

Controllability of a nonlinear hybrid system

NICOLAE CÎNDEA, SORIN MICU, IONEL ROVENȚA, AND MARIUS TUCSNAK

ABSTRACT. In this paper we study a controllability problem for a simplified 1-d nonlinear system which models the self-propelled motion of a rigid body in a fluid located on the real axis. The control variable is the difference of the velocities of the fluid and the solid and depends only on time. The main result of the paper asserts that any final position and velocity of the rigid body can be reached by a suitable input function.

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1. Introduction

We consider the following nonlinear system which models the interaction of a one-dimensional fluid and a solid represented by a point mass which floats in the fluid

$$\begin{cases} u'(t, x) - u_{xx}(t, x) + u(t, x)u_x(t, x) = 0 \\ x < h(t), x > h(t), t \in (0, T) \\ h'(t) = u(t, h(t)), t \in (0, T) \\ mh''(t) = [u_x](t, h(t)) \quad t \in (0, T) \\ u(0, x) = u^0(x) \quad x \in \mathbb{R} \\ h(0) = h^0, \quad h'(0) = h^1. \end{cases} \quad (1)$$

In (1), $u = u(t, x)$ denotes the velocity of the fluid located on the whole real axis whereas $h = h(t)$ indicates the position of the point mass. We assume that the velocity u of the fluid is governed by the viscous Burgers equation at both sides of the mass, $x < h(t)$ and $x > h(t)$. The velocities of the fluid and the solid are supposed to be equal, $h'(t) = u(t, h(t))$. Moreover, the mass is accelerated by the difference of pressure on its sides, given by $[u_x](t, h(t))$. Here and in the sequel, we denote by $[f](x)$ the jump of the function f at the point x . The mass of the solid is m and for simplicity we have supposed that the density of the fluid is equal to one.

A lot of works have addressed the problem of fluid-structure interaction in the last years. Most of them concern the 2-D incompressible Navier-Stokes equation coupled with the motion of a finite number of rigid masses (see, for instance, [4, 5, 6, 7, 11, 12]). (3) is a simplified version of those models for two reasons: the solid structure is a point mass with a scalar motion and the fluid is one-dimensional. This 1-D model has been introduced in [14] where the asymptotic behavior of the solutions for large time is studied. In [15] multiple masses are considered and their lack of collision is proved. Latter on, in [8] a finite interval version of the model has been introduced in the context of a boundary controllability problem.

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The change of variable $y(t, x) = u(t, x + h(t))$ allows us to rewrite the problem as follows

$$\begin{cases} y'(t, x) - y_{xx}(t, x) - h'(t)y_x(t, x) + y(t, x)y_x(t, x) = 0 & x < 0 \quad x > 0 \quad t \in (0, T) \\ y(t, 0) = h'(t) & t \in (0, T) \\ mh''(t) = [y_x](t, 0) & t \in (0, T) \\ y(0, x) = y^0 & x \in \mathbb{R} \\ h(0) = h^0, \quad h'(0) = h^1. \end{cases} \quad (2)$$

The advantage of (2) consists of the fact that the first equation does not hold anymore in variable domains as in (1).

Our aim is to control the position h and the velocity $g := h'$ of the body by introducing a scalar force f which represents the difference between the velocities of the fluid and solid. More precisely, we study the controllability properties of the following system

$$\begin{cases} y'(t, x) - y_{xx}(t, x) - g(t)y_x(t, x) + y(t, x)y_x(t, x) = 0 & x < 0 \quad x > 0 \quad t \in (0, T) \\ y(t, 0) = g(t) + f(t) & t \in (0, T) \\ mg'(t) = [y_x](t, 0) & t \in (0, T) \\ h'(t) = g(t) & t \in (0, T) \\ y(0, x) = y^0 & x \in \mathbb{R} \\ h(0) = h^0, \quad g(0) = g^0. \end{cases} \quad (3)$$

The main result of the paper is the following.

Theorem 1.1. *There exists a constant $Q > 0$ such that, for any $(y^0, f^0, g^0, h^0) \in L^2(\mathbb{R}) \times \mathbb{R}^3$ and $(g^T, h^T) \in \mathbb{R}^2$ with*

$$\max\{\|(y^0, f^0, g^0, h^0)\|_{L^2(\mathbb{R}) \times \mathbb{R}^3}, \|(h^T, g^T)\|_{\mathbb{R}^2}\} \leq Q \quad (4)$$

there exists a control $f \in C[0, T]$ such that $f(0) = f^0$ and the solution (y, g, h) of the nonlinear system (3) satisfies $(g(T), h(T)) = (g^T, h^T)$.

Note that Theorem 1.1 says that we can control the position h and the velocity g of the body but gives no clue about the behavior of the fluid velocity y . In fact, as it is proven in [9], we cannot control the linear heat equation in an infinite domain from the boundary. Therefore, one does not expect to control the fluid state y from (3) either. The proof of Theorem 1.1 is based on a fixed point argument similar to the one used in [3, 10].

The rest of the paper is organized as follows. Section 2 studies the linearized version of (3) and Section 3 presents its controllability properties. Finally, Section 4 is devoted to the proof of Theorem 1.1.

2. Study of the initial value problem

This section presents the elementary properties of existence, uniqueness and regularity of solutions for the linearized version on (3).

2.1. The linearized system. The following system is a nonhomogeneous linearized version of (3)

$$\begin{cases} y'(t, x) - y_{xx}(t, x) = q(t, x) & x < 0 & x > 0 & t > 0 \\ y(t, 0) = p(t) + r(t) & t > 0 \\ mp'(t) = [y_x](t, 0) + w(t) & t > 0 \\ mr'(t) = [y_x](t, 0) & t > 0 \\ y(0, x) = y^0(x) & x \in \mathbb{R} \\ p(0) = p^0, r(0) = r^0 \end{cases} \quad (5)$$

where (y^0, p^0, r^0) is the initial datum and (q, w) are the nonhomogeneous terms.

Let us write (5) in an abstract Cauchy form. To do that, define the functional spaces

$$X = L^2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}$$

$$V = \{Y = (y, p, r) \in H^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \mid y(0) = p + r\}$$

$$D(A) = \left\{ Y = (y, p, r) \in V \mid y|_{(-\infty, 0)} \in H^2(-\infty, 0), y|_{(0, \infty)} \in H^2(0, \infty) \right\}$$

and the unbounded linear operator in X , $(D(A), A)$ given by

$$A \begin{pmatrix} y \\ p \\ r \end{pmatrix} = \begin{pmatrix} -y_{xx} \\ -\frac{1}{m}[y_x](0) \\ -\frac{1}{m}[y_x](0) \end{pmatrix}. \quad (6)$$

In X and V we consider the following inner products

$$(Y_1, Y_2) = \int_{\mathbb{R}} y_1(x)y_2(x)dx + mp_1p_2 + mr_1r_2,$$

$$(Y_1, Y_2)_V = \int_{\mathbb{R}} y_{1,x}(x)y_{2,x}(x)dx + \int_{\mathbb{R}} y_1(x)y_2(x)dx + mp_1p_2 + mr_1r_2, \quad (7)$$

$$Y_i = (y_i, p_i, r_i), \quad i = 1, 2.$$

Moreover, let $F(t) : X \rightarrow X$ be defined by

$$F(t) = \begin{pmatrix} q(t, \cdot) \\ \frac{1}{m}w(t) \\ 0 \end{pmatrix}. \quad (8)$$

With this notation system (5) is equivalently written as follows

$$\begin{cases} Y'(t) + AY(t) = F(t) & t > 0 \\ Y(0) = Y^0, \end{cases} \quad (9)$$

where $Y(t) = (y(t), p(t), r(t))^*$.

Proposition 2.1. *The unbounded operator $(D(A), A)$ is maximal monotone self-adjoint operator in X .*

Proof. (See also [14]). We show that $(D(A), A)$ is maximal monotone in X . Indeed, if $Y \in D(A)$, we have

$$\begin{aligned} (AY, Y) &= - \int_{-\infty}^0 y_{xx}(x)y(x)dx - \int_0^{\infty} y_{xx}(x)y(x)dx - m\frac{1}{m}[y_x](0)p - m\frac{1}{m}[y_x](0)r = \\ &= \int_{\mathbb{R}} |y_x|^2(x)dx + y(0)[y_x](0) - [y_x](0)(p + r) = \int_{\mathbb{R}} |y_x|^2(x)dx. \end{aligned}$$

Hence, if $Y = (y, p, r)^* \in D(A)$, then

$$(AY, Y) = \int_{\mathbb{R}} |y_x|^2(x) dx \quad (10)$$

which shows that $(D(A), A)$ is monotone.

In order to prove that $(D(A), A)$ is maximal, let $\Pi = (\mu, \theta, \zeta) \in X$. System $AY + Y = \Pi$ is equivalent to

$$\begin{cases} y(x) - y_{xx}(x) = \mu(x) & x < 0 & x > 0 \\ y(0) = p + r \\ p - \frac{1}{n}[y_x](0) = \theta \\ r - \frac{1}{m}[y_x](0) = \zeta. \end{cases} \quad (11)$$

It is easy to see that the elliptic equation

$$\begin{cases} z(x) - z_{xx}(x) = \mu(x) - \theta - \zeta & x < 0 & x > 0 \\ z(0) = \frac{2}{m}[z_x](0) \end{cases} \quad (12)$$

has a unique solution $z \in H^1(\mathbb{R})$ and $Y = (z + \theta + \zeta, \frac{1}{m}[z_x](0) + \theta, \frac{1}{m}[z_x](0) + \zeta)$ is a solution in $D(A)$ of $AY + Y = \Pi$.

Hence, $(D(A), A)$ is maximal in X .

In order to show that we have a self-adjoint operator, it remains to prove that $(D(A), A)$ is symmetric. Indeed, for any $Y_i = (y_i, p_i, r_i) \in D(A)$, $i = 1, 2$, we have that

$$\begin{aligned} (AY_1, Y_2) &= - \int_{-\infty}^0 y_{1xx}(x)y_2(x)dx - \int_0^{\infty} y_{1xx}(x)y_2(x)dx - [y_{1x}](0)p_2 - [y_{1x}](0)r_2 = \\ &= \int_{\mathbb{R}} y_{1x}(x)y_{2x}(x)dx = (Y_1, AY_2). \end{aligned}$$

This concludes the proof of the Proposition. \square

Let $D(A^{\frac{1}{2}})$ be the domain of the square root of the operator $(D(A), A)$ defined as the completion of $D(A)$ with respect to the norm $Y \rightarrow \sqrt{(AY, Y) + (Y, Y)}$. We have the following characterization of $D(A^{\frac{1}{2}})$.

Proposition 2.2. *The space $D(A^{\frac{1}{2}})$ coincides with V .*

Proof. First of all, note that, from (10), $(AY, Y) + (Y, Y) = \|Y\|_V^2$. Moreover, $D(A)$ is dense in V . In this case, the density is reduced to the proof of the existence of a family of functions from $H^2(\mathbb{R})$ which have a prescribed value in 0 and arbitrarily small H^1 -norms. For instance, the family of functions $(f_h)_{h>0}$, $f_h(x) = ae^{-\frac{h^3}{h^2-x^2}}$, if $|x| \leq h$ and $f_h(x) = 0$, elsewhere, fulfill these conditions. Hence $D(A^{\frac{1}{2}}) = \overline{D(A)}^{\|\cdot\|_V} = V$. \square

The operator $(D(A), A)$, being maximal monotone, then $(D(A), -A)$ is the infinitesimal generator of a contraction semigroup in X , denoted by $(S(t))_{t \geq 0}$. The following results are consequences of the classical theory of differential equations.

Theorem 2.1. *For each initial data $Y^0 \in X$ and nonhomogeneous term $F \in L^1(0, T; X)$, there exists a unique weak solution $Y \in C([0, T]; X)$ of (9) given by the variation of constants formula*

$$Y(t) = S(t)Y^0 + \int_0^t S(t-s)F(s)ds. \quad (13)$$

Moreover there exists a positive constant $C > 0$ such that

$$\|Y(t)\|_X \leq C (\|Y^0\|_X + \|F\|_{L^1(0,T;X)}) \quad \forall t \in [0, T]. \quad (14)$$

If $Y^0 \in D(A)$ and $F \in W^{1,1}(0, T; X)$, there exists a unique solution of (9) such that

$$Y \in C^1([0, T]; X) \cap C([0, T]; D(A)). \quad (15)$$

Finally, if $Y^0 \in V$ and $F \in L^1(0, T; V)$ there exists a unique solution of (9) such that

$$Y \in C([0, T]; V). \quad (16)$$

Proof. See, for instance, Tucsnak and Weiss [13]. \square

Also, we have the following additional properties for the solutions of (9) (see [2], Lemma 3.3 and Theorem 3.1).

Theorem 2.2. *For each initial data $Y^0 \in X$ and $F \in L^1(0, T; X)$ the unique solution Y of (9) belongs to the space $L^2(0, T; V)$ and there exists a positive constant $C > 0$ such that*

$$\|Y\|_{L^2(0,T;V)} \leq C (\|Y^0\|_X + \|F\|_{L^1(0,T;X)}). \quad (17)$$

For each $Y^0 \in V$ and $F \in L^2(0, T; X)$, the unique solution $Y \in C([0, T], X)$ of (9) belongs to the space $H^1(0, T; X) \cap L^2(0, T; D(A))$ and exists a constant $C > 0$ such that

$$\max\{\|Y\|_{H^1(0,T;X)}, \|Y\|_{L^2(0,T;D(A))}\} \leq C (\|Y^0\|_V + \|F\|_{L^2(0,T;X)}). \quad (18)$$

Proof. For the sake of completeness, let us give the main ideas. The last term in the above inequality is bounded by (14) and (17) follows immediately.

By multiplying (9) by Y and integrating in time we obtain

$$\frac{1}{2} \int_0^T \frac{d}{dt} \|Y(t)\|^2 dt + \int_0^T (AY(t), Y(t)) dt = \int_0^T (F(t), Y(t)) dt$$

from which we deduce that

$$\frac{1}{2} \|Y(T)\|^2 - \frac{1}{2} \|Y(0)\|^2 + \int_0^T (A^{1/2}Y(t), A^{1/2}Y(t)) dt = \int_0^T (F(t), Y(t)) dt$$

and

$$\int_0^T \|A^{1/2}Y(t)\|^2 dt \leq \frac{1}{2} \|Y^0\|^2 + \|Y\|_{L^\infty(0,T;X)} \|F\|_{L^1(0,T;X)}.$$

By multiplying in (9) by Y' and integrating in time we obtain

$$\int_0^T \|Y'(t)\|^2 dt + \int_0^T (AY(t), Y'(t)) dt = \int_0^T (F(t), Y'(t)) dt$$

from which we deduce that

$$\int_0^T \|Y'(t)\|^2 dt \leq \|A^{1/2}Y^0\|^2 + \|F\|_{L^2(0,T;X)}^2.$$

Finally, by multiplying in (9) by AY and integrating in time we obtain

$$\int_0^T (Y'(t), AY(t)) dt + \int_0^T \|AY(t)\|^2 dt = \int_0^T (F(t), AY(t)) dt$$

from which we deduce that

$$\int_0^T \|AY(t)\|^2 dt \leq \|A^{1/2}Y^0\|^2 - \|A^{1/2}Y(T)\|^2 + \|AY\|_{L^2(0,T;X)} \|F\|_{L^2(0,T;X)}$$

and the proof ends. \square

2.2. The extended system. Since our aim is to control both the position and velocity of the mass in (3), we need to study the following system which introduces a new variable o , representing the position of the body,

$$\begin{cases} y'(t, x) - y_{xx}(t, x) = 0 & x < 0 \quad x > 0 \quad t > 0 \\ y(t, 0) = p(t) + r(t) & t > 0 \\ mp'(t) = [y_x](t, 0) + v(t) & t > 0 \\ mr'(t) = [y_x](t, 0) & t > 0 \\ o'(t) = r(t) & t > 0 \\ y(0, x) = 0 & x \in \mathbb{R} \\ p(0) = p^0, r(0) = r^0, o(0) = o^0. \end{cases} \quad (19)$$

Remark 2.1. To solve (19) is equivalent to solve (5) with initial data (y^0, p^0, r^0) and to put $o(t) = o^0 + \int_0^t r(s)ds$. Therefore, from Theorems 2.1 and 2.2, we may deduce immediately existence and regularity results for system (19). However, since in the controllability problem additional properties of (19) will be needed, we have to study it with more details.

Let us introduce the “extended” operators $(D(\tilde{A}), \tilde{A})$ and $\tilde{B} \in \mathcal{L}(\mathbb{R}, \tilde{X})$, corresponding to (19), as follows

$$\begin{aligned} \tilde{X} &= X \times \mathbb{R}, \quad \tilde{V} = V \times \mathbb{R}, \quad D(\tilde{A}) = D(A) \times \mathbb{R} \\ \tilde{A} \begin{pmatrix} y \\ p \\ r \\ o \end{pmatrix} &= \begin{pmatrix} -y_{xx} \\ -\frac{1}{m}[y_x](0) \\ -\frac{1}{m}[y_x](0) \\ -r \end{pmatrix} \quad \text{and} \quad \tilde{B}v = \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (20)$$

With these notation (19) is equivalent to

$$\begin{cases} \tilde{Y}' + \tilde{A}\tilde{Y} = \tilde{B}v, \\ \tilde{Y}(0) = 0. \end{cases} \quad (21)$$

The operator $(D(\tilde{A}), \tilde{A})$ is not maximal monotone in \tilde{X} . However, as we have said before, from Theorem 2.1 we deduce immediately that (21) has a unique solution $\tilde{Y} \in C([0, T]; \tilde{X})$ for any $v \in L^2(0, T)$. Moreover, if $v = 0$, the solution of (21) is given by $\tilde{Y}(t) = \tilde{S}(t)\tilde{Y}^0$, where $(\tilde{S}(t))_{t \geq 0}$ is a strongly continuous semigroup in \tilde{X} . In fact, $(\tilde{S}(t))_{t \geq 0}$ is defined by

$$\tilde{S}(t)(\tilde{Y}^0) = \tilde{S}(t)(Y^0, o^0) = \left(S(t)Y^0, o^0 + \int_0^t \mathbb{P}_3 S(s)Y^0 ds \right)$$

where $(S(t))_{t \geq 0}$ is the semigroup of contractions generated by the operator $(D(A), -A)$ in X . Here and in the sequel \mathbb{P}_i denotes the i -th component projection operator.

Let us now compute the adjoint semigroup $(\tilde{S}^*(t))_{t \geq 0}$.

Proposition 2.3. For any $\tilde{Z}^0 = (\varphi^0, \eta^0, \ell^0, \varsigma^0) \in \tilde{X}$, we have that

$$\tilde{S}^*(t)\tilde{Z}^0 = (\varphi, \eta, \ell, \varsigma^0) \quad (22)$$

where $Z = (\varphi, \eta, \ell)$ is the solution of

$$\begin{cases} \varphi'(t, x) - \varphi_{xx}(t, x) = 0 & x < 0 & x > 0 & t > 0 \\ \varphi(t, 0) = \eta(t) + \ell(t) & t > 0 \\ m\eta'(t) = [\varphi_x](t, 0) & t > 0 \\ m\ell'(t) = [\varphi_x](t, 0) + m\varsigma^0 & t > 0 \\ \varphi(0, x) = \varphi^0 & x \in \mathbb{R} \\ \eta(0) = \eta^0 \\ \ell(0) = \ell^0. \end{cases} \quad (23)$$

Proof. By definition $\tilde{S}^*(t)\tilde{Z}^0 = (\varphi(t), \eta(t), \ell(t), \varsigma(t))$ is the solution of the linear system

$$\begin{cases} \tilde{Z}'(t) + \tilde{A}^*\tilde{Z}(t) = 0, \\ \tilde{Z}(0) = \tilde{Z}^0. \end{cases} \quad (24)$$

We have that

$$\begin{aligned} \left((\varphi, \eta, \ell, \varsigma)^*, \tilde{A}(y, p, r, o)^* \right) &= - \int_{-\infty}^0 y_{xx} \varphi dx - \int_0^{\infty} y_{xx} \varphi dx - [y_x](0)(\eta + \ell) - m\varsigma r = \\ &= \varphi(0)[y_x](0) - y(0)[\varphi_x](0) - \int_{-\infty}^0 \varphi_{xx} y dx - \int_0^{\infty} \varphi_{xx} y dx - [y_x](0)(\eta + \ell) - m\varsigma r = \\ &= - \int_{-\infty}^0 \varphi_{xx} y dx - \int_0^{\infty} \varphi_{xx} y dx + [y_x](0)(\eta + \ell - \varphi(0)) - p[\varphi_x](0) + r(-m\varsigma - [\varphi_x](0)). \end{aligned}$$

We deduce that $D(\tilde{A}^*) = D(\tilde{A})$ and

$$\tilde{A}^* \begin{pmatrix} \varphi \\ \eta \\ \ell \\ \varsigma \end{pmatrix} = \begin{pmatrix} -\varphi_{xx} \\ -\frac{1}{m}[\varphi_x](0) \\ -\varsigma - \frac{1}{m}[\varphi_x](0) \\ 0 \end{pmatrix}. \quad (25)$$

From (24) it follows that $\varsigma(t) = \varsigma^0$ and $Z = (\varphi, \eta, \ell)$ is the solution of (23). \square

3. The linear control problem

The aim of this section is to study the controllability properties of (19). We have the following result.

Theorem 3.1. *Let $T > 0$. For each $(g^T, h^T) \in \mathbb{R}^2$ there exists a control $v \in L^2(0, T)$ such that the solution (y, f, g, h) of*

$$\begin{cases} y'(t, x) - y_{xx}(t, x) = 0 & x < 0 & x > 0 & t > 0 \\ y(t, 0) = f(t) + g(t) & t > 0 \\ mf'(t) = [y_x](t, 0) + v(t) & t > 0 \\ mg'(t) = [y_x](t, 0) & t > 0 \\ h'(t) = g(t) & t > 0 \\ y(0, x) = 0 & x \in \mathbb{R} \\ f(0) = f^0, g(0) = g^0, h(0) = h^0 \end{cases} \quad (26)$$

verifies

$$g(T) = g^T, h(T) = h^T. \quad (27)$$

Proof. We define the linear operator

$$\mathcal{L} : L^2(0, T) \rightarrow \mathbb{R}^2, \quad \mathcal{L}(v) = (g(T), h(T)) \quad (28)$$

where (y, f, g, h) is the solution of (26). We remark that the controllability of (26) is equivalent to the fact that \mathcal{L} is onto. Since the range of \mathcal{L} is finite dimensional, to show that \mathcal{L} is onto it suffices to check that \mathcal{L}^* is one to one. Let us compute $\mathcal{L}^* : \mathbb{R}^2 \rightarrow L^2(0, T)$. We have that

$$\begin{aligned} (\mathcal{L}^*(\alpha, \beta), v)_{L^2(0, T)} &= \alpha g(T) + \beta h(T) = ((\alpha, \beta), \mathcal{L}(v)) = \\ &= \left((\alpha, \beta), \mathbb{P}_{3,4} \int_0^T \tilde{S}(T-s) B v(s) ds \right) = \left(\mathbb{P}_{3,4}^*(\alpha, \beta), \int_0^T \tilde{S}(T-s) B v(s) ds \right) = \\ &= \int_0^T B^* \tilde{S}^*(T-s) \mathbb{P}_{3,4}^*(\alpha, \beta) v(s) ds. \end{aligned}$$

Now, we deduce that

$$\mathcal{L}^*(\alpha, \beta)(t) = B^* \tilde{S}_{T-t}^* \mathbb{P}_{3,4}^*(\alpha, \beta) = \mathbb{P}_2 \tilde{S}_{T-t}^* \mathbb{P}_{3,4}^*(\alpha, \beta) = \eta(t) \quad (29)$$

where $\tilde{Z} = (\varphi, \eta, \ell, \varsigma)$ represents the solution of

$$\begin{cases} -\tilde{Z}' + \tilde{A}^* \tilde{Z} = 0, \\ \tilde{Z}(T) = (0, 0, \frac{\alpha}{m}, \frac{\beta}{m}). \end{cases} \quad (30)$$

From Proposition 2.3 it follows that $\tilde{Z}(t) = (\varphi(t), \eta(t), \ell(t), \beta)$ where $(\varphi(t), \eta(t), \ell(t))$ is the solution of

$$\begin{cases} -\varphi'(t, x) - \varphi_{xx}(t, x) = 0 & x < 0 & x > 0 & t > 0 \\ \varphi(t, 0) = \eta(t) + \ell(t) & t > 0 \\ -m\eta'(t) = [\varphi_x](t, 0) & t > 0 \\ -m\ell'(t) = [\varphi_x](t, 0) + \beta & t > 0 \\ \varphi(T, x) = 0 & x \in \mathbb{R} \\ \eta(T) = 0 \\ \ell(T) = \frac{\alpha}{m}. \end{cases} \quad (31)$$

To show that \mathcal{L}^* is one to one let us suppose that $\mathcal{L}^*(\alpha, \beta) = \eta = 0$.

System (31) may be written as

$$\begin{cases} -\phi' + A\phi = F \\ \phi(T) = (0, 0, l^T)^* \end{cases}$$

with $F = (0, 0, \beta)^*$. If we multiply by $\psi \in D(A^*) = D(A)$ we have the following variational characterization of the solutions of (31):

$$-(\phi, \psi)|_t^T + \int_t^T (\phi(s), A\psi) ds = \int_t^T (F(s), \psi) ds.$$

If we take $\Psi = (\psi_1, \psi_1(0), 0) \in D(A)$ with $\psi_1 \in H^2(\mathbb{R})$ we deduce that

$$\begin{aligned} &\int_{\mathbb{R}} \varphi(t, x) \psi_1(x) dx + m\eta(t) \psi_1(0) + \\ &+ \int_t^T \left(\int_{\mathbb{R}} \varphi(s, x) (-\psi_{1xx}) - m\eta(s) \frac{1}{m} [\psi_{1x}](0) - ml(s) \frac{1}{m} [\psi_{1x}](0) \right) ds = 0 \end{aligned}$$

Since $\eta = 0$ we have

$$\int_{\mathbb{R}} \left(\varphi(t) \psi_1(x) - \int_t^T \varphi(s, x) \psi_{1xx} ds \right) dx = 0, \quad \forall \psi_1 \in H^2(\mathbb{R}).$$

From Ball [1] we deduce that φ is the unique weak solution, in distributional sense, of the problem

$$\begin{cases} -\varphi' - \varphi_{xx} = 0, \\ \varphi(T) = 0. \end{cases} \quad (32)$$

It follows that $\varphi = 0$ and \mathcal{L}^* is one to one. Consequently, \mathcal{L} is onto and the controllability property holds. \square

Now, we present a systematic way to obtain controls for (26). Given any $(g^T, h^T) \in \mathbb{R}^2$, we define the following map

$$\mathcal{J}(\alpha, \beta) = \frac{1}{2} \|\mathcal{L}^*(\alpha, \beta)\|_{L^2(0,T)}^2 - \alpha g^T - \beta h^T \quad (\alpha, \beta) \in \mathbb{R}^2. \quad (33)$$

We remark that, if $(\hat{\alpha}, \hat{\beta}) \in \mathbb{R}^2$ is a minimizer of \mathcal{J} , then $\mathcal{L}^*(\hat{\alpha}, \hat{\beta})$ gives a control for (26)-(27) by solving (26) with $v = \mathcal{L}^*(\hat{\alpha}, \hat{\beta})$. The method presented above and the particular control $\mathcal{L}^*(\hat{\alpha}, \hat{\beta})$ are usually called HUM (Hilbert Uniqueness Method) method and control respectively.

In order to ensure the existence of a minimizer for \mathcal{J} it is sufficient to remark that, the fact that \mathcal{L}^* is into implies that $\|\mathcal{L}^*(\alpha, \beta)\|_{L^2(0,T)}$ is a norm in \mathbb{R}^2 and, consequently, there exists a constant $C > 0$ such that

$$\alpha^2 + \beta^2 \leq C \|\mathcal{L}^*(\alpha, \beta)\|_{L^2(0,T)}^2. \quad (34)$$

From (34) it follows that \mathcal{J} is coercive and the existence of a minimizer $(\hat{\alpha}, \hat{\beta})$ for it. Moreover, since $v = \mathcal{L}^*(\hat{\alpha}, \hat{\beta}) = \hat{\eta}$, it follows from Theorem 2.2 that $v \in H^1(0, T)$. Thus, we have proved the following

Theorem 3.2. *Let $T > 0$. A control $v \in L^2(0, T)$ is given by*

$$v(t) = \hat{\eta}(t) \quad t \in [0, T] \quad (35)$$

where $(\hat{\alpha}, \hat{\beta})$ is the minimizer of \mathcal{J} and $(\hat{\varphi}, \hat{\eta}, \hat{\ell})$ is the weak solution of (31) with initial data $\ell^T = \frac{\hat{\alpha}}{m}$ and $b = \hat{\beta}$.

We define the following map $\Gamma : \mathbb{R}^2 \rightarrow L^2(0, T)$,

$$\Gamma(g^T, h^T) = v = \hat{\eta} = \mathcal{L}^*(\hat{\alpha}, \hat{\beta}), \quad (36)$$

where v is the HUM control given by Theorem 3.2.

To show the continuity of the application Γ we evaluate the norm of $\Gamma(h^T, g^T)$

$$\begin{aligned} \|\Gamma(h^T, g^T)\|_{L^2(0,T)}^2 &= \|\hat{\eta}\|_{L^2(0,T)}^2 = \|\mathcal{L}^*(\hat{\alpha}, \hat{\beta})\|_{L^2(0,T)}^2 = 2J(\hat{\alpha}, \hat{\beta}) + 2\hat{\alpha}g^T + 2\hat{\beta}h^T \\ &\leq 2J(0, 0) + 2\hat{\alpha}g^T + 2\hat{\beta}h^T \leq 2 \left(\hat{\alpha}^2 + \hat{\beta}^2 \right)^{\frac{1}{2}} \left((g^T)^2 + (h^T)^2 \right)^{\frac{1}{2}} \\ &\leq C \|\Gamma(h^T, g^T)\|_{L^2(0,T)} \|(h^T, g^T)\|_{\mathbb{R}^2} \end{aligned}$$

where the last inequality follows from the observability inequality (34). Hence there exists a constant $C > 0$ such that

$$\|\Gamma(h^T, g^T)\|_{L^2(0,T)} \leq C \|(h^T, g^T)\|_{\mathbb{R}^2}. \quad (37)$$

4. The nonlinear control problem

The aim of this section is to provide the proof of the Theorem 1.1. Firstly, let us study the controllability of the following nonhomogeneous system

$$\begin{cases} y'(t, x) - y_{xx}(t, x) = q(t, x), & x < 0, x > 0 \quad t > 0 \\ y(t, 0) = f(t) + g(t), & t > 0 \\ mf'(t) = [y_x](t, 0) + v(t), & t > 0 \\ mg'(t) = [y_x](t, 0), & t > 0 \\ h'(t) = g(t), & t > 0 \\ y(0, x) = y^0(x), & x \in \mathbb{R} \\ f(0) = f^0, g(0) = g^0, h(0) = h^0. \end{cases} \quad (38)$$

We have the following result

Theorem 4.1. *Given any $q \in L^1(0, T; L^2(\mathbb{R}))$, $(y^0, f^0, g^0, h^0) \in \tilde{X}$ and $(g^T, h^T) \in \mathbb{R}^2$, there exists a control $v \in L^2(0, T)$ such that the corresponding solution (y, f, g, h) of (38) verifies*

$$(g(T), h(T)) = (g^T, h^T). \quad (39)$$

Moreover, there exists a constant $C > 0$ such that

$$\|v\|_{L^2(0, T)} \leq C (\|(g^T, h^T)\|_{\mathbb{R}^2} + \|(y^0, f^0, g^0, h^0)\|_{\tilde{X}} + \|q\|_{L^1(0, T; L^2(\mathbb{R}))}). \quad (40)$$

Proof. Let (y_1, f_1, g_1, h_1) be the solution of the nonhomogeneous system

$$\begin{cases} y_1'(t, x) - y_{1xx}(t, x) = q(t, x), & x < 0, x > 0 \quad t > 0 \\ y_1(t, 0) = f_1(t) + g_1(t), & t > 0 \\ mf_1'(t) = [y_{1x}](t, 0), & t > 0 \\ mg_1'(t) = [y_{1x}](t, 0), & t > 0 \\ h_1'(t) = g_1(t), & t > 0 \\ y_1(0, x) = y^0, & x \in \mathbb{R} \\ f_1(0) = f^0, g_1(0) = g^0, h_1(0) = h^0. \end{cases} \quad (41)$$

From Theorem 2.1 and Remark 2.1, we deduce that $(y_1, f_1, g_1, h_1) \in C([0, T]; X \times \mathbb{R})$. Now, let $v \in L^2(0, T)$ be the control given by Theorem 3.2 for which the solution of

$$\begin{cases} y_2'(t, x) - y_{2xx}(t, x) = 0, & x < 0, x > 0 \quad t > 0 \\ y_2(t, 0) = f_2(t) + g_2(t), & t > 0 \\ mf_2'(t) = [y_{2x}](t, 0) + v(t), & t > 0 \\ mg_2'(t) = [y_{2x}](t, 0), & t > 0 \\ h_2'(t) = g_2(t), & t > 0 \\ y_2(0, x) = 0, & x \in \mathbb{R} \\ f_2(0) = g_2(0) = h_2(0) = 0 \end{cases} \quad (42)$$

verifies $(g_2(T), h_2(T)) = (g^T - g_1(T), h^T - h_1(T))$.

It follows that the solution of system (38) is given by

$$(y, f, g, h) = (y_1, f_1, g_1, h_1) + (y_2, f_2, g_2, h_2)$$

and verifies (39).

Inequality (40) is a consequence of (37) and Theorem 2.1. \square

Now, for each $(y^0, f^0, g^0, h^0) \in \tilde{X}$, we define the map

$$\Lambda : L^1(0, T; L^2(\mathbb{R})) \longrightarrow L^1(0, T; L^2(\mathbb{R})), \quad \Lambda(q) = -yy_x + gy_x \quad (43)$$

where (y, f, g, h) is the controlled solution of (38) given by Theorem 4.1.

Firstly, let us show that Λ is well-defined. Indeed, if (y, f, g, h) is the solution of (38) with $q \in L^1(0, T; L^2(\mathbb{R}))$ and $v \in L^2(0, T)$, from Theorems 2.1 and 2.2 it follows that $(y, f, g) \in C([0, T]; X) \cap L^2(0, T; V)$. Now, we have that

$$\begin{aligned} \|yy_x\|_{L^1(0, T; L^2(\mathbb{R}))} &= \int_0^T \|y(t)y_x(t)\|_{L^2(\mathbb{R})} \leq \\ &\leq \int_0^T \|y(t)\|_{L^\infty(\mathbb{R})} \|y_x(t)\|_{L^2(\mathbb{R})} \leq C \|y\|_{L^2(0, T; H^1(\mathbb{R}))}^2 \end{aligned}$$

and

$$\|gy_x\|_{L^1(0, T; L^2(\mathbb{R}))} \leq C \|y\|_{L^2(0, T; H^1(\mathbb{R}))} \|g\|_{L^2(0, T)}.$$

It follows that

$$\| -yy_x + gy_x \|_{L^1(0, T; L^2(\mathbb{R}))} \leq C \left(\|y\|_{L^2(0, T; H^1(\mathbb{R}))}^2 + \|g\|_{L^2(0, T)}^2 \right) \quad (44)$$

which ensures the fact that Λ is well-defined.

Note that, if q is a fixed point of Λ , then the corresponding solution of (38) is a controlled solution of the nonlinear system (49). Thus, our aim is to prove that Λ has a fixed point. This will be the consequence of the following result.

Theorem 4.2. *With the above notations, there exist two constants $Q > 0$ and $R > 0$ such that, for any $(y^0, f^0, g^0, h^0) \in \tilde{X}$, $(g^T, h^T) \in \mathbb{R}^2$ with*

$$\max \{ \|(y^0, f^0, g^0, h^0)\|_{\tilde{X}}, \|(g^T, h^T)\|_{\mathbb{R}^2} \} < r \quad (45)$$

the application Λ defined by (43) is a contraction in the ball $B(0, R) \subset L^1(0, T; L^2(\mathbb{R}))$ of center 0 and radius R , i.e., there exists a constant $0 < \mathcal{Q} < 1$ such that

$$\|\Lambda(q) - \Lambda(\tilde{q})\|_{L^1(0, T; L^2(\mathbb{R}))} \leq \mathcal{Q} \|q - \tilde{q}\|_{L^1(0, T; L^2(\mathbb{R}))}, \quad (46)$$

for every q and \tilde{q} in $B(0, R)$.

Proof. Firstly, we show that there are $Q > 0$ and $R > 0$ such that $\Lambda(B(0, R)) \subseteq B(0, R)$, where $B(0, R)$ is the ball of center 0 and radius R from $L^1(0, T; L^2(\mathbb{R}))$. Indeed, from (44)

$$\|\Lambda(q)\|_{L^1(0, T; L^2(\mathbb{R}))} \leq C \left(\|y\|_{L^2(0, T; H^1(\mathbb{R}))}^2 + \|g\|_{L^2(0, T)}^2 \right) \leq C \|(y, f, g)\|_{L^2(0, T; V)}^2.$$

Theorem 2.2 and (40) ensure that

$$\begin{aligned} \|\Lambda(q)\|_{L^1(0, T; L^2(\mathbb{R}))} &\leq C \left(\|(y^0, f^0, g^0, h^0)\|_{\tilde{X}}^2 + \|q\|_{L^1(0, T; L^2(\mathbb{R}))}^2 + \|v\|_{L^1(0, T)}^2 \right) \leq \\ &\leq C \left(\|(g^T, h^T)\|_{\mathbb{R}^2}^2 + \|(y^0, f^0, g^0, h^0)\|_{\tilde{X}}^2 + \|q\|_{L^1(0, T; L^2(\mathbb{R}))}^2 \right). \end{aligned}$$

Now, if $q \in B(0, R)$, from (45) we deduce that

$$\|\Lambda(q)\|_{L^1(0, T; L^2(\mathbb{R}))} \leq C(Q^2 + R^2).$$

By taking Q and R sufficiently small, we obtain that $C(Q^2 + R^2) < R$ and $\Lambda(q) \in B(0, R)$ for all $q \in B(0, R)$.

We pass now to prove that Λ is a contraction in $B(0, R)$. In order to prove that we need two more estimates. Let (y, f, g, h) and $(\tilde{y}, \tilde{f}, \tilde{g}, \tilde{h})$ be the controlled solutions of

(38) with initial data (y^0, f^0, g^0, h^0) and nonhomogeneous terms q and \tilde{q} respectively. We have that

$$\begin{aligned}
& \|yy_x - \tilde{y}\tilde{y}_x\|_{L^1(0,T; L^2(\mathbb{R}))} = \\
& = \int_0^T \left(\int_{\mathbb{R}} |yy_x - \tilde{y}\tilde{y}_x|^2 dx \right)^{\frac{1}{2}} dt = \frac{1}{2} \int_0^T \left(\int_{\mathbb{R}} ((y^2 - \tilde{y}^2)_x)^2 dx \right)^{\frac{1}{2}} dt \\
& = \frac{1}{2} \int_0^T \left(\int_{\mathbb{R}} ((y_x - \tilde{y}_x)(y + \tilde{y}) + (y - \tilde{y})(y_x + \tilde{y}_x))^2 dx \right)^{\frac{1}{2}} dt \\
& \leq \frac{1}{2} \int_0^T \left(\int_{\mathbb{R}} (y_x - \tilde{y}_x)^2 (y + \tilde{y})^2 dx \right)^{\frac{1}{2}} dt + \frac{1}{2} \int_0^T \left(\int_{\mathbb{R}} (y - \tilde{y})^2 (\tilde{y}_x + \tilde{y}_x)^2 dx \right)^{\frac{1}{2}} dt \leq \\
& \leq \frac{C}{2} \int_0^T \| (y + \tilde{y}) \|_{H^1(\mathbb{R})} \| (y - \tilde{y}) \|_{H^1(\mathbb{R})} dt + \frac{C}{2} \int_0^T \| (y - \tilde{y}) \|_{H^1(\mathbb{R})} \| y + \tilde{y} \|_{H^1(\mathbb{R})} dt \leq \\
& \leq C \| (y - \tilde{y}) \|_{L^2(0,T;H^1(\mathbb{R}))} \| (y + \tilde{y}) \|_{L^2(0,T;H^1(\mathbb{R}))} \leq \\
& \leq C (\| (q - \tilde{q}) \|_{L^1(0,T;L^2(\mathbb{R}))} + \| (v - \tilde{v}) \|_{L^1(0,T)}) \\
& (2\| (y^0, f^0, g^0, h^0) \|_{\tilde{X}} + \| (q + \tilde{q}) \|_{L^1(0,T;L^2(\mathbb{R}))} + \| (v + \tilde{v}) \|_{L^1(0,T)}) \\
& \leq C \| (q - \tilde{q}) \|_{L^1(0,T;L^2(\mathbb{R}))} (\| (y^0, f^0, g^0, h^0) \|_{\tilde{X}} + \| (q + \tilde{q}) \|_{L^1(0,T;L^2(\mathbb{R}))} + \| (g^T, h^T) \|_{\mathbb{R}^2}) \leq \\
& \leq C(Q + R) \| (q - \tilde{q}) \|_{L^1(0,T;L^2(\mathbb{R}))}.
\end{aligned}$$

Therefore, we obtain the following estimate

$$\|yy_x - \tilde{y}\tilde{y}_x\|_{L^1(0,T; L^2(\mathbb{R}))} \leq C(Q + R) \| (q - \tilde{q}) \|_{L^1(0,T;L^2(\mathbb{R}))}. \quad (47)$$

Moreover,

$$\begin{aligned}
& \| -gy_x + \tilde{g}\tilde{y}_x \|_{L^1(0,T;L^2(\mathbb{R}))} \leq \| (\tilde{g} - g)y_x \|_{L^1(0,T;L^2(\mathbb{R}))} + \| \tilde{g}(\tilde{y}_x - y_x) \|_{L^1(0,T;L^2(\mathbb{R}))} \leq \\
& \leq \| y \|_{L^2(0,T;H^1(\mathbb{R}))} \| g - \tilde{g} \|_{L^2(0,T)} + \| \tilde{g} \|_{L^2(0,T)} \| y - \tilde{y} \|_{L^2(0,T;H^1(\mathbb{R}))} \leq \\
& \leq C (\| (y^0, f^0, g^0, h^0) \|_{\tilde{X}} + \| q \|_{L^1(0,T;L^2(\mathbb{R}))} + \| \tilde{q} \|_{L^1(0,T;L^2(\mathbb{R}))} + \| v \|_{L^2(0,T)} + \| \tilde{v} \|_{L^2(0,T)}) \times \\
& (\| q - \tilde{q} \|_{L^1(0,T;L^2(\mathbb{R}))} + \| v - \tilde{v} \|_{L^2(0,T)}) \leq C(Q + R) \| q - \tilde{q} \|_{L^1(0,T;L^2(\mathbb{R}))}.
\end{aligned}$$

Hence, we have that

$$\| -gy_x + \tilde{g}\tilde{y}_x \|_{L^1(0,T;L^2(\mathbb{R}))} \leq C(Q + R) \| q - \tilde{q} \|_{L^1(0,T;L^2(\mathbb{R}))}. \quad (48)$$

By using (47) and (48) we obtain that

$$\| \Lambda(q) - \Lambda(\tilde{q}) \|_{L^1(0,T; L^2(\mathbb{R}))} \leq C(Q + R) \| q - \tilde{q} \|_{L^1(0,T;L^2(\mathbb{R}))}$$

which shows that, for sufficiently small R and Q , Λ is a contraction in $B(0, R)$ with $\mathcal{Q} = C(Q + R) < 1$. \square

Now we have all the ingredients needed to prove the main result of our paper which shows the controllability property of (3).

Proof of Theorem 1.1: Let us take Q and R like in Theorem 4.2. Since the application Λ is a contraction, it follows that it has a unique fixed point $q \in L^1(0, T; L^2(\mathbb{R}))$.

From the definition of Λ , we deduce that the corresponding solution $(y, f, g, h) \in C([0, T]; \tilde{X})$ of (38) is a solution of

$$\begin{cases} y'(t, x) - y_{xx}(t, x) - g(t)y_x(t, x) + y_x(t, x)y(t, x) = 0, & x < 0, x > 0 \quad t > 0 \\ y(t, 0) = f(t) + g(t), & t > 0 \\ mf'(t) = [y_x](t, 0) + v(t), & t > 0 \\ mg'(t) = [y_x](t, 0), & t > 0 \\ h'(t) = g(t), & t > 0 \\ y(0, x) = y^0(x), & x \in \mathbb{R} \\ f(0) = f^0, g(0) = g^0, h(0) = h^0. \end{cases} \quad (49)$$

which verifies $(g(T), h(T)) = (g^T, h^T)$. Consequently, (y, g, h) is a controlled solution of (3), with a control $f \in C[0, T]$ such that $f(0) = f^0$. \square

Remark 4.1. *If the initial data of (3) is more regular, then controls may be found in $H^1(0, T)$. Indeed, if we suppose that $(y^0, 0, 0) \in V$, we deduce from Theorem 2.2 that the solution of the controlled equation (38) belongs to $H^1(0, T; X)$. Consequently, the control f for (3) is in $H^1(0, T)$.*

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(Nicolae Cîndea) INSTITUT ELIE CARTAN, NANCY UNIVERSITÉ/CNRS/INRIA, BP 70239, 54506
VANDOEUVRE-LÈS-NANCY, FRANCE
E-mail address: `cindea@iecn.u-nancy.fr`

(Sorin Micu) DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF CRAIOVA, 13 A.I. CUZA
STREET, CRAIOVA, 200585, ROMANIA
E-mail address: `sd.micu@yahoo.com`

(Ionel Rovența) DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF CRAIOVA, 13 A.I. CUZA
STREET, CRAIOVA, 200585, ROMANIA
E-mail address: `roventaionel@yahoo.com`

(Marius TucsnaK) INSTITUT ELIE CARTAN, NANCY UNIVERSITÉ/CNRS/INRIA, BP 70239, 54506
VANDOEUVRE-LÈS-NANCY, FRANCE
E-mail address: `Marius.TucsnaK@iecn.u-nancy.fr`