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Nonlinear elliptic equations with divergence term and without sign condition

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Abstract. This paper aims at bounded solutions for nonlinear Dirichlet problems with divergence term in a bounded domains. Our results is obtained without imposing any sign condition on the term which growth quadratically to the gradient. For a given source term in a suitable Lebesgue spaces and with less regularity on the divergence term, We establish an existence and regularity results of solutions. Our approach falls within the scope of Schauder fixed point theorem, some properties of a priori estimates and Stampacchia's L^{∞}-regularity.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N > 2$. Considering the non linear Dirichlet problem whose simplest model is

$$-div(\nabla u - \Theta) + a(x)u|u|^{r-1} + g(x,u)|\nabla u|^2 = f(x) \text{ in } \Omega, \qquad (1.1)$$
$$u|_{\partial\Omega} = 0,$$

under suitable conditions on the source term f and the divergence term with Θ. Without imposing any sign condition and any limitation on the growth of the Carathéodory function g we are interested in existence of bounded solutions.

Considering a Carathéodory function $g(.,.) : \Omega \times (0,+\infty) \to \mathbb{R}$ which my change of sign and may have a singularity at $s = 0$. Let us note that the real function $a(.)$ is nonnegative and bounded in $L^{\infty}(\Omega)$. Using suitable conditions on the data, we establish existence and regularity of solutions for problem (P). In general the boundedness and then the existence of u cannot be obtained if one does not put any restriction on Θ, a, g, f and $|\Omega|$. Indeed, one can exhibit problems like (1.1) which do not have any solution (see [2, 17, 27]).

This work is considered without any restriction on the growth of g and with the presence of the less regular elements $\Theta(.)$ and $a(.)$. We prove existence of bounded weak solutions by assuming that

$$f \in L^m, \ m > \frac{N}{2},$$

and the function $\Theta : \mathbb{R} \to \mathbb{R}^N$ is such that

$$\Theta \in (L^q)^N, \ q > N.$$

In addition of the complications introduced by Θ which provokes several difficulties in the controllability of the integrals, other difficulties will be introduced by $g(x,s)$

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which is a Carathéodory function on $\Omega \times (0, +\infty)$ changing of sign having a singularity at $s = 0$, and a is a nonnegative bounded real function. We shall obtain solution by approximating process. Using a priori estimates, Schauder fixed point theorem and Stampacchia's L^∞-regularity results we shall show that the approximated solutions converges to a solution of problem (P).

In some papers (see, for example, [5] and the references contained therein), it is proved existence of solutions when the source therm f is small in a suitable norm. On the other hand, condition on the function g have been considered in order to get a solution for f in a given Lebesgue space. This last case is considered in [13] under a restricted hypotheses: g is a Carathéodory function non increasing in u and $g(x, 0) = 0$.

There is many works starting from the classical references of Ladyzenskaja and Lions (see [23] and [24]), many other works have been devoted to elliptic problems with lower order terms having quadratic growth with respect to the gradients (see e.g. [7], [9], [10], [18], [20], [21], [22], [27] and the references therein).

This kind of problems though being physically natural, does not seem to have been studied in the literature. The particular situations where on has the following condition

$$g(x, s)\, s \geq 0,$$

for almost every x in Ω, for every s in $\mathbb{R}$, existence results in $H_0^1(\Omega) \cap L^\infty(\Omega)$ have been given (see, for example, [5] and the references contained therein). In our case there is now sign condition on g.

Finally, we give some remarks for problem in the general form

$$\begin{aligned} -div\, a(x, u, \nabla u) &= H(x, u, \nabla u) - div\, f, \text{ in } D'(\Omega), \\ u &\in H_0^1(\Omega) \cap L^\infty(\Omega), \end{aligned} \tag{1.2}$$

where $a(x, u, \xi)$ and $H(x, u, \xi)$ are Carathéodory functions satisfying suitable growth conditions on $|\xi|$. The question of finding estimates for solution of the general problems like (1.2) has been studied by many authors, under various assumptions on H (see e.g. [27, 2, 17]). We would like to remark that there exist various papers where estimates and existence results are proved for problems of the form (1.2) when H satisfies a sign condition (see e.g. [5, 7, 8, 11, 30]). The uniqueness results have been shown in [3]. Estimates, existence and regularity results (like Hölder-continuity) for variational problems are contained for example in [7, 6, 10, 28, 29, 31].

2. Notations assumptions and main result

Let us consider the following elliptic problem

$$a(x)u|u|^{r-1} - div(\nabla u - \Theta) + g(x, u)|\nabla u|^2 = f(x) \text{ in } \Omega,$$

$$u|_{\partial\Omega} = 0.$$

Where Ω is a bounded domain of $\mathbb{R}^N$ with $N > 2$, with boundary $\Gamma = \partial\Omega$ and r is a constant, such that $r > 1$.

The elements f and Θ satisfies the following hypotheses

$$f \in L^m(\Omega),\ m > \frac{N}{2},$$

and

$$\Theta \in (L^q(\Omega))^N,\ q > N.$$

Let us assume that there exist an increasing function $b:(0\times+\infty)\to(0\times+\infty)$ and $\beta\in(0,1)$ such that the Carathéodory function g satisfies the following hypothesis

$$-\beta\le s\,g(x,s)\le b(s),\ \ \forall s>0,\ \text{and}\ a.e.\ x\in\Omega. \tag{2.1}$$

There is no sign condition imposed on g and any condition on the upper growth of $g(x,s)$ as s goes to infinity is imposed.

The function $a(.)$ satisfies

$$\begin{cases} a\in\ L^{\infty}(\Omega), \\ \exists\ a_0 \text{ such that } a\ge a_0>0,\ a.e \text{ in } \Omega. \end{cases} \tag{2.2}$$

We recall the definition of a truncated function $T_k(s)$ defined, for all $k\in\mathbb{R}^+$, by

$$T_k(z)=\begin{cases} z & \text{if} \quad |z|\le k \\ k & \text{if} \quad z>k \\ -k & \text{if} \quad z<-k \end{cases},$$

and the corresponding tail function

$$G_k(z)=z-T_k(z)=(|z|-k)^+\,sign\,(z).$$

We begin by proving an existence result when the source term is regular.

Theorem 2.1. *Let f in $L^{\infty}(\Omega)$, and Θ be a function in $(L^q(\Omega))^N$, $q>N$. We assume that $r>1$, then there exist a solution $u\in H_0^1(\Omega)\cap L^{\infty}(\Omega)$ of*

$$a(x)u|u|^{r-1}-div(\nabla u-\Theta)+g(x,u)|\nabla u|^2=f(x)\ \text{in}\,\Omega, \tag{2.3}$$

$$u|_{\partial\Omega}=0.$$

In the sense that

$$\int_{\Omega}a(x)u|u|^{r-1}\phi+\int_{\Omega}(\nabla u-\Theta)\nabla\phi+\int_{\Omega}g(x,u)|\nabla u|^2\phi=\int_{\Omega}f\,\phi, \tag{2.4}$$

for any test function ϕ in $C_0^{\infty}(\Omega)$.

Remark 2.1. *The result of the precedent theorem is still new on the literature. Indeed, existence results in $H_0^1(\Omega)\cap L^{\infty}(\Omega)$ have been given under a sign condition on g : namely,*

$$g(x,s)\,s\ge 0,$$

for almost every x in Ω, for every s in $\mathbb{R}$ (see, for example, [5] *and the references contained therein). In our case there is now sign condition on g.*

Theorem 2.2. *Let $r>1$. If we assume that $f\in L^m(\Omega)$, $m>\frac{N}{2}$, and $\Theta\in(L^q(\Omega))^N$, $q>N$. Then there exist a solution u of*

$$a(x)u|u|^{r-1}-div(\nabla u-\Theta)+g(x,u)|\nabla u|^2=f(x)\ \text{in}\,\Omega,$$

$$u|_{\partial\Omega}=0.$$

In the sense that u belongs to $H_0^1(\Omega)$, $g(x,u)\,|\nabla u|^2$ and $a(x)u|u|^{r-1}$ are integrable, and the following equality holds

$$\int_{\Omega}a(x)u|u|^{r-1}\phi+\int_{\Omega}(\nabla u-\Theta)\nabla\phi+\int_{\Omega}g(x,u)|\nabla u|^2\phi=\int_{\Omega}f\,\phi, \tag{2.5}$$

for any test function ϕ in $C_0^{\infty}(\Omega)$.

Furthermore, any solution of the problem (P) belongs to $H_0^1(\Omega)\cap L^{\infty}(\Omega)$.

3. Proof of the main results

We will give a proof of the first result. We denote by c a positive constant which may only depend on the parameters of our problem, its value my vary from line to line.We consider the following operator

$$A(x,s) = a(x)s|s|^{r-1}.$$

For any $n \in \mathbb{N}$ we set

$$A_n(x,s) = a_n(x,s)\frac{|s|^{r-1}}{1+\frac{1}{n}|s|^{r-1}},$$

where

$$a_n(x,s) = a(x)T_n(s).$$

Let us define the Carathéodory function $G : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ as follows:

$$G(x,s,\zeta) = g(x,s)|\zeta|^2.$$

We define also the following Carathéodory function

$$G_n(x,s,\zeta) = g_n(x,s)\frac{|\zeta|^2}{1+\frac{1}{n}|\zeta|^2},$$

where

$$g_n(x,s) = \left\{\begin{array}{ccc} 0 & \text{if} & s \le 0 \\ n^2 s^2 T_n g(x,s) & \text{if} & 0 < s < \frac{1}{n} \\ T_n g(x,s) & \text{if} & \frac{1}{n} \le s \end{array}\right. .$$

Let us remark that $g_n(x,s)$ is bounded and

$$g_n(x,s) \to g(x,s), \text{ as } n \text{ tend to infinity, for all } x \in \Omega \text{ and all } s > 0.$$

We consider now the operator defined by

$$h_n(x,s,\zeta) = G_n(x,s,\zeta) + A_n(x,s).$$

This operator is bounded. Indeed, $g_n(x,s)$, $a(x)$ and $\frac{|\zeta|^2}{1+\frac{1}{n}|\zeta|^2}$ are bounded.

Then we have the following problem

$$-div(\nabla u - \Theta) + h_n(x,u,\nabla u) = f(x) \text{ in } \Omega. \tag{3.1}$$

By classical results (see for example [24]) there exists a solution u_n in $H_0^1(\Omega)$ of this problem, in the sense that

$$\int_\Omega (\nabla u - \Theta).\nabla\vartheta + \int_\Omega h_n(x,u,\nabla u)\vartheta = \int_\Omega f\,\vartheta,$$

for any test function ϑ in $H_0^1(\Omega)$.

Let us now consider the function $G_k(s) = s - T_k(s)$ and we teste the approximating problem by $G_k(u_n)$ we obtain

$$(1-\beta)\int_\Omega |\nabla G_k(u_n)|^2 + \int_\Omega [g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}G_k(u_n) + \beta|\nabla G_k(u_n)|^2] \le \int_\Omega f\,G_k(u_n) + \int_\Omega \Theta.\nabla\,G_k(u_n).$$

Denoting by $ß_{k,n}$ the set

$$ß_{k,n} = \{|u_n| \ge k\}.$$

From Young inequality, on has

$$\int_\Omega \Theta.\nabla G_k(u_n) \leq c\int_{ß_{k,n}} |\Theta|^{2'} - \beta\int_\Omega |\nabla G_k(u_n)|^2.$$

Using the fact that $|\Theta|$ belongs to $L^q(\Omega)$, with $q > N > 2$, we have by Hölder inequality that

$$\int_{ß_{k,n}} |\Theta|^{2'} \leq ||\Theta||_{(L^q(\Omega))^N} \, |ß_{k,n}|^{1-\frac{2}{q}},$$

where $|ß_{k,n}|$ denote the measure of $ß_{k,n}$. Using the Sobolev embedding, on obtain

$$\int_\Omega |\nabla G_k(u_n)|^2 \geq c\,(\int_\Omega |\nabla G_k(u_n)|^{2^*})^{\frac{2}{2^*}}.$$

It follows that

$$(\int_\Omega |\nabla G_k(u_n)|^{2^*})^{\frac{2}{2^*}} \leq c|ß_{k,n}|^{1-\frac{2}{q}}.$$

Using the fact that $|G_k(u_n)| \geq h-k$ on $ß_{h,n}$, for h such that $h > k$, on obtain

$$(h-k)^2|ß_{h,n}|^{\frac{2}{q}} \leq c\,|ß_{k,n}|^{1-\frac{2}{q}}, \quad \text{for all } h > k \geq \sigma.$$

Then on has

$$|ß_{h,n}| \leq \frac{c}{(h-k)^2}|ß_{k,n}|^{\frac{2^*}{2}(1-\frac{2}{q})} \quad \text{for all } h > k \geq \sigma.$$

By a well-known result of G. Stampacchia on has, there exist a constant c such that

$$|ß_{k,n}| = 0, \text{ for all } k \geq c+\sigma.$$

Which means that

$$||u_n||_\infty \leq c. \tag{3.2}$$

Then, the proof of the first theorem is concluded. In deed, it is possible to extract a subsequence which converges strongly in $H_0^1(\Omega)$ to a solution u of (2.3).

To prove the next result, for $f \in L^m(\Omega)$ with $m > N/2$ and $\Theta \in (L^q(\Omega))^N$, $q > N$, we consider two sequences $f_n \subset L^\infty(\Omega)$ and $\Theta_n \subset (L^\infty(\Omega))^N$ such that

$$f_n \to f \text{ strongly in } L^m(\Omega),$$

and

$$\Theta_n \to \Theta \text{ strongly in } (L^q(\Omega))^N.$$

The following mapping

$$H_n : H_0^1(\Omega) \to L^m(\Omega),$$

is defined by

$$H_n(u) = \Upsilon_n(x,u,\nabla u) + F_n.$$

Where

$$\Upsilon_n(x,u,\nabla u) = -a_n(x,u)\frac{|u|^{r-1}}{1+\frac{1}{n}|u|^{r-1}} - g_n(x,u)\frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2}),$$

and

$$F_n = f_n(x) - div(\Theta_n).$$

Denoting

$$T_k^1 = \int_\Omega \left|\left[g_n(x,u_k)\frac{|\nabla u_k|^2}{1+\frac{1}{n}|\nabla u_k|^2} - g_n(x,u)\frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2}\right]\right|^m,$$

and

$$T_k^2 = \int_\Omega \left|\left[a_n(x,u)\frac{|u_k|^{r-1}}{1+\frac{1}{n}|u_k|^{r-1}} - a_n(x,u)\frac{|u|^{r-1}}{1+\frac{1}{n}|u|^{r-1}}\right]\right|^m.$$

As an application of the dominated convergence theorem, if $u_k \to u$ in $H_0^1(\Omega)$ we infer, from the convergence $u_k \to u$ and $\nabla u_k \to \nabla u$ a.e. $x \in \Omega$ and the boundedness of the operators $g_n(x,u)$, $a_n(x,u)$ that

$$\lim_{k\to\infty} T_k^1 = 0,$$

and

$$\lim_{k\to\infty} T_k^2 = 0.$$

It follows that

$$\Upsilon_n(x,u_k,\nabla u_k) - \Upsilon_n(x,u,\nabla u) \to 0, \quad \text{as } k \text{ tend to infinity.}$$

Then,

$$H_n\, u_k \to H_n\, u \ \text{ in } \ L^m(\Omega), \text{ as } k \text{ tend to infinity.}$$

Therefor, H_n is continuous.

We consider now the operator

$$(-\Delta)^{-1} : L^m(\Omega) \to H_0^1(\Omega).$$

Then, the solutions of (3.1) are the fixed points of the composition operator

$$\Gamma_n \equiv (-\Delta)^{-1} \circ H_n.$$

Since $m > N/2$, we deduce that the operator $(-\Delta)^{-1}$ is compact and hence the composition of it with the continuous operator H_n i.e. that T_n is also compact.

let us now observe that,

$$\begin{aligned}||H_n\, u||_m \le ||g_n(x,u)\frac{|\nabla u|^2}{1+\frac{1}{n}|\nabla u|^2}||_m + ||a_n(x,u)\frac{|u|^{r-1}}{1+\frac{1}{n}|u|^{r-1}}||_m \\ + ||f||_m + ||\Theta||_{(L^q(\Omega))^N} \\ \le ||f||_m + ||\Theta||_{(L^q(\Omega))^N} + 2n^2|\Omega|.\end{aligned}$$

By the continuity of $(-\Delta)^{-1}$, this implies that there exist $R > 0$ such that

$$||\Gamma_n\, u||_{H_0^1(\Omega)} \le R.$$

So that, The operator Γ_n maps the ball in $H_0^1(\Omega)$ centrad at zero and with radius R into it self. Finally from the Schauder fixed point theorem there exists a fixed point $u_n \in H_0^1(\Omega)$ of T_n.

Denoting by $2^* = \frac{2N}{N-2}$ and by λ the Sobolev constant:

$$\lambda = \inf_{w\in H_0^1(\Omega)-\{0\}} \frac{||w||^2}{||w||_{2^*}^2}.$$

Next, proving that there exist a constant $c > 0$ such that

$$\int_\Omega |u_n|^r + ||\nabla u_n||^2 \le c\,||f||_m\,||u_n||_{m'} + c||\Theta||_{(L^q(\Omega))^N}, \tag{3.3}$$

where $m\prime$ is the conjugate exponent of m ($m\prime = \frac{m}{m-1}$).

We denote now by

$$\gamma = 2^*\left[\frac{1}{(2^*)'} - \frac{1}{q}\right],$$

and

$$\eta = \lambda^{-2^*} 2^{2^*\frac{\gamma}{\gamma-1}}\, \lambda^{-2^*}(1-\beta)^{-2^*}||f||_m^{2^*}|\Omega|^{\gamma-1}. \tag{3.4}$$

Let us note that, there exist $\kappa > 1$ such that

$$sg_n(x,s) \le \kappa, \ \forall s \in]0,\eta], \ \text{a.e}\, x \in \Omega, \tag{3.5}$$

and from (2.1) that

$$sg_n(x,s)+\beta \geq 0,\ \text{a.e}\, x\in\Omega,\ \forall s\in\mathbb{R}, \tag{3.6}$$

for every n.

From the approximated problem it follows that

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 + \int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n \\ + \int_\Omega a_n(x,u_n)\frac{|u_n|^{r-1}}{1+\frac{1}{n}|u_n|^{r-1}}u_n \\ \leq \int_\Omega f_{\ n} u_n + \int_\Omega \Theta.\nabla u_n.\end{aligned}$$

Which is equivalent to

$$\begin{aligned}(1-\beta)\int_\Omega |\nabla u_n|^2 + \int_\Omega [g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n + \beta|\nabla u_n|^2] \\ + \int_\Omega a_n(x,u_n)\frac{|u_n|^{r-1}}{1+\frac{1}{n}|u_n|^{r-1}}u_n \\ \leq \int_\Omega f_n\, u_n + \int_\Omega \Theta.\nabla u_n.\end{aligned}$$

Using Young and Hlder inequality as above, on obtain

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 + \int_\Omega [g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n + \beta|\nabla u_n|^2] \\ + \int_\Omega a_n(x,u_n)\frac{|u_n|^{r-1}}{1+\frac{1}{n}|u_n|^{r-1}}u_n \\ \leq c\,||f||_m\,||u_n||_{m'} + c||\Theta||_{(L^q(\Omega))^N}.\end{aligned}$$

Taking a conte to (3.6) on has

$$u_n g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} + \beta\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \geq 0,\ a.e\, x\in\Omega.$$

By consequence

$$u_n g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} + \beta|\nabla u_n|^2 \geq 0.$$

From the estimate (3.3) and by the use of Sobolev embedding theorem the sequence u_n is bounded in $H_0^1(\Omega)\cap L^r(\Omega)$. Then there exist a function $u\in H_0^1(\Omega)\cap L^r(\Omega)$ and a subsequence, still denoted by u_n, such that

$$u_n \to u \text{ weakly in } H_0^1(\Omega), \tag{3.7}$$

$$u_n \to u \text{ almost every where in } \Omega\ , \tag{3.8}$$

and

$$a_n(x,u_n) \to a(x,u) \text{ almost every where in } \Omega. \tag{3.9}$$

From the construction of f_n and $\Theta_n$we have for n tending to infinity

$$\Theta_n \to \Theta \ \text{ in }\ (L^q(\Omega))^N, \tag{3.10}$$

and

$$f_n \to f \quad \text{in } L^1(\Omega). \tag{3.11}$$

Taking into account the equi-integrability of u_n in $L^r(\Omega)$, it follows that of $a_n(x,u_n)|u_n|^{r-1}$ in $L^1(\Omega)$. Hence, we have

$$a_n(x,u_n)|u_n|^{r-1} \to a(x,u)|u|^{r-1} \text{ in } L^1(\Omega). \tag{3.12}$$

Since on has up to a subsequence u_n, that

$$\nabla u_n \to \nabla u \text{ almost every where in } \Omega, \tag{3.13}$$

and ∇u_n is bounded in $L^2(\Omega)$, then on has

$$\nabla u_n \to \nabla u \text{ in } L^2(\Omega).$$

We conclude that

$$\Delta u_n \to \Delta u \text{ in } L^1(\Omega).$$

Considering now the following function

$$\phi = \psi_\mu(u_n - u)\varphi,$$

where φ is a positive function in $C_0^\infty(\Omega)$ and

$$\psi_\mu(s) = s\, e^{\mu s^2}, \ \mu \text{ is a positive constant.}$$

We have

$$\psi'_\mu(s) - c|\psi_\mu(s)| = e^{\mu s^2}[1 + 2\mu s^2 - c|s|].$$

Then for large value of μ we have

$$e^{\mu s^2}[1 + 2\mu s^2 - c|s|] \geq \frac{1}{2}, \ \forall s \in \mathbb{R}. \tag{3.14}$$

Testing the approximated problem by $\phi = \psi_\mu(u_n - u)\varphi$ we obtain

$$\int_\Omega \nabla u_n \nabla(u_n - u)\psi'_\mu(u_n - u)\varphi + \int_\Omega \nabla u_n \nabla\varphi\, \psi_\mu(u_n - u) + I_{1n} \leq \int_\Omega f\, \psi_\mu(u_n - u)\varphi + I_{2n} + I_{3n},$$

where

$$I_{1,n} = \int_\Omega g_n(x,u_n)\frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2}\psi_\mu(u_n - u)\varphi,$$

$$I_{2,n} = \int_\Omega \Theta_n\,.\nabla(u_n - u)\psi'_\mu(u_n - u)\varphi,$$

and

$$I_{3,n} = \int_\Omega \Theta_n.\nabla\varphi\, \psi_\mu(u_n - u).$$

It follows

$$\int_\Omega |\nabla(u_n - u)|^2\psi'_\mu(u_n - u)\varphi + \int_\Omega \nabla u\, \nabla(u_n - u)\psi'_\mu(u_n - u)\varphi + \int_\Omega \nabla u_n \nabla\varphi\, \psi_\mu(u_n - u) + I_{1,n} \leq \int_\Omega f_n\, \psi_\mu(u_n - u)\varphi + I_{2n} + I_{3n}.$$

Since on has

$$g_n(x,u_n)\frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2}\psi_\mu(u_n - u)\varphi \geq -c|\nabla u_n|^2|\psi_\mu(u_n - u)|\varphi, \ \text{ a.e. } x \in \Omega.$$

Then

$$I_{1,n} \geq -c|\nabla u_n|^2 |\psi_\mu(u_n - u)|\varphi, \text{ a.e. } x \in \Omega.$$

It follows, that

$$\begin{aligned}\int_\Omega |\nabla(u_n - u)|^2 \psi'_\mu(u_n - u)\varphi - c|\nabla u_n|^2 |\psi_\mu(u_n - u)|\varphi \\ \leq -\int_\Omega \nabla u \, \nabla(u_n - u)\psi'_\mu(u_n - u)\varphi - \int_\Omega \nabla u_n \nabla\varphi \, \psi_\mu(u_n - u) \\ + \int_\Omega f \, \psi_\mu(u_n - u)\varphi + I_{2n} + I_{3n}.\end{aligned}$$

Observing that

$$\begin{aligned}\int_\Omega |\nabla(u_n - u)|^2 \psi_\mu(u_n - u)\varphi = \int_\Omega |\nabla u_n|^2 |\psi_\mu(u_n - u)|\varphi \\ + \int_\Omega |\nabla u|^2 |\psi_\mu(u_n - u)|\varphi \\ - 2\int_\Omega \nabla u_n \nabla u| \, \psi_\mu(u_n - u)|\varphi,\end{aligned}$$

and taking a count to (3.14) we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla(u_n - u)|^2 \varphi \leq -c\int_\Omega |\nabla u|^2 |\psi_\mu(u_n - u)|\varphi \\ + 2c\int_\Omega \nabla u_n \nabla u| \, \psi_\mu(u_n - u)|\varphi \\ - \int_\Omega \nabla u \, \nabla(u_n - u)\psi'_\mu(u_n - u)\varphi \\ - \int_\Omega \nabla u_n \nabla\varphi \, \psi_\mu(u_n - u) \\ + \int_\Omega f \, \psi_\mu(u_n - u)\varphi.\end{aligned}$$

In this stage, we will use the weak* topology of $L^\infty(\Omega)$ and the almost every where convergence in Ω, for n tending to infinity on has

$$\psi'_\mu(u_n - u) \to 0,$$

and

$$\psi_\mu(u_n - u) \to 0.$$

Since, f_n is strongly compact in $L^1(\Omega)$, then

$$\int_\Omega f_n \, \psi_\mu(u_n - u) \to 0.$$

Since, u_n converges to u weakly in $H_0^1(\Omega)$, and is strongly compact in $(L^q(\Omega))^N$, then for n tending to infinity on has

$$I_{1,n} \to 0,$$

and

$$I_{2,n} \to 0.$$

Using the dominated convergence theorem, we obtain for n tending to $+\infty$

$$\int_\Omega |\nabla(u_n - u)|^2 \varphi \to 0.$$

Since this convergence is satisfied for all $\varphi \in C_0^\infty(\Omega)$. We cane deduce now that there exist $h \in L^2(\Omega)$ such that up to a subsequence, on has

$$|\nabla u_n(x)| \leq h(x), \quad \text{a.e. } x \in \Omega, \tag{3.15}$$

and

$$\nabla u_n(x) \to \nabla u(x), \quad \text{a.e. } x \in \Omega.$$

From (3.5) and (3.6) we conclude that fore some $c > 0$ we have

$$|g_n(x, u_n)| \leq c \quad \text{a.e. } x \in \Omega.$$

Taking a count to (3.15) we obtain

$$|g_n(x, u_n)| \frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} \leq ch^2 \text{ a.e. } x \in \Omega.$$

Using the definition of $g_n(x, u_n)$, and passing now to the limit on n, we obtain

$$g_n(x, u_n) \frac{|\nabla u_n|^2}{1 + \frac{1}{n}|\nabla u_n|^2} \to g(x, u)|\nabla u|^2 \quad \text{a.e. } x \in \Omega.$$

Then

$$G_n(x, u_n, \nabla u_n) \to G(x, u, \nabla u) \quad \text{a.e. } x \in \Omega.$$

Finally, the dominated convergence theorem yields to

$$\int_\Omega G_n(x, u_n, \nabla u_n)\phi \to \int_\Omega G(x, u, \nabla u)\phi.$$

Then, on has

$$\int_\Omega h_n(x, u_n, \nabla u_n)\phi \to \int_\Omega h(x, u, \nabla u)\phi.$$

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