition 1.2. *A Banach space E is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, as $n \to \infty$, then*

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||, \quad \forall y \in E, y \neq x. \tag{3}$$

Lemma 1.1. ([9], Lemma 1). *Let $\{t_n\}$ be a real sequence and τ be a real number such that $\liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n$. Suppose that either of the following holds:*
i) $\limsup_{n\to\infty}(t_{n+1} - t_n) \leq 0$, *or*
ii) $\liminf_{n\to\infty}(t_{n+1} - t_n) \geq 0$.
Then τ is a cluter point of $\{t_n\}$. Moreover, for $\epsilon > 0, k, m \in \mathbb{N}$, there exists $m_0 \geq m$ such that $|t_j - \tau| < \epsilon$ for every integer j with $m_0 \leq j \leq m_0 + k$.

Lemma 1.2. (Zhou [13]). *Let E be a real reflexive Banach space with the Opial condition. Let C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous pseudocontractive mapping. Then T is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup y$ and $\|(I - T)x_n\| \to 0$, then $(I - T)y = 0$.*

2. Main results

Theorem 2.1. *Let E be a reflexive Banach space which satisfies Opial's condition, suppose K is a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on K such that $F := \bigcap_{t\geq 0} Fix(T(t)) \neq \emptyset$, and $f : K \to K$ be a fixed contractive mapping with contractive coefficient $\alpha \in (0, 1)$. Define a sequence $\{x_n\}$ in K by*

$$x_n = \alpha_n f(x_n) + \beta_n T(t_n)x_n + \gamma_n u_n. \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, $t_n > 0$ and $\{u_n\}$ is a bounded sequence in K. Assume that the following conditions are satisfied:
a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1;$

b) $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n+\gamma_n}{t_n} = 0.$
Then $\{x_n\}$ *converges weakly to a common fixed point of the semigroup* $\{T(t) : t \geq 0\}$.

Proof. First, it is easy to see that $\{x_n\}$ is well defined. Now fix $p \in F$ we have

$$\begin{aligned}\|x_n - p\| &\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|T(t_n)x_n - p\| \\ &\quad + \gamma_n\|u_n - p\| \\ &\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|u_n - p\|.\end{aligned}$$

Therefore

$$\|x_n - p\| \leq \frac{1}{1-\alpha}\|f(p) - p\| + \|u_n - p\|.$$

This implies the sequence $\{x_n\}$ is bounded, and so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. Since $\{x_n\}$ is bounded, without loss of generality we assume that a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $q \in K$. Now, we prove that $q = T(t)q$ for a fixed $t > 0$. Indeed,

$$\begin{aligned}\|x_{n_j} - T(t)q\| &\leq \sum_{k=0}^{\left[\frac{t}{t_{n_j}}\right]-1} \|T((k+1)t_{n_j})x_{n_j} - T(kt_{n_j})x_{n_j}\| \\ &\quad + \left\|T\left(\left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)x_{n_j} - T\left(\left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)q\right\| \\ &\quad + \left\|T\left(\left[\frac{t}{t_{n_j}}\right]t_{n_j}\right)q - T(t)q\right\| \\ &\leq \frac{t}{t_{n_j}}\|T(t_{n_j})x_{n_j} - x_{n_j}\| + \|x_{n_j} - q\| \\ &\quad + \left\|T\left(t - \left[\frac{t}{t_{n_j}}\right]s_j\right)q - q\right\| \\ &\leq t\frac{\alpha_{n_j}+\gamma_{n_j}}{t_{n_j}}\left(\|f(x_{n_j} - T(t_{n_j})x_{n_j}\| + \|u_{n_j} - T(t_{n_j})x_{n_j}\|\right) \\ &\quad + \|x_{n_j} - x\| + \max_{0\leq s\leq t_{n_j}}\{\|T(s)q - q\|\}\end{aligned}$$

for all $j \in \mathbb{N}$, we have

$$\limsup_{n\to\infty}\|x_{n_j} - T(t)q\| \leq \limsup_{n\to\infty}\|x_{n_j} - q\|.$$

Therefore $T(t)q = q$ i.e., $q \in F$. Since the space E satisfies Opial's condition, we see that $\omega_w(x_n)$ is a singleton. This completes the proof. □

Theorem 2.2. *Let E be a reflexive Banach space which satisfies Opial's condition, suppose K is a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a Lipschitzian pseudocontractive semigroup on K such that $F := \bigcap_{t\geq 0} Fix(T(t)) \neq \emptyset$, and $f : K \to K$ be a fixed contractive mapping with contractive coefficient $\alpha \in (0,1)$. Suppose that for any bounded subset $C \subset K$,*

$$\lim_{s\to 0}\sup_{x\in C}\|T(s)x - x\| = 0. \tag{5}$$

Define a sequence $\{x_n\}$ in K by

$$x_n = \alpha_n f(x_n) + \beta_n T(t_n)x_n + \gamma_n u_n. \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$*,* $t_n > 0$ *and* $\{u_n\}$ *is a bounded sequence in* K*. Assume that the following conditions are satisfied:*
a) $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \geq 1;$
b) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = 0;$
c) $\liminf_{n\to\infty} t_n = 0, \ \limsup_{n\to\infty} t_n > 0, \ \lim_{n\to\infty}(t_{n+1} - t_n) = 0.$
Then $\{x_n\}$ *converges weakly to a common fixed point of the semigroup* $\{T(t) : t \geq 0\}$*.*

Proof. First, we show that $\{x_n\}$ is well defined. For each $n \geq 1$, define a mapping $S_n : K \to K$ by

$$S_n x = \alpha_n f(x) + \beta_n T(t_n)x + \gamma_n u_n, \quad \forall x \in K.$$

We see that S_n is a continuous strong pesudocontraction for each $n \geq 1$. Indeed, for every $x, y \in K$, we have

$$\langle S_n x - S_n y, j(x-y)\rangle = \beta_n \langle T(t_n)x - T(t_n)y, j(y-x)\rangle \leq \beta_n \|x-y\|^2.$$

By [2, Corollary 2] then there exists a unique fixed point x_n for each $n \geq 1$ such that

$$x_n = \alpha_n f(x_n) + \beta_n T(t_n)x_n + \gamma_n u_n.$$

That is, the sequence $\{x_n\}$ is well defined. Fix $p \in F$, we have

$$\begin{aligned}
\|x_n - p\|^2 &= \alpha_n\langle f(x_n) - p, j(x_n - p)\rangle + \beta_n\langle T(t_n)x_n - p, j(x_n-p)\rangle \\
&\quad + \gamma_n\langle u_n - p, j(x_n - p)\rangle \\
&\leq \alpha_n\langle f(x_n) - f(p), j(x_n - p)\rangle + \alpha_n\langle f(p) - p, j(x_n - p)\rangle \\
&\quad + \beta_n\|x_n - p\|^2 + \gamma_n\|u_n - p\|\|x_n - p\| \\
&\leq \alpha\alpha_n\|x_n - p\|^2 + \alpha_n\|f(p) - p\|\|x_n - p\| \\
&\quad + \beta_n\|x_n - p\|^2 + \gamma_n\|u_n - p\|\|x_n - p\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x_n - p\| &\leq \frac{\alpha_n}{(1-\alpha)\alpha_n + \gamma_n}\|f(p) - p\| + \frac{\gamma_n}{(1-\alpha)\alpha_n + \gamma_n}\|u_n - p\| \\
&\leq \frac{1}{1-\alpha}\|f(p) - p\| + \|u_n - p\|.
\end{aligned}$$

This implies the sequence $\{x_n\}$ is bounded, and so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. We have

$$\|x_n - T(t_n)x_n\| \leq \alpha_n\|f(x_n) - T(t_n)x_n\| + \gamma_n\|u_n - T(t_n)x_n\|.$$

Therefore

$$\|x_n - T(t_n)x_n\| \to 0 \text{ as } n \to \infty. \tag{7}$$

We choose a sequence $\{t_{n_j}\}$ of positive real number such that

$$t_{n_j} \to 0, \frac{1}{t_{n_j}}\|x_{n_j} - T(t_{n_j})x_{n_j}\| \to 0. \tag{8}$$

We now show that how such a special subsequence can be constructed. Fixed $\delta > 0$ such that

$$\liminf_{n\to\infty} t_n = 0 < \delta < \limsup_{n\to\infty} t_n.$$

From (7), there exists $m_1 \in \mathbb{N}$ such that $\|T(t_n)x_n - x_n\| < \frac{1}{3^2}$ for all $n \geq m_1$. By Lemma 1.1, $\frac{\delta}{2}$ is a cluster point of $\{t_n\}$. In particular, there exists $n_1 > m_1$ such that $\frac{\delta}{3} < t_{n_1} < \delta$. Next, we choose $m_2 > n_1$ such that $\|T(t_n)x_n - x_n\| < \frac{1}{4^2}$ for all $n \geq m_2$. Again, by Lemma 1.1, $\frac{\delta}{3}$ is a cluster point of $\{t_n\}$ and this implies that there exists $n_2 > m_2$ such that $\frac{\delta}{4} < t_{n_2} < \frac{\delta}{2}$. Continuing in this way, we obtain a subsequence $\{n_j\}$ of n satisfying

$$\|T(t_{n_j})x_{n_j} - x_{n_j}\| < \frac{1}{(j+2)^2}, \quad \frac{\delta}{j+2} < t_{n_j} < \frac{\delta}{j} \text{ for all } j \in \mathbb{N}.$$

Consequently, (8) is satisfied.

Since $\{x_n\}$ is bounded, without loss of generality we assume that a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $q \in K$. Now, we prove that $q = T(t)q$ for a fixed $t > 0$. Indeed,

$$\begin{aligned}
\|x_{n_j} - T(t)x_{n_j}\| \leq & \sum_{k=0}^{\left[\frac{t}{t_{n_j}}\right]-1} \|T((k+1)t_{n_j})u_j - T(kt_{n_j})x_{n_j}\| \\
& + \left\| T\left(\left[\frac{t}{t_{n_j}}\right] t_{n_j}\right) x_{n_j} - T(t)x_{n_j} \right\| \\
& \leq \left[\frac{t}{t_{n_j}}\right] M \|T(t_{n_j})x_{n_j} - x_{n_j}\| \\
& + M \left\| T\left(t - \left[\frac{t}{t_{n_j}}\right] t_{n_j}\right) x_{n_j} - x_{n_j} \right\| \\
& \leq Mt \frac{\|T(t_{n_j})x_{n_j} - x_{n_j}\|}{t_{n_j}} + M \max_{0 \leq s \leq t_{n_j}} \{\|T(s)x_{n_j} - x_{n_j}\|\}
\end{aligned}$$

for all $j \in \mathbb{N}$. From (8) and the continuity of mapping $t \mapsto T(t)x$, $x \in K$, we get

$$\lim_{j \to \infty} \|x_{n_j} - T(t)x_{n_j}\| = 0.$$

By Lemma 1.2, then $T(t)q = q$, therefore $q \in F$. On the other hand, since the space E satisfies Opial's condition, we see that $\omega_w(x_n)$ is a singleton. This completes the proof. □

References

[1] R. Chen, Y. Song and H. Zhou, Convergence theorems for implicit iteration process for a finite family of continuous pseudocontractive mappings, *J. Math. Anal. Appl.* **314** (2006), 701–709.

[2] K. Delmling, Zero of accretive operators, *Manuscripta Math.* **13** (1974), 365–374.

[3] Y. Hao, Convergence theorems of common fixed points for pseudocontractive mappings, *Fixed Point Theory Appl.* **2008** (2008) Art. ID 902985.

[4] T.H. Kim and H.-K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal. Appl.* **64** (2006), 1140–1152.

[5] M.O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, *J. Math. Anal. Appl.* **294** (2004), 73–81.

[6] X. Qin, Y.J. Cho and M. Shang, Convergence analysis of implicit iterative algorithms for asymptotically nonexpansive mappings, *Appl. Math. Comput.* **210** 2009, 542–550.

[7] X. Qin and Y.J. Cho, Implicit iterative algorithms for treating strongly continuous semigroups of Lipschitz pseudocontractions, *Appl. Math. Lett.* **23** (2010), 1252–1255.

[8] S. Saejung, Strong convergence theorem for nonexpansive semigroups without Bochner integrals, *Fixed Point Theorey and Applications* **2008** (2008), Article ID 745010, 7 papers.

[9] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.* **305** (2005), 227–239.
[10] H.K. Xu and M.G. Ori, An implicit iterative process for nonexpansive mappings, *Numer. Funct.Anal. Optimiz.* **22** (2001), 767–773.
[11] S.S. Zhang, Convergence theorem of common fixed points for Lipschitzian pseudo-contraction semi-groups in Banach spaces, *Appl. Math. Mech. -Engl. Ed.* **30** (2009), 145–152.
[12] S.S. Zhang, Weak convergence theorem for Lipschitzian pseudocontraction semigroups in Banach spaces, *Acta Mathematica Sinica, Einglish Series* **26** (2010), 337–344.
[13] H. Zhou, Convergence theorems of common fixed points for a finite family of Lipschitz pseudo-contractions in Banach spaces, *Nonlinear Anal.* **68** (2008), 2977–2983.

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