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Analytic solution of fractional integro-differential equations

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Abstract. This paper is focused on deriving an analytic solution for the fractional integro-differential equations, commonly used in the mathematical modeling of various physical phenomena. In this contribution, based on the homotopy analysis method, a new solution strategy for the fractional integro-differential equations is proposed. Different from all other analytic techniques, this approach provides a simple way to ensure the convergence of series of solution so that one can always get accurate enough approximations.

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1. Introduction

The fractional calculus has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ has been described by Leibniz [10, 12]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grünwald, Letnikov and Riemann. There are many interesting books about fractional calculus and fractional differential equations [10, 12, 3, 14]. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc.

Derivatives of non-integer order can be defined in different ways, e.g. Riemann–Liouville, Grünwald–Letnikow, Caputo and Generalized Functions Approach [12]. In this paper we focus attention on Caputo's definition which turns out to be more useful in real-life applications since it can be coupled with initial conditions having a clear physical meaning.

There are only a few techniques for the solution of fractional integro-differential equations, since it is relatively a new subject in mathematics. Some of these methods are; Adomian decomposition method (ADM) [9], fractional differential transform method (FDTM) [11] and the collocation method [13].

In this paper, we will suggest an approach to the search for an explicit analytical solution for the fractional integro-differential equation of the type:

$$\begin{aligned} D^{\alpha}y(t) &= p(t)y(t) + f(t) + \int_0^t K(t,s)y(s)ds, \qquad t \in I = [0,1], \\ y(0) &= a, \end{aligned} \tag{1}$$

by homotopy analysis method (HAM) [1, 2, 4, 5, 6, 7, 8]. In particular, the derived explicit expression for the problem is mathematically and computationally friendly.

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2. Preliminaries and notations

In this section, let us recall essentials of fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. For the purpose of this paper the Caputo's definition of fractional differentiation will be used, taking the advantage of Gaputo's approach that the initial conditions for fractional differential equations with Caputo's derivatives take on the traditional form as for integer-order differential equations.

Definition 2.1. *Caputo's definition of the fractional-order derivative is defined as*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,$$

where $n-1<\alpha\leq n$, $n\in\mathbb{N}$, α is the order of the derivative and a is the initial value of function f.

For the Caputo's derivative we have:

$$D^{\alpha}C = 0, \qquad C \text{ is constant},$$

$$D^{\alpha}t^{\beta} = \{.0 \qquad \beta\leq\alpha-1\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha} \qquad \beta>\alpha-1$$

Caputo's fractional differentiation is a linear operation and if $f(\tau)$ is continuous in $[a,t]$ and $g(\tau)$ has $n+1$ continuous derivatives in $[a,t]$, it satisfies the so-called Leibnitz rule:

$$D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty}\binom{\alpha}{k}g^{(k)}(t)D^{\alpha-k}f(t)$$

For establishing our results, we also necessarily introduce the following Riemann–Liouville fractional integral operator.

Definition 2.2. *The Riemann–Liouville fractional integral operator of order $\alpha\geq 0$, of a function $f\in C_{\mu}$, $\mu\geq -1$, is defined as*

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}f(\tau)d\tau$$

We mention only some properties of the operator J^{α}: For $f\in C_{\mu}$, $\mu,\gamma\geq -1$, $\alpha,\beta\geq 0$:

$$J^{0}f(t) = f(t), \quad J^{\alpha}J^{\beta}f(t) = J^{\alpha+\beta}f(t), \quad J^{\alpha}J^{\beta}f(t) = J^{\beta}J^{\alpha}f(t),$$

$$J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha}$$

Also, we need here two of its basic properties. $m-1<\alpha\leq m$, $m\in\mathbb{N}$, and $f\in C_{\mu}^{m}$, $\mu\geq -1$, then

$$D^{\alpha}J^{\alpha}f(t) = f(t), \qquad J^{\alpha}D^{\alpha}f(t) = f(t) - \sum_{i=0}^{m-1}f^{(i)}(0^{+})\frac{t^{i}}{i!}, \quad t>0.$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult [14].

3. Numerical schemes

3.1. Approach based on the HAM. In this paper, we present analytic solution of an integro-differential equation with fractional derivative of the type:

$$D^\alpha y(t) = p(t)y(t) + f(t) + \int_0^t K(t,s)y(s)ds, \qquad t \in I = [0,1], \tag{2a}$$

$$y(0) = a. \tag{2b}$$

Here, the given functions $f, p : I \to \mathbb{R}$ and $K : S \to \mathbb{R}$ (with $S = \{(t,s) : 0 \leq s \leq t \leq 1\}$) are supposed to be sufficiently smooth, with $0 < \alpha \leq 1$. According to Eqs. 2a and 2b, the solution $y(t)$ can be expressed by the base functions

$$\{t^{\alpha n} : n \geq 1,\ n \in \mathbb{N}\} \tag{3}$$

as

$$y(t) = \sum_{n=1}^{\infty} b_n t^{\alpha n}, \tag{4}$$

where b_n is a coefficient. It provides us the so-called Solution Expression of $y(t)$.

According to Eq. 2a, we define a nonlinear operator

$$N[\phi(t;q)] = D^\alpha \phi(t;q) - p(t)\phi(t;q) - f(t) - \int_0^t K(t,s)\phi(s;q)ds, \tag{5}$$

where $q \in [0,1]$ denotes the embedding parameter. Let $y_0(t)$ denote an initial guess of the exact solution $y(t)$ which satisfies the initial condition 2b. Also, $h \neq 0$ an auxiliary parameter and L an auxiliary linear operator. All of $y_0(x)$, L and h will be chosen later with great freedom. Then, we construct a one-parameter family of differential equations

$$(1-q)L[\phi(t;q) - y_0(t)] = qhN[\phi(t;q)] \tag{6}$$

subject to the boundary conditions

$$\phi(0;q) = a. \tag{7}$$

Obviously, when $q = 0$, because of the property $L(0) = 0$ of any linear operator L, Eqs. 6 and 7 have the solution

$$\phi(t;0) = y_0(t), \tag{8}$$

and when $q = 1$, since $h \neq 0$, Eqs. 6 and 7 are equivalent to the original ones, 2a and 2b, provided

$$\phi(t;1) = y(t). \tag{9}$$

Thus, according to 8 and 9, as the embedding parameter q increases from 0 to 1, $\phi(t;q)$ varies continuously from the initial approximation $y_0(t)$ to the exact solution $y(t)$. This kind of deformation $\phi(t;q)$ is totally determined by the so-called zeroth-order deformation equations 6 and 7.

By Taylor's theorem, $\phi(t;q)$ can be expanded in a power series of q as follows

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{10}$$

where

$$y_m(t) = D_m[\phi(t;q)] = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}.$$

D_m is called the mth-order homotopy-derivative of ϕ.

Fortunately, the homotopy-series 10 contains an auxiliary parameter h, and besides we have great freedom to choose the auxiliary linear operator L, as illustrated by Liao [6]. If the auxiliary linear parameter L and the nonzero auxiliary parameter h are properly chosen so that the power series 10 of $\phi(t;q)$ converges at $q = 1$. Then, we have under these assumptions the the so-called homotopy-series solution

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{11}$$

According to the fundamental theorems in calculus, each coefficient of the Taylor series of a function is unique. Thus, $y_m(t)$ is unique, and is determined by $\phi(t;q)$. Therefore, the governing equations and boundary conditions of $y_m(t)$ can be deduced from the zeroth-order deformation equations 6 and 7. For brevity, define the vectors

$$\overrightarrow{y}_n(t) = \{y_0(t), y_1(t), y_2(t), \ldots, y_n(t)\}.$$

Differentiating the zero-order deformation equation 6 m times with respective to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$\begin{aligned} L[y_m(t) - \chi_m y_{m-1}(t)] &= h\Re_m(\overrightarrow{y}_{m-1}(x)), \\ y_m(0) &= 0, \end{aligned} \tag{12}$$

where

$$\Re_m(\overrightarrow{y}_{m-1}(x)) = D_{m-1}(N[\phi]) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}|_{q=0} \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

In this line we have that,

$$\Re_m(\overrightarrow{y}_{m-1}(t)) = D^{\alpha}y_{m-1}(t) - p(t)y_{m-1}(t) - f(t)(1-\chi_m) - \int_0^t K(t,s)y_{m-1}(s)ds. \tag{14}$$

So, by means of symbolic computation software such as Mathematica, Maple, Matlab and so on, it is not difficult to get $\Re_m(\overrightarrow{y}_{m-1}(t))$ for large value of m.

Note that the high-order deformation equations 12 are linear ODEs with fractional derivatives. So, according to 12, the original nonlinear problem is transferred into an infinite number of linear ODEs. However, unlike perturbation techniques, we do not need any small physical parameters to do such a kind of transformation. Besides, unlike the traditional "non-perturbation techniques", we have great freedom to choose the auxiliary linear operator L and the initial guess $y_0(t)$.

Both the auxiliary linear operator L and the initial guess $y_0(t)$ are chosen under the so-called Rule of Solution Expression: the auxiliary linear operator L and the initial guess $y_0(t)$ must be chosen so that the solutions of the high-order deformation equations 12 exist and besides they obey the Solution Expression 4. So, for the solutions to obey the Solution Expression 4 and the boundary conditions 2b, we choose the initial guess of the solution:

$$y_0(t) = a. \tag{15}$$

Because the original equation 2a is of order α, we simply choose such an auxiliary linear operator

$$Ly = D^{\alpha}y, \tag{16}$$

with the property

$$L[C] = 0,$$

where C is an integral constant. By taking the inverse of the linear operator L in 12, then we get for $m \geq 1$,

$$y_m(t) = \chi_m y_{m-1}(t) - \chi_m \sum_{k=0}^{n-1} y_{m-1}^{(k)}(0)\frac{t^k}{k!} + hJ^{\alpha}[\Re_m(\overrightarrow{y}_{m-1}(t))], \tag{17}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. In this way, we get $y_m(t)$ one by one in the order $m = 1, 2, 3, \ldots$. Thus, it is easy to get approximations at high enough order, especially by means of the symbolic computation software.

3.2. Series solution.

Theorem 3.1. *As long as the series 11 converges, it must be the exact solution of the integral equation 2a-2b.*

Proof. If the series 11 converges, we can write

$$S(t) = \sum_{m=0}^{\infty} y_m(t),$$

and it holds that

$$\lim_{m\to\infty} y_m(t) = 0. \tag{18}$$

We can verify that

$$\begin{aligned} \sum_{m=1}^{n} [y_m(t) - \chi_m y_{m-1}(t)] &= y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) \\ &= y_n(t), \end{aligned}$$

which gives us, according to 18,

$$\sum_{m=1}^{\infty} [y_m(t) - \chi_m y_{m-1}(t)] = \lim_{n\to\infty} y_n(t) = 0. \tag{19}$$

Furthermore, using 19 and the definition of the linear operator L, we have

$$\sum_{m=1}^{\infty} L[y_m(t) - \chi_m y_{m-1}(t)] = L[\sum_{m=1}^{\infty} [y_m(t) - \chi_m y_{m-1}(t)]] = 0.$$

In this line, we can obtain that

$$\sum_{m=1}^{\infty} L[y_m(t) - \chi_m y_{m-1}(t)] = h \sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(t)) = 0$$

which gives, since $h \neq 0$, that

$$\sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(t)) = 0. \tag{20}$$

Substituting $\Re_{m-1}(\overrightarrow{y}_{m-1}(x))$ into the above expression and simplifying it, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_{m-1} &= \sum_{m=1}^{\infty}[D^{\alpha}y_{m-1}-p(t)y_{m-1}-f(t)(1-\chi_m)-\int_0^t K(t,s)y_{m-1}(s)ds] \\ &= \sum_{m=0}^{\infty}D^{\alpha}y_m-p(t)\sum_{m=0}^{\infty}y_m-f(t)-\int_0^t K(t,s)\sum_{m=0}^{\infty}y_m(s)ds \\ &= D^{\alpha}S(t)-p(t)S(t)-f(t)-\int_0^t K(t,s)S(s)ds \qquad (21)\end{aligned}$$

From 20 and 21, we have

$$D^{\alpha}S(t)=p(t)S(t)+f(t)+\int_0^t K(t,s)S(s)ds,$$

and so, $S(t)$ must be the exact solution of 2a, 2b. □

Note that we have great freedom to choose the value of the auxiliary parameter h. Mathematically the value of $y(t)$ at any finite order of approximation is dependent upon the auxiliary parameter h, because the zeroth and highorder deformation equations contain h. Let R_h denote the set of all values of h which ensure the convergence of the HAM series solution 11 of $y(t)$. According to Theorem 3, all of these series solutions must converge to the solution of the original equations 2a and 2b. Let h be the variable of the horizontal axis and the limit of the series solution 11 of $y(t)$ be the variable of vertical axis. Plot the curve $y(t)$ *vs* h, where $y(t)$ denotes the limit of the series 11. Because the limit of all convergent series solutions 11 is the same for a given a, there exists a horizontal line segment above the region $h \in R_h$. So, by plotting the curve $y(t)$ *vs* h at a high enough order approximation, one can find an approximation of the set R_h.

4. Applications

In this part, we introduce some applications on HAM to solve integro-differential equations with fractional derivatives.

4.1. Example 1.

First we consider the following fractional integro-differential equation, for $t \in I = [0,1]$

$$\begin{aligned}D^{(0.5)}y(t) &= y(t)+\frac{8}{3\Gamma(0.5)}t^{1.5}-t^2-\frac{1}{3}t^3+\int_0^t y(s)ds, \qquad (22a)\\ y(0) &= 0,\end{aligned}$$

which has the exact solution $y(t)=t^2$. From 22a, it is sraightforward to use the set of base functions

$$\{t^{0.5n} : n \geq 1,\ n \in \mathbb{N}\},$$

to represent $y(t)$,

$$y(t)=\sum_{k=1}^{\infty}b_k t^{0.5k}, \qquad (23)$$

where b_k is a coefficient to be determined later. According to 6, the zeroth-order deformation can be given by

$$\begin{aligned}(1-q)L[\phi(t;q)-y_0(t)] &= qh(D^{(0.5)}\phi(t;q)-\phi(t;q)+ \\ &\frac{8}{3\Gamma(0.5)}t^{1.5}-t^2-\frac{1}{3}t^3+\int_0^t \phi(s;q)ds).\end{aligned}$$

Under the rule of solution expression denoted by 23 and according to the initial condition in 22a, we can choose the initial guess of $y(t)$ as follows:

$$y_0(t)=0,$$

and we choose the auxiliary linear operator

$$L[\phi(t;q)]=D^{(0.5)}\phi(t;q),$$

with the property

$$L[C]=0,$$

where C is an integral constant. Hence, the mth-order deformation equation can be given by

$$\begin{aligned}y_m(t) &= \chi_m y_{m-1}(t)+hJ^{0.5}[D^{(0.5)}y_{m-1}(t)-y_{m-1}(t)-\int_0^t y_{m-1}(s)ds- \\ &(\frac{8}{3\Gamma(0.5)}t^{1.5}-t^2-\frac{1}{3}t^3)(1-\chi_m)].\end{aligned}$$

Consequently, the HAM series solution is

$$y(t)=y_0(t)+\sum_{m=1}^{K} y_m(t), \tag{24}$$

where K is the number of terms. Eq. 24 is a family of solution expression in the auxiliary parameter h. As Liao suggested [7], to investigate the influence of h on the convergent of the solution series 24, we plot the so-called h-curve of $y\prime(0.2)$ as shown in Fig. 4.1.

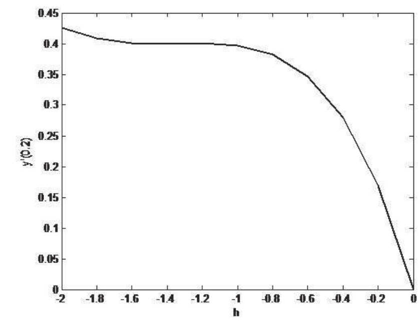

FIGURE 1. Curve $y'(0.2)$~h at the 5th order of approximation.

According to this h-curve, it is easy to decide that $(-1.6,-1)$ is the valid region of h. which corresponds to the line segments nearly parallel to the horizontal axis.

A proper value of $h=-1.4$ is found from the h-curve shown in Fig. 4.1. Then the ten terms from the series solution expression by HAM is

$$y(t)\approx t^2,$$

which is in good agreement with the exact solution as shown in Fig. 4.1.

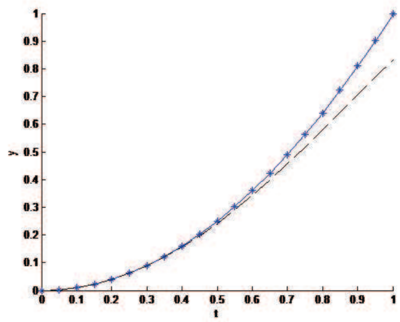

FIGURE 2. Comparison of the numerical solution. Hollow dots: 5th-order HAM approximation; solid stars: 10th-order HAM approximation; continued solid lines: exact solution.

4.2. Example 2. Consider the fractional integro-differential equation for $t \in I = [0, 1]$,

$$\begin{aligned} D^{(0.75)}y(t) &= \frac{1}{\Gamma(1.25)}t^{0.25} + (t\cos t - \sin t)y(t) + \int_0^t t \sin s\ y(s)ds, \qquad (25a) \\ y(0) &= 0, \end{aligned}$$

which has the exact solution $y(t) = t$. From 25a, we use the set of base functions

$$\{t^{0.75n} : n \geq 1,\ n \in \mathbb{N}\},$$

to represent $y(t)$,

$$y(t) = \sum_{k=1}^{\infty} b_k t^{0.75k},$$

where b_k is a coefficient to be determined later. We choose

$$y_0(t) = 0,$$

as our initial approximation of $y(t)$. Besides that we select the the auxiliary linear operator

$$L[\phi(t;q)] = D^{(0.75)}\phi(t;q),$$

with property

$$L[C] = 0,$$

in which C is an integral constant. Using 14, we have

$$\begin{aligned} \Re_m(\overrightarrow{y}_{m-1}(t)) &= D^{(0.75)}y_{m-1}(t) - (t\cos t - \sin t)y_{m-1}(t) - \\ &\int_0^t t\sin s\ y_{m-1}(s)ds - (1-\chi_m)\frac{1}{\Gamma(1.25)}t^{0.25}, \end{aligned}$$

so that the mth order deformation equation is

$$y_m(t) = \chi_m y_{m-1}(t) + hJ^{0.75}[\Re_m(\overrightarrow{y}_{m-1}(t))],$$

subject to the initial condition

$$y_m(0) = 0.$$

Now we successfully obtain

$$y_1(t) = h(t^2 - \frac{\Gamma(3)}{\Gamma(3.5)}t^{2.5} - \frac{\Gamma(4)}{3\Gamma(4.5)}t^{3.5})$$

By the same manner we can get $y_1(t), y_2(t), \dots, y_m(t)$. In order to find range of admissible values of h, the h-curve is plotted in Fig. 4.2 for 5th-order approximation. We can see that the range of values for h is between $-1.5 \leq h \leq -0.5$.

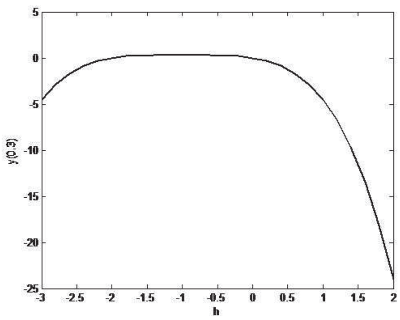

FIGURE 3. Curve $y(0.3)$~h at the 5th order of approximation.

Then, we may conclude that we have achieved a good approximation with the numerical solution of the equation by using the first few terms only of the linear equations derived above. It is evident that the overall errors can be made smaller by adding new terms of the HAM series solution. Fig. 4.2 presents a Comparison of the numerical solution of 10th-order HAM approximation and the exact solution.

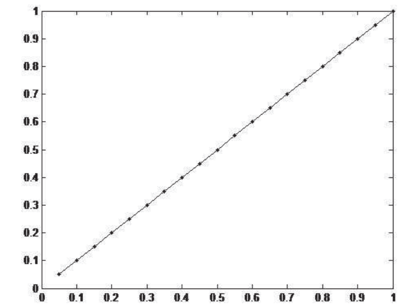

FIGURE 4. Comparison of the numerical solution. Hollow dots: 10th-order HAM approximation; continued solid lines: exact solution.

5. Conclusion

Solving fractional integro-differential problems lack of analytical or closed form solutions. Based on the fact, this study has focused on developing a simple procedure to obtain an explicit analytical solution concerning the fractional integro-differential equations. The method presented was applied to problems that exist in the literature.

The results evaluated are in very good agreement with the already existing ones, besides that even much more accurate. A series solution is evaluated in a very fast convergence rate where the accuracy is improved by increasing the number of terms considered. Shortly, from now on, with proven theorems, HAM can be used as a powerful solver for the solution of fractional integro-differential equations.

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