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# A Hermite-Hadamard type inequality for multiplicatively convex functions

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Abstract. In this paper we discuss an analogue of the Hermite-Hadamard inequality for multiplicatively convex functions.

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The purpose of this paper is to discuss an analogue of the Hermite-Hadamard inequality for the multiplicatively convex functions. Our results improve the recent paper of X. Zhang and N. Zheng [2, Theorem 1].

Recall that a positive function $f$ defined on a subinterval $I$ of $(0,+\infty)$ is called multiplicatively convex if

$$f\left(x^{1-\alpha}y^{\alpha}\right) \leq f\left(x\right)^{1-\alpha} f\left(y\right)^{\alpha} \tag{$MC$}$$

for every $x, y \in I$ and every $\alpha \in [0,1]$.

Examples of such functions are exp, sinh, cosh, Gamma, Lobacevski's function and the logarithmic integral (each on an appropriate interval). See the monograph of C. P. Niculescu and L.-E. Persson [1, pp. 83-85] for details.

The theory of multiplicatively convex functions can be deduced from the theory of usual convex functions. Indeed, a function $f$ is multiplicatively convex if and only if $\log \circ f \circ \exp$ is convex. See [1, Lemma 2.1.1, pp. 67]. This lemma allows us to translate easily results functioning for convex functions into their counterparts for multiplicatively convex functions. An illustration of our assertion is offered by the case of classical Hermite - Hadamard inequality:

**Theorem 0.1.** *If $\mu$ is a Borel probability measure on an interval $[a,b]$ with barycenter*

$$b_{\mu} = \int_a^b x \mathrm{d}\mu\left(x\right),$$

*then for every continuous convex function $f : [a,b] \rightarrow \mathbb{R}$,*

$$f\left(b_{\mu}\right) \leq \int_a^b f\left(x\right) \mathrm{d}\mu\left(x\right) \leq \frac{b-b_{\mu}}{b-a} f\left(a\right) + \frac{b_{\mu}-a}{b-a} f\left(b\right). \tag{$HH$}$$

See [1], Corollary 4.4.4, pp. 195-196, for details.

As a consequence we obtain:

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**Corollary 0.1.** *Suppose that $f : I \to (0, +\infty)$ is a multiplicatively convex function. Then for every compact subinterval $[a, b]$ of $I$ and for every Borel probability measure $\mu$ on $[a, b]$ we have*

$$f(\xi) \le \exp \int_a^b \log f(x)\, \mathrm{d}\mu(x) \le f(a)^{\frac{\mathcal{M}(\xi)-a}{b-a}}\, f(b)^{\frac{b-\mathcal{M}(\xi)}{b-a}}. \qquad (MHH)$$

*Here $\xi = \exp \int_a^b \log x \mathrm{d}\mu(x)$ is the identric mean of $\mu$, and $\mathcal{M}(\xi) = \frac{b \log \frac{b}{\xi} - a \log \frac{a}{\xi}}{\log b - \log a}$.*

Notice that Corollary 0.1 (unlike Theorem 0.1) is a statement about the integral geometric mean.

*Proof.* We will need the change of variable $\exp : [\log a, \log b] \to [a, b]$ and the Borel probability measure $\nu$ on $[\log a, \log b]$ given by

$$\nu(A) = \mu(\{e^a : a \in A\}) \text{ for every } A \subset [\log a, \log b].$$

Clearly,

$$\mu(B) = \nu(\{\log b : b \in B\}) \text{ for every } B \subset [a, b].$$

We have

$$\int_a^b \log f(x)\, \mathrm{d}\mu(x) = \int_{\log a}^{\log b} \log f(\exp t)\, \mathrm{d}\nu(t).$$

According to Theorem 0.1,

$$\begin{aligned} \log f(b_\nu) &\le \int_{\log a}^{\log b} \log f(\exp t)\, \mathrm{d}\nu(t) \\ &\le \frac{\log b - b_\nu}{\log b - \log a} \log f(a) + \frac{b_\nu - \log a}{\log b - \log a} \log f(b), \end{aligned}$$

where

$$b_\nu = \int_{\log a}^{\log b} t \mathrm{d}\nu(t) = \int_a^b \log x \mathrm{d}\mu(x)$$

represents the barycenter of $\nu$. Thus

$$b_\nu = \log \xi,$$

where $\xi = \exp \int_a^b \log x d\mu(x)$ is the identric mean of $\mu$.

The proof ends by noticing that

$$\begin{aligned} \exp \int_a^b \log f(x)\, \mathrm{d}\mu(x) &\le \exp\left( \frac{\log b - \log \xi}{\log b - \log a} \log f(a) + \frac{\log \xi - \log a}{\log b - \log a} \log f(b) \right) \\ &= f(a)^{\frac{\mathcal{M}(\xi)-a}{b-a}}\, f(b)^{\frac{b-\mathcal{M}(\xi)}{b-a}}. \end{aligned}$$

due to the identity

$$\frac{\log \xi - \log a}{\log b - \log a} = \frac{b - \mathcal{M}(\xi)}{b - a}. \qquad (Id)$$

□

**Remark 0.1.** *In the particular case where $\mu = \frac{dx}{b-a}$, the identric mean is*

$$\xi = I(a, b) = \frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/(b-a)}$$

*and $\mathcal{M}(\xi)$ coincides with the logarithmic mean*

$$L(a, b) = \frac{b - a}{\log b - \log a}.$$

Coming back to a remark above concerning the relationship between the convex functions and the multiplicatively convex functions, let us notice that the multiplicatively convex functions $f$ admit finite lateral derivatives at each interior point $z$ and

$$f'_{-}(z) \leq f'_{+}(z).$$

The theory of support lines for usual convex functions reads in the multiplicative context as follows: if $f : [a, b] \to (0, +\infty)$ is multiplicatively convex and $z \in (a, b)$, then there exists a real number $\lambda$ such that

$$f(t) \geq f(z) \left(\frac{t}{z}\right)^{\lambda} \tag{CS}$$

for all $t \in [a, b]$. We will call the function $t \to f(z)\left(\frac{t}{z}\right)^{\lambda}$ a *support* of $f$ at $z$. The discussion above shows that $\lambda$ can be any number such that

$$\frac{zf'_{-}(z)}{f(z)} \leq \lambda \leq \frac{zf'_{+}(z)}{f(z)}.$$

A multiplicatively convex function admits a support even at an endpoint provided the corresponding lateral derivative exists and is finite.

By integrating the inequality $(CS)$ we obtain the following result.

**Lemma 0.1.** *Suppose that $f : [a, b] \subseteq (0, +\infty) \to (0, +\infty)$ is a multiplicatively convex function, $z$ is a point in $[a, b]$ and $\mu$ is a Borel probability measure on $[a, b]$. Then*

$$\int_a^b f(x)\,\mathrm{d}\mu(x) \geq f(z) \int_a^b \left(\frac{x}{z}\right)^{\frac{zf'_{+}(z)}{f(z)}} \mathrm{d}\mu(x)$$

*if $f'_{+}(z)$ exists and has a finite value, and*

$$\int_a^b f(x)\,\mathrm{d}\mu(x) \geq f(z) \int_a^b \left(\frac{x}{z}\right)^{\frac{zf'_{-}(z)}{f(z)}} \mathrm{d}\mu(x)$$

*if $f'_{-}(z)$ exists and has a finite value.*

X. Zhang and N. Zheng [2, Theorem 1] have proved Lemma 0.1 in the particular case where $\mathrm{d}\mu(x) = \frac{1}{b-a}\mathrm{d}x$ and $z$ is one of the endpoints of $[a, b]$. However their argument is unnecessarily complicated.

We end our paper by outlining the connection between Lemma 0.1 and Corollary 0.1:

**Theorem 0.2.** *Suppose that $f : [a, b] \subset (0, +\infty) \to (0, +\infty)$ is a multiplicatively convex function and $\mu$ is a Borel probability measure on $[a, b]$. Then*

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}\mu(x) &\geq f\left(\exp \int_a^b \log x \mathrm{d}\mu(x)\right) \\ &= \sup_{\Phi \text{ is a support of } f} \left\{\exp \int_a^b \log \Phi(x) \mathrm{d}\mu(x)\right\}. \end{aligned}$$

*Proof.* The first inequality is motivated by the arithmetic mean-geometric mean inequality and the left hand side of formula $(MHH)$.

In order to prove the equality in our statement, put $g = \log \circ f \circ \exp$, $\nu = \log \#\mu$ and consider the same change of variable and the same corresponding measure $\nu$ as in the proof of Corollary 0.1. The barycenter of $\nu$ is

$$b_\nu = \int_{\log a}^{\log b} t\mathrm{d}\nu(t) \in [\log a, \log b]$$

and we may encounter two cases.

*Case 1:* $b_\nu \in (\log a, \log b)$. Since $g$ is convex,

$$\sup_{\Psi \text{ is a support line of } g} \left\{ \int_{\log a}^{\log b} \Psi(t) \mathrm{d}\nu\,(t) \right\} = \sup_{\Psi \text{ is a support line of } g} \{\Psi\,(b_\nu)\} = g\,(b_\nu)\,.$$

Since $b_\nu = \int_a^b \log x \mathrm{d}\mu\,(x)$, this yields

$$\log f \left( \exp \int_a^b \log x \mathrm{d}\mu\,(x) \right) = \sup_{\Phi \text{ is a support of } f} \left\{ \int_a^b \log \Phi(x) \mathrm{d}\mu\,(x) \right\}.$$

*Case* 2 : $b_\nu$ is an endpoint, say $b_\nu = \log a$. We have to prove that

$$g\,(\log a) = \sup_{\substack{y \in (\log a, \log b) \\ g'_-(y) \leq \psi(y) \leq g'_+(y)}} \{g\,(y) + \psi\,(y)\,(\log a - y)\}\,. \tag{$SD$}$$

We notice that

$$g(\log a + t) - g(\log a) \leq t\psi(\log a + t) \leq g(\log a + 2t) - g(\log a + t)$$

for $t > 0$ small enough, which yields $\lim_{t\to 0+} t\psi(\log a + t) = 0$. Given $\varepsilon > 0$, there is $\delta > 0$ such that $|g(\log a + t) - g(\log a)| < \varepsilon/2$ and $|t\psi(\log a + t)| < \varepsilon/2$ for $0 < t < \delta$. This yields $g(\log a + t) - t\psi(\log a + t) < g(\log a) + \varepsilon$ for $0 < t < \delta$ and the inequality $(SD)$ follows.

This concludes the proof. □

## References


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[2] X. Zhang, N. Zheng, *Geometrically convex functions and estimation of remainder terms for Taylor expansion of some functions*, J. Math. Inequal., **4** (2010), No. **1**, 15–25.



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