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The Voronovskaja type theorem for an extension of Kantorovich operators

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Abstract. Recently, D. Bărbosu, O. T. Pop and D. Miclăuş defined a class of linear and positive operators depending on a certain function φ. These operators generalize the well known Szász-Mirakjan-Kantorovich operators. In the present paper we establish a Voronovskaja theorem, the uniform convergence and the order of approximation using the modulus of continuity for the same operators.

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this section we recall some results from [13], which we shall use in the present paper. Let I, J be real intervals with the property $I \cap J \neq \emptyset$. For any $n, k \in \mathbb{N}$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \to \mathbb{R}$, with the property that $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{n,k} : E(I) \to \mathbb{R}$.

For any $n \in \mathbb{N}$ define the operator $L_n : E(I) \to F(J)$, by

$$(L_n f)(x) = \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(f), \tag{1}$$

where $E(I)$ is a linear space of real-valued functions defined on I, for which the operators (1) are convergent and $F(J)$ is a subset of the set of real-valued functions defined on J.

Remark 1.1. *The operators $(L_n)_{n\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.*

For $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define $T^*_{n,i}$ by

$$(T^*_{n,i} L_n)(x) = n^i (L_n \psi^i_x)(x) = n^i \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(\psi^i_x),\ x \in I \cap J. \tag{2}$$

In what follows $s \in \mathbb{N}_0$ is even and we assume that the next two conditions:

- there exists the smallest $\alpha_s,\ \alpha_{s+2} \in [0, +\infty[$, so that

$$\lim_{n\to\infty} \frac{(T^*_{n,j} L_n)(x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{3}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{4}$$

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• $I \cap J$ is an interval
hold.

Theorem 1.1. *([13]) Let $f \in E(I)$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x, then*

$$\lim_{n\to\infty} n^{s-\alpha_s}\left((L_n f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{n^i i!}(T^*_{n,i}L_n)(x)\right) = 0. \tag{5}$$

Assume that f is s times differentiable on I and there exists an interval $K \subset I \cap J$, such that, there exists $n(s) \in \mathbb{N}$ and the constants $k_j \in \mathbb{R}$ depending on K, so that for $n \geq n(s)$ and $x \in K$, the following

$$\frac{(T^*_{n,j}L_n)(x)}{n^{\alpha_j}} \leq k_j, \tag{6}$$

hold, for $j \in \{s, s+2\}$.

Then, the convergence expressed by (5) is uniform on K and

$$n^{s-\alpha_s}\left|(L_n f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{n^i i!}(T^*_{n,i}L_n)(x)\right| \tag{7}$$
$$\leq \frac{1}{s!}(k_s + k_{s+2})\omega_1\left(f^{(s)}; \frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in K$, $n \geq n(s)$, where $\omega_1(f;\delta)$ denotes the modulus of continuity [1], of the functions f.

In [12], C. Mortici defined the sequence of operators

$$\varphi S_n : C^2([0,+\infty[) \to C^\infty([0,+\infty[),$$

given by

$$(\varphi S_n f)(x) = \frac{1}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(nx)^k f\left(\frac{k}{n}\right), \tag{8}$$

for any $x \in [0,+\infty[$ and any $n \in \mathbb{N}$, where $\varphi : \mathbb{R} \to]0,+\infty[$ is an analytic function.

These are called the φ-Szász-Mirakjan operators, because in the case when $\varphi(y) = e^y$, they reduced to the classical Mirakjan-Favard-Szász operators [5], [11] and [16].

Remark 1.2. *Similar generalizations of this type are the operators defined and studied by Jakimovski and Leviatan [6] or the operators defined by Baskakov in 1957 (see, e.g., the book [2], subsection 5.3.11, page 344, where they are attributed to Mastroianni).*

Remark 1.3. *The classical Mirakjan-Favard-Szász operators $S_n : C_2([0,+\infty[) \to C([0,+\infty[)$ are defined by*

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

for any $x \in [0,+\infty[$ and any $n \in \mathbb{N}$, where

$$C_2([0,+\infty[) := \left\{ f \in C([0,+\infty[) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

In the following we shall use the classical definition of Mirakjan-Favard-Szász operators, i.e. $f \in C_2([0,+\infty[)$.

2. The φ-Szász-Mirakjan-Kantorovich operators

Let $\varphi : \mathbb{R} \to]0, +\infty[$ be an analytic function.

In [3], D. Bărbosu, O.T. Pop and D. Miclăuş defined the operators

$$\varphi K_n : C_2([0, +\infty[) \to C([0, +\infty[),$$

given by

$$(\varphi K_n f)(x) = \frac{n}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}(nx)^k \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{9}$$

for $x \in [0, +\infty[$ and any $n \in \mathbb{N}$.

The sequence $(\varphi K_n)_{n\in\mathbb{N}}$ is called φ-Szász-Mirakjan-Kantorovich operators, because in the case when $\varphi(y) = e^y$, they reduce to the classical Szász-Mirakjan-Kantorovich operators [4], [7], [15].

The following two results are established in [3].

Lemma 2.1. *The φ-Szász-Mirakjan-Kantorovich operators satisfy*

i) $(\varphi K_n e_0)(x) = 1$,

ii) $(\varphi K_n e_1)(x) = \dfrac{\varphi^{(1)}(nx)}{\varphi(nx)}x + \dfrac{1}{2n}$,

iii) $(\varphi K_n e_2)(x) = \dfrac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \dfrac{2}{n}\dfrac{\varphi^{(1)}(nx)}{\varphi(nx)}x + \dfrac{1}{3n^2}$,

for any $x \in [0, +\infty[$ and any $n \in \mathbb{N}$.

If the function φ verifies

$$\lim_{y\to\infty} \frac{\varphi^{(1)}(y)}{\varphi(y)} = \lim_{y\to\infty} \frac{\varphi^{(2)}(y)}{\varphi(y)} = 1, \tag{10}$$

then the following convergence theorem holds:

Theorem 2.1. *For any function $f \in C_2([0, +\infty[)$ the following*

$$\lim_{n\to\infty} (\varphi K_n f)(x) = f(x) \tag{11}$$

holds, uniformly on any compact interval $[a, b] \subset [0, +\infty[$ and any $x \in [a, b]$.

3. Main results

We recall from [14], that the calculation of test functions by φ-Szász-Mirakjan operators is given by the following identities:

$$\begin{aligned}
(\varphi S_n e_0)(x) &= 1,\\
(\varphi S_n e_1)(x) &= \frac{\varphi^{(1)}(nx)}{\varphi(nx)}x,\\
(\varphi S_n e_2)(x) &= \frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \frac{1}{n}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x, \qquad (12)\\
(\varphi S_n e_3)(x) &= \frac{\varphi^{(3)}(nx)}{\varphi(nx)}x^3 + \frac{3}{n}\frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \frac{1}{n^2}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x,\\
(\varphi S_n e_4)(x) &= \frac{\varphi^{(4)}(nx)}{\varphi(nx)}x^4 + \frac{6}{n}\frac{\varphi^{(3)}(nx)}{\varphi(nx)}x^3 + \frac{7}{n^2}\frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \frac{1}{n^3}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x.
\end{aligned}$$

Lemma 3.1. *The φ-Szász-Mirakjan-Kantorovich operators satisfy*

$$(\varphi K_n e_3)(x) = \frac{\varphi^{(3)}(nx)}{\varphi(nx)}x^3 + \frac{9}{2n}\frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \frac{7}{2n^2}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x + \frac{1}{4n^3}, \quad (13)$$

$$(\varphi K_n e_4)(x) = \frac{\varphi^{(4)}(nx)}{\varphi(nx)}x^4 + \frac{8}{n}\frac{\varphi^{(3)}(nx)}{\varphi(nx)}x^3 + \frac{15}{n^2}\frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \frac{6}{n^3}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x + \frac{1}{5n^4}, \quad (14)$$

for any $x \in [0, +\infty[$ and any $n \in \mathbb{N}$.

Proof. Because the function φ is analytic, it follows

$$\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!}y^k = \varphi(y)$$

and next, by differentiation we get

$$\varphi^{(1)}(y) = \sum_{k=1}^{\infty} \frac{\varphi^{(k)}(0)}{(k-1)!}, \quad \varphi^{(2)}(y) = \sum_{k=2}^{\infty} \frac{\varphi^{(k)}(0)}{(k-2)!},$$

$$\varphi^{(3)}(y) = \sum_{k=3}^{\infty} \frac{\varphi^{(k)}(0)}{(k-3)!}y^{k-3}, \quad \varphi^{(4)}(y) = \sum_{k=4}^{\infty} \frac{\varphi^{(k)}(0)}{(k-4)!}y^{k-4}.$$

For the test functions $e_j(x) = x^j$, $j \in \{0, 1, 2, 3, 4\}$, the following identities

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} e_0(t)dt = \frac{1}{n}, \quad \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_1(t)dt = \frac{2k+1}{2n^2}, \quad \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_2(t)dt = \frac{3k^2+3k+1}{3n^3},$$

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} e_3(t)dt = \frac{4k^3+6k^2+4k+1}{4n^4}, \quad \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_4(t)dt = \frac{5k^4+10k^3+10k^2+5k+1}{5n^5}$$

hold. Taking the identities (12) into account, it follows

$$\begin{aligned}
(\varphi K_n e_3)(x) &= \frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_3(t)dt \\
&= \frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k \frac{4k^3+6k^2+4k+1}{4n^4} \\
&= \frac{1}{n^3}\left(\frac{n^3}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{k^3}{n^3} + \frac{3n^2}{2\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{k^2}{n^2}\right. \\
&\quad \left. + \frac{n}{\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\frac{k}{n} + \frac{1}{4\varphi(nx)}\sum_{k=0}^{\infty}\frac{\varphi^{(k)}(0)}{k!}(nx)^k\right) \\
&= \frac{1}{n^3}\left(n^3(\varphi S_n e_3)(x) + \frac{3n^2(\varphi S_n e_2)(x)}{2} + n(\varphi S_n e_1)(x) + \frac{(\varphi S_n e_0)(x)}{4}\right) \\
&= \frac{\varphi^{(3)}(nx)}{\varphi(nx)}x^3 + \frac{9}{2n}\frac{\varphi^{(2)}(nx)}{\varphi(nx)}x^2 + \frac{7}{2n^2}\frac{\varphi^{(1)}(nx)}{\varphi(nx)}x + \frac{1}{4n^3},
\end{aligned}$$

$$(\varphi K_n e_4)(x) = \frac{n}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} e_4(t)dt$$

$$= \frac{n}{\varphi(nx)} \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} (nx)^k \frac{5k^4 + 10k^3 + 10k^2 + 5k + 1}{5n^5} =$$

$$\frac{1}{n^4} \left(n^4 (\varphi S_n e_4)(x) + 2n^3 (\varphi S_n e_3)(x) + 2n^2 (\varphi S_n e_2)(x) + n(\varphi S_n e_1)(x) + \frac{(\varphi S_n e_0)(x)}{5} \right)$$

$$= \frac{\varphi^{(4)}(nx)}{\varphi(nx)} x^4 + \frac{8}{n} \frac{\varphi^{(3)}(nx)}{\varphi(nx)} x^3 + \frac{15}{n^2} \frac{\varphi^{(2)}(nx)}{\varphi(nx)} x^2 + \frac{6}{n^3} \frac{\varphi^{(1)}(nx)}{\varphi(nx)} x + \frac{1}{5n^4}.$$

□

Lemma 3.2. *For any $x \in [0, +\infty[$ and any $n \in \mathbb{N}$, the following*

$$(T^*_{n,0} \varphi K_n)(x) = 1, \tag{15}$$

$$(T^*_{n,1} \varphi K_n)(x) = n \left(\left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x + \frac{1}{2n} \right), \tag{16}$$

$$(T^*_{n,2} \varphi K_n)(x) = n^2 \left(\left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) x^2 + \frac{1}{n} \left(2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x + \frac{1}{3n^2} \right), \tag{17}$$

$$(T^*_{n,4} \varphi K_n)(x) = n^4 \left(\left(\frac{\varphi^{(4)}(nx)}{\varphi(nx)} - 4 \frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 6 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 4 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) x^4 \right.$$
$$+ \frac{2}{n} \left(\frac{\varphi^{(3)}(nx)}{\varphi(nx)} - 9 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 6 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x^3$$
$$\left. + \frac{1}{n^2} \left(15 \frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 14 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 2 \right) x^2 + \frac{1}{n^3} \left(6 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x + \frac{1}{5n^4} \right) \tag{18}$$

hold.

Proof. Taking relation (2), Lemma 2.1 and Lemma 3.1 into account, we get

$$(T^*_{n,0} \varphi K_n)(x) = (\varphi K_n e_0)(x) = 1,$$

$$(T^*_{n,1} \varphi K_n)(x) = n(\varphi K_n \psi_x)(x) = n \left((\varphi K_n e_1)(x) - x(\varphi K_n e_0)(x) \right)$$
$$= n \left(\left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x + \frac{1}{2n} \right),$$

$$(T^*_{n,2} \varphi K_n)(x) = n^2 (\varphi K_n \psi_x^2)(x)$$
$$= n^2 \left((\varphi K_n e_2)(x) - 2x(\varphi K_n e_1)(x) + x^2 (\varphi K_n e_0)(x) \right)$$
$$= n^2 \left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} x^2 + \frac{2}{n} \frac{\varphi^{(1)}(nx)}{\varphi(nx)} x + \frac{1}{3n^2} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} x^2 - \frac{x}{n} + x^2 \right)$$
$$= n^2 \left(\left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1 \right) x^2 + \frac{1}{n} \left(2 \frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x + \frac{1}{3n^2} \right),$$

$$(T^*_{n,4}\varphi K_n)(x) = n^4(\varphi K_n \psi_x^4)(x) = n^4\left((\varphi K_n e_4)(x) - 4x(\varphi K_n e_3)(x)\right.$$
$$\left. +6x^2(\varphi K_n e_2)(x) - 4x^3(\varphi K_n e_1)(x) + x^4(\varphi K_n e_0)(x)\right)$$
$$= n^4\left(\left(\frac{\varphi^{(4)}(nx)}{\varphi(nx)} - 4\frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 6\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 4\frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1\right)x^4\right.$$
$$+\frac{2}{n}\left(4\frac{\varphi^{(3)}(nx)}{\varphi(nx)} - 9\frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 6\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1\right)x^3$$
$$\left.+\frac{1}{n^2}\left(15\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 14\frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 2\right)x^2 + \frac{1}{n^3}\left(6\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1\right)x + \frac{1}{5n^4}\right).$$

□

In the following, we assume that γ, δ exist, so that $0 < \gamma \le 1$, $\delta \le 2$, $\gamma < \delta$ and the analytic function φ satisfies the conditions (10) and verifies

$$\lim_{n\to\infty} n^{\gamma}\left(\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 2\frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1\right) = \beta_2(x), \tag{19}$$

$$\lim_{n\to\infty} n^{\delta-1}\left(4\frac{\varphi^{(3)}(nx)}{\varphi(nx)} - 9\frac{\varphi^{(2)}(nx)}{\varphi(nx)} + 6\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1\right) = \beta_3(x) \tag{20}$$

and

$$\lim_{n\to\infty} n^{\delta}\left(\frac{\varphi^{(4)}(nx)}{\varphi(nx)} - 4\frac{\varphi^{(3)}(nx)}{\varphi(nx)} + 6\frac{\varphi^{(2)}(nx)}{\varphi(nx)} - 4\frac{\varphi^{(1)}(nx)}{\varphi(nx)} + 1\right) = \beta_4(x), \tag{21}$$

for any $x \in [0, +\infty[$, where $\beta_2, \beta_3, \beta_4$ are functions, $\beta_2, \beta_3, \beta_4 : [0, +\infty[\to \mathbb{R}$.

Lemma 3.3. *For any $x \in [0, +\infty[$, the following identities*

$$\lim_{n\to\infty} (T^*_{n,0}\varphi K_n)(x) = 1, \tag{22}$$

$$\lim_{n\to\infty} \frac{(T^*_{n,2}\varphi K_n)(x)}{n^{2-\gamma}} = \beta_2(x)\cdot x^2 + \epsilon(\gamma)\cdot x, \tag{23}$$

$$\lim_{n\to\infty} \frac{(T^*_{n,4}\varphi K_n)(x)}{n^{4-\delta}} = \beta_4(x)\cdot x^4 + 2\beta_3(x)\cdot x^3 + 3\eta(\delta)\cdot x^2 \tag{24}$$

hold and there exists $n_0 \in \mathbb{N}$, so that

$$(T^*_{n,0}\varphi K_n)(x) = 1 = k_0, \tag{25}$$

$$\frac{(T^*_{n,2}\varphi K_n)(x)}{n^{2-\gamma}} \le m_2(K)\cdot b^2 + b + 1 = k_2, \tag{26}$$

$$\frac{(T^*_{n,4}\varphi K_n)(x)}{n^{4-\delta}} \le m_4(K)\cdot b^4 + 2m_3(K)\cdot b^3 + 3b^2 + 1 = k_4, \tag{27}$$

for any $x \in K = [0, b]$, $b > 0$, $n \in \mathbb{N}$, $n \ge n_0$ and $m_2(K) = \sup\limits_{x\in K} |\beta_2(x)|$,

$$m_3(K) = \sup_{x\in K}|\beta_3(x)|,\ m_4(K) = \sup_{x\in K}|\beta_4(x)|,\ \textit{where } \epsilon(\gamma) = \begin{cases}1, & \gamma = 1\\ 0, & 0 < \gamma < 1\end{cases}$$

$$\textit{and } \eta(\delta) = \begin{cases}1, & \delta = 2\\ 0, & \delta < 2.\end{cases}$$

Proof. The identities (22)-(24) follow from Lemma 3.2, while (25)-(27) yield from (22)-(24) by taking the definition of the limit into account. □

Now we assume that $I = J = [0, +\infty[$, $E(I) = C_2([0, +\infty[)$, $F(J) = C([0, +\infty[)$, then the function $\varphi_{n,k} : [0, +\infty[\to \mathbb{R}$ be defined by $\varphi_{n,k}(x) = \dfrac{n}{\varphi(nx)} \dfrac{\varphi^{(k)}(0)}{k!}(nx)^k$, for any $x \in [0, +\infty[$, any $n, k \in \mathbb{N}_0$, $n \neq 0$ and the functionals $A_{n,k} : C_2([0, +\infty[) \to \mathbb{R}$ be defined by $A_{n,k}(f) = \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt$ for any $n, k \in \mathbb{N}_0$, $n \neq 0$. In this case we get the φ-Szász-Mirakjan-Kantorovich operators.

Theorem 3.1. *Let $f \in C_2([0, +\infty[)$ be a function. If $x \in [0, +\infty[$ and f is s times differentiable in a neighborhood of x, then*

$$\lim_{n\to\infty} (\varphi K_n f)(x) = f(x), \tag{28}$$

for $s = 0$;

$$\lim_{n\to\infty} n^{\gamma} \left((\varphi K_n f)(x) - f(x) - \left(\left(\frac{\varphi^{(1)}(nx)}{\varphi(nx)} - 1 \right) x + \frac{1}{2n} \right) f^{(1)}(x) \right) \tag{29}$$
$$= \frac{1}{2} \left(\beta_2(x) \cdot x^2 + \epsilon(\gamma) \cdot x \right) f^{(2)}(x),$$

for $s = 2$.

If f is s times differentiable on $[0, +\infty[$, then the convergence from (28) *and* (29) *is uniform on any compact interval $K = [0, b] \subset [0, +\infty[$. Moreover, we get*

$$|(\varphi K_n f)(x) - f(x)| \leq (1 + k_2)\omega_1 \left(f; \frac{1}{\sqrt{n^{\gamma}}} \right), \tag{30}$$

for any $x \in K$ and $n \in \mathbb{N}$, $n \geq n_0$.

Proof. One applies Theorem 1.1, with $\alpha_0 = 0$, $\alpha_2 = 2 - \gamma$ and $\alpha_4 = 4 - \delta$, Lemma 3.2 and Lemma 3.3. □

Remark 3.1. *In the case when $\varphi(y) = e^y$, for any $y \in [0, +\infty[$, we get the well-known results of the classical Szász-Mirakjan-Kantorovich operators. For this case $\gamma = 1$, $\delta = 2$, $\beta_2(x) = \beta_3(x) = \beta_4(x) = 0$, for any $x \in [0, +\infty[$ and $k_0 = 1$, $k_2 = b+1$, $k_4 = 3b^2 + 1$, see* [10].

We give an example for φ-Szász-Mirakjan-Kantorovich type operators.

Application 3.1. If $\varphi(x) = (x + 1)e^x$, then

$$\varphi^{(k)}(x) = (x + k + 1)e^x,$$

for any $k \in \mathbb{N}_0$ and any $x \in [0, +\infty[$. In this case, we get

$$(\varphi K_n f)(x) = \frac{n}{(nx + 1)e^{nx}} \sum_{k=0}^{\infty} \frac{k + 1}{k!} (nx)^k \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{31}$$

for any $f \in C_2([0, +\infty[)$, any $x \in [0, +\infty[$ and any $n \in \mathbb{N}$.

Remark 3.2. *For these φ-Szász-Mirakjan-Kantorovich type operators, the conclusions of the Theorem 3.1 hold.*

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