Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 40(1), 2013, Pages 45–51
ISSN: 1223-6934

Solution of first iterative differential equations

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Abstract. In this paper we shall establish an existence result for a first order differential equation in C_L. The main tool used in our study is the nonexpansive operator technique and Browder-Ghode-Kirk's fixed point theorem.

2010 Mathematics Subject Classification. 45B05, 45D05, 47H10.
Key words and phrases. Differential equation, existence of solutions, nonexpansive mapping, deviating argument.

1. Introduction

Several works deal with first iterative initial value problems, see [1], [3], [4], [6-9], [11-14]. The general form of these equations is

$$y'(t) = f(x, y(y(t))). \tag{1.1}$$

Starting from this equations in [11] we prove an existence result from the following equations

$$y'(x) = f(x, y(x),\, y(\lambda x)) \tag{1.2}$$

with initial condition

$$y(x_0) = y_0,$$

where $x_0, y_0 \in [a, b]$ and $f \in C([a, b] \times [a, b] \times [a, b])$.

Our main aim in this paper is to use the technique of nonexpansive operators introduced in [3] for more general iterative first order differential equations of type

$$y'(x) = f(x, y(x), y(\lambda_1 x), y(\lambda_2 x)) \tag{1.3}$$

and

$$y'(x) = f(x, y(x), y(\lambda_1 y(x)), y(\lambda_2 y(x)) \tag{1.4}$$

respectively.

2. Preliminaries

We introduce the definitions and a fixed point theorem for nonexpansive mappings which will play an important role in this paper, see [2].

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an $\alpha-$contraction if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \;\; \forall x, y \in X.$$

Received November 30, 2010; Revised October 16, 2012.

In the case when $\alpha = 1$, the mapping T is said to be nonexpansive. Let K be a nonempty subset of a real normed linear space E and $T : K \to K$ be a map. In this setting, T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in K.$$

Although the nonexpansive mappings are generalizations of α- contractions, they do not inherit properties of contractive mappings. One of the most important fixed point theorems for nonexpansive mappings, due to Browder, Ghode and Kirk, see e.g. [3], states as follows.

Theorem 2.1. ([3]) *Let K be a nonempty closed convex and bounded subset of an uniformly Banach space E. Then any nonexpansive mapping $T : K \to K$ has at least a fixed point.*

Remark 2.2. The fixed points of T can be approximated by Krasnoselskij sequence, defined as follows.

Let K be a convex subset of a normed linear space E and let $T : K \to K$ be a self-mapping. Given an $x_0 \in K$ and a real numbers $\lambda \in [0, 1]$, the sequence x_n defined by the formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, ...$$

is usually called *Krasnoselskij iteration* or *Krasnoselskij-Mann iteration.*
For $x_0 \in K$ the sequence x_n defined by

$$x_{n+1} = (1 - \lambda_n) \cdot x_n + \lambda_n \cdot Tx_n, n = 0, 1, 2... \tag{2.1}$$

where $(\lambda_n)_n \subset [0, 1]$ is a sequence of real numbers satisfying some appropriate condition, is called *Mann iteration.* Edelstein [7] proved that strict convexity of E is sufficient for the Krasnoselskij iteration to converge to a fixed point of T. The question of whether or not strict convexity can be removed has been answered in the affirmative by Ishikawa [10] by the following result.

Theorem 2.3. ([10]) *Let K be a subset of a Banach E and let $T : K \to K$ be a nonexpansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process x_n given by (2.1) under the following assumptions:*
(a) $x_n \in K$ for all positive integers n;
(b) $0 \leq \lambda_n \leq b < 1$ for all positive integers n;
(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$. If x_n is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.

The following corollaries of Theorem 2.3 will be particularly important for the application part of our paper.

Corollary 2.4. ([5]) *Let K be a convex and compact subset of a Banach space E and let $T : K \to K$ be a nonexpansive mapping. If the Mann iteration process x_n satisfies assumptions (a)-(c) in Theorem 2.3, then x_n converges strongly to a fixed point of T.*

Proof. See Theorem 6.17 in Chidume [5]. □

Corollary 2.5. ([5]) *Let K be a closed bounded convex subset of a real normed space E and $T : K \to K$ be a nonexpansive mapping. If $I - T$ maps closed bounded subsets of E into closed subsets of E and x_n is the Mann iteration, with λ_n satisfying assumptions (a)-(c) in Theorem 2.3, then x_n converges strongly to a fixed point of T in K.*

Proof. See Corollary 6.19 in Chidume [5]. □

3. Main results

Starting from equation (2) we study the following problem:

$$\begin{cases} y'(x) = f(x, y(x), y(\lambda_1 x), y(\lambda_2 x)) \\ y(x_0) = y_0 \end{cases} \tag{3.1}$$

where $x_0, y_0 \in [a,b], \lambda_1, \lambda_2 \in (0,1)$ and $f \in C([a,b]\times[a,b]\times[a,b]\times[a,b])$. This problem extends equation (2). We formulate the first result for the existence of solutions to initial value problem (3.1).
For $x \in [a,b]$ denote

$$C_x = \max\{x - a, b - x\},$$

and

$$(*)\ \mathcal{C}_L = \{y \in C([a,b],[a,b]) : |y(t_1) - y(t_2)| \leq L \cdot |t_1 - t_2|, \forall t_1, t_2 \in [a,b]\},$$

where $L > 0$ is given.

Theorem 3.1. *Assume that the following conditions are satisfied for initial value problem (3.1)*

(i) $f \in C([a,b] \times [a,b] \times [a,b] \times [a,b])$;

(ii) there exists $L_1 > 0$ *such that*

$$|f(s,u_1,v_1,w_1) - f(s,u_2,v_2,w_2)| \leq L_1(|u_1 - v_2| + |v_1 - v_2| + |w_1 - w_2|)$$

for any $s, u_i, v_i, w_i \in [a,b]$, $i = 1,2$;

(iii) if L *is the Lipschitz constant involved in* (*), *then*

$$M = \max\{|f(s,u,v,w)| : (s,u,v,w) \in [a,b]\} \leq L;$$

(iv) one of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$;

b) $x_0 = 0,\ M(b-a) \leq b - y_0, f(s,u,v,w) \geq 0,\ \forall s,u,v,w \in [a,b]$;

c) $x_0 = b,\ M(b-a) \leq y_0 - a, f(s,u,v,w) \geq 0,\ \forall s,u,v,w \in [a,b]$;

(v) $3L_1 \cdot C_{x_0} \leq 1$.

Then the problem (3.1) has at least one solution in $\mathcal{C}_L$, *which can be approximated by the Krasnoselskij iteration*

$$y_{n+1}(t) = (1-\mu)y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(s), y_n(\lambda_1 s), y_n(\lambda_2 s))ds,\ t \in [a,b],\ n \geq 1,$$

where $\mu \in (0,1)$ *and* $y_1 \in \mathcal{C}_L$ *is arbitrary.*

Proof. As a consequence of Arzela-Ascoli or from [4, Lemma 1], $\mathcal{C}_L$ is a nonempty convex and compact subset of the Banach space $(C[a,b], \|\cdot\|)$ where $\|x\| = \sup_{t\in[a,b]} |x(t)|$.
Consider the integral operator $F : \mathcal{C}_L \to C[a,b]$ defined by

$$(Fy)(t) = y_0 + \int_{x_0}^{t} f(s, y(s), y(\lambda_1 s), y(\lambda_2 s))ds, \quad t \in [a,b].$$

Any fixed point of the equation $y = Fy$ is a solution of initial value problem (3.1).

We prove that $\mathcal{C}_L$ is an invariant set with respect to F, i.e., we have $F(\mathcal{C}_L) \subset \mathcal{C}_L$.

If condition (a) holds, then for any $y \in \mathcal{C}_L$ and $t \in [a,b]$ we have

$$|(Fy)(t)| \leq |y_0| + \left|\int_{x_0}^{t} f(s, y(s), y(\lambda_1 s), y(\lambda_2 s))ds\right| \leq |y_0| + M \cdot |x_0 - t| \leq b,$$

$$\begin{aligned}|(Fy)(t)| &\geq |y_0| - \left|\int_{x_0}^{t} f(s, y(s), y(\lambda_1 s), y(\lambda_2 s))ds\right| \geq |y_0| - M \cdot |x_0 - t| \\ &\geq |y_0| - M \cdot C_{x_0} \geq |y_0| - C_{y_0} \geq a.\end{aligned}$$

So, $Fy \in [a, b]$ for any $y \in \mathcal{C}_L$.

Now, for any $t_1, t_2 \in [a, b]$ we have

$$\begin{aligned}|(Fy)(t_1) - (Fy)(t_2)| &\leq \left|\int_{t_1}^{t_2} f(s, y(s), y(\lambda_1 s), y(\lambda_2 s))ds\right| \\ &\leq M \cdot |t_1 - t_2| \leq L \cdot |t_1 - t_2|.\end{aligned}$$

Thus, $F_y \in \mathcal{C}_L$ for any $y \in \mathcal{C}_L$. In a similar way we treat the cases (b) and (c). Therefore $F : \mathcal{C}_L \to \mathcal{C}_L$ (i.e. F is a self-mapping of $\mathcal{C}_L$).

We prove that F is nonexpansive operator. Let $y, z \in \mathcal{C}_L$ and $t \in [a, b]$. Then

$$\begin{aligned}|(Fy)(t) - (Fz)(t)| &\leq \left|\int_{x_0}^{t} f(s, y(s), y(\lambda_1 s), y(\lambda_2 s)) - f(s, z(s), z(\lambda_1 s), z(\lambda_2 s))\right| ds \\ &\leq \int_{x_0}^{t} L_1(|y(s) - z(s)| + |y(\lambda_1 s) - z(\lambda_1 s)| + |y(\lambda_2 s) - z(\lambda_2 s)|)ds \\ &\leq 3 \cdot L_1 \cdot C_{x_0} \cdot \|y - z\|.\end{aligned}$$

Now, by taking the norm, we get

$$\|Fy - Fz\| \leq 3L_1 \cdot C_{x_0} \cdot \|y - z\|,$$

which in view of condition (v), proves that F is nonexpansive operator hence continuous.

It now remains to apply the Browder-Ghode-Kirk's fixed point theorem and obtain the first part of the conclusion and Corollary 2.4 or 2.5 to get the second one. □

Now we are applying the same technique for an extra-iterative differential equation which extends problem (3.1), namely

$$y'(x) = f(x, y(x), y(\lambda_1 y(x)), y(\lambda_2 y(x))) \tag{3.2}$$

with initial condition

$$y(x_0) = y_0, \tag{3.3}$$

where $x_0, y_0 \in [a, b], \lambda_1, \lambda_2 \in (0, 1)$ and $f \in C([a, b] \times [a, b] \times [a, b] \times [a, b])$ are given.

We formulate the second result on the existence of solutions to initial value problem (3.2)+(3.3) in $\mathcal{C}_L$.

Theorem 3.2. *Assume that*

(i) $f \in C([a, b] \times [a, b] \times [a, b] \times [a, b])$;

(ii) there exists $L_1 > 0$ *such that*

$$(**)\ \ |f(s, u_1, v_1, w_1) - f(s, u_2, v_2, w_2)| \leq L(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|),$$

for any $s, u_i, v_i, w_i \in [a, b]$, $i = 1, 2$;

(iii) if L *is the Lipschitz constant involved in* (*), *then*

$$M = \max\{|f(s, u, v, w)| : (s, u, v, w) \in [a, b]\} \leq L$$

(iv) one of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$;

b) $x_0 = a,\ M(b - a) \leq b - y_0, f(s, u, v, w) \geq 0,\ \forall s, u, v, w \in [a, b]$;

c) $x_0 = b,\ M(b - a) \leq y_0 - a, f(s, u, v, w) \geq 0,\ \forall s, u, vw \in [a, b]$;

(v) $L_1 \cdot [1 + L(\lambda_1 + \lambda_2)] \cdot C_{x_0} \leq 1$.

Then the initial value problem (3.2)+(3.3) has at least one solution in $\mathcal{C}_L$, which can be approximated by the Krasnoselskij iteration

$$y_{n+1}(t) = (1-\mu)y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(s), y_n(\lambda_1 y_n(s)), y_n(\lambda_2 y_n(s)))ds,$$

$t \in [a,b],\ n \geq 1$, *where* $\mu \in [a,b]$ *and* $y_1 \in \mathcal{C}_L$ *are arbitrary.*

Proof. We define the integral operator $F : \mathcal{C}_L \to C[a,b]$, by

$$(Fy)(t) = y_0 + \int_{x_0}^{t} f(s, y(s), y(\lambda_1 y(s)), y(\lambda_2 y(s)))ds, \quad t \in [a,b].$$

In the same way as Theorem 3.1 we prove that $\mathcal{C}_L$ is an invariant set with respect to F, which means $F(\mathcal{C}_L) \subset \mathcal{C}_L$. We deduce

$$\begin{aligned}
|(Fy)(t)| &\leq |y_0| + \left| \int_{x_0}^{t} f(s, y(s), y(\lambda_1 y(s)), y(\lambda_2 y(s)))ds \right| \leq |y_0| + M \cdot |t - x_0| \leq b, \\
|(Fy)(t)| &\geq |y_0| - \left| \int_{x_0}^{t} f(s, y(s), y(\lambda_1 y(s)), y(\lambda_2 y(s)))ds \right| \geq |y_0| - M \cdot |t - x_0| \\
&\geq |y_0| - M \cdot C_{x_0} \geq y_0 - C_{y_0} \geq a.
\end{aligned}$$

Thus, $Fy \in [a,b]$ for any $y \in \mathcal{C}_L$. For any $t_1, t_2 \in [a,b]$ we have:

$$\begin{aligned}
|(Fy)(t_1) - (Fy)(t_2)| &\leq \left| \int_{t_1}^{t_2} f(s, y(s), y(\lambda_1 y(s)), y(\lambda_2 y(s)))ds \right| \\
&\leq M \cdot |t_1 - t_2| \leq L \cdot |t_1 - t_2| .
\end{aligned}$$

So, $Fy \in \mathcal{C}_L$ for any $y \in \mathcal{C}_L$. In a similar way we treat the cases (b) and (c).

We consider $y, z \in \mathcal{C}_L$ and $t \in [a,b]$ in order to prove that F is nonexpansive operator.

$$\begin{aligned}
&|(Fy)(t) - (Fz)(t)| \\
&\leq \int_{x_0}^{t} |f(s, y(s), y(\lambda_1 y(s), y(\lambda_2 y(s))) - f(s, z(s), z(\lambda_1 z(s)), z(\lambda_2 z(s)))| \, ds \\
&\leq \int_{x_0}^{t} L_1(|y(s) - z(s)| + |y(\lambda_1 y(s)) - z(\lambda_1 z(s))| + |y(\lambda_2 y(s)) - z(\lambda_2 z(s))|)ds \\
&\leq L_1 \int_{x_0}^{t} (|y(s) - z(s)| + |\lambda_1| \cdot L \cdot |y(s) - z(s)| + |\lambda_2| \cdot L \cdot |y(s) - z(s)|)ds \\
&\leq L_1 \cdot [1 + L(\lambda_1 + \lambda_2)] \cdot |t - x_0| \cdot \|y - z\| \leq [1 + L(\lambda_1 + \lambda_2)] \cdot C_{x_0} \cdot \|y - z\| .
\end{aligned}$$

Now, by taking the maximum in the last inequality, we get

$$\|Fy - Fz\| \leq L_1 \cdot [1 + L(\lambda_1 + \lambda_2)] \cdot C_{x_0} \cdot \|y - z\| ,$$

which in view of condition (v), proves that F is nonexpansive operator hence continuous.

Applying the Browder-Ghode-Kirk or Schauder's fixed point theorems we obtain the first part of conclusion and Corollary 2.4 or 2.5 to get the second part of conclusion. □

4. An example

We conclude the paper by presenting an example to illustrate the generality of our results.

Example 4.1. Consider the following initial value problem associated to an extra-iterative differential equation:

$$\begin{cases} y'(x) = -3 + y(x) + y(\frac{1}{2}y(x)) + y(\frac{1}{2}y(x)) \\ y(\frac{1}{2}) = 1 \end{cases} \tag{4.1}$$

where $x \in [0,1], y \in C^1([0,1],[0,1])$, $\lambda_1 = \lambda_2 = \frac{1}{2}$. We are interested to study the solutions $y \in C^1([0,1],[0,1])$ belonging to the set

$$\mathcal{C}_1 = \{y \in C([0,1],[0,1]) : |y(t_1) - y(t_2)| \leq |t_1 - t_2|\},$$

for any $t_1, t_2 \in [0,1]$ which, in view of our notations, means that $L = 1$. We have

$$a = 0, b = 1, x_0 = \frac{1}{2} \text{ hence } C_{x_0} = \max\{x_0 - a, b - x_0\} = \frac{1}{2}.$$

The function $f(x,u,v,w) = -3+u+v+w$ is Lipschitzian in the sense of (**) with respect to u, v and w, with Lipschitz constant $L_1 = 1$. This shows that $L_1[1 + L(\lambda_1 + \lambda_2)] \cdot C_{x_0} = 1$, so the condition (v) in Theorem 3.2 is satisfied. Note also that $y(x) = 1,\ x \in [0,1]$ is a solution to initial value problem (4.1). By Theorem 3.2 initial value problem (4.1) has at least a solution in $\mathcal{C}_1$ that can be approximated by Krasnoselskji iteration

$$y_{n+1}(t) = (1-\mu)y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} \left[-3 + y_n(s) + 2 \cdot y_n(\frac{1}{2}y_n(s))\right] ds,\ t \in [0,1], n \geq 1,$$

where $\mu \in (0,1)$ and $y_1 \in \mathcal{C}_1$ are arbitrary.

Particular case

If $f = f(t,u,v)$, we find the differential equation studied in [11].

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