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On Volterra's Population Growth Models

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Abstract. A class of population growth problems is considered. We consider the case when the problems employ delay kernels reflecting the presence of some instantaneous effect on growth rate response, with delayed maximum effect. Under appropriate assumptions on the data of the problems, we construct an analytical solution with the help of an optimal homotopy analysis approach. This optimal approach contains a convergence-control parameter which can be estimated by minimizing the square residual error. Finally, numerical examples are presented that illustrate the approach efficiency.

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1. Introduction

Assume that we have n species whose populations are denoted by

$$N_1, N_2, \dots, N_N$$

where their coefficients of increase are called $\epsilon_1, \epsilon_2, \dots, \epsilon_n$. Taking these constants as positive or negative according to whether the species tend to increase or to die out when not interfered with, we will have the following equations expressing the variations of the populations:

$$\frac{dN_i}{dt} = \epsilon_i N_i \ , \qquad i = 1, 2, \dots, n. \tag{1}$$

By t we denote the actual instant and by τ a preceding instant. The number of individuals of the species s at time τ will be $N_s(\tau)$. Let us suppose that the species s exercises over the coefficient of increase of the species r an action which will be manifested in the future and which varies with the distance in time. We shall denote such a (unitary) action by $F_{sr}(t-\tau)$ when it is exercised by the species s in the infinitesimal interval of time $(\tau, \tau + d\tau)$ and is manifested on the species r at time t. Then the action corresponding to the population $N_s(\tau)$ will be

$$N_s(\tau) F_{sr}(t-\tau) d\tau.$$

If we take into account all these actions beginning from the origin of times at which they are supposed to have begun, up to the present moment t we shall have

$$\int_0^t N_s(\tau) F_{sr}(t-\tau) d\tau.$$

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Considering historical actions for all n species on the coefficient of increase of the species r, we will have

$$\sum_{s=1}^{n} \int_{0}^{t} N_s(\tau)F_{sr}(t-\tau)d\tau.$$

The coefficient of increase of the species r, taking into account all immediate and historical actions exercised upon it, will therefore become

$$\epsilon_r + \sum_{s=1}^{n} A_{sr}N_s(\tau) \int_{0}^{t} N_s(\tau)F_{sr}(t-\tau)d\tau. \tag{2}$$

Here, the coefficient A_{sr} measures that unitary action (per individual) which the species s exercises upon the species r, while A_{rs} denotes the inverse action that species r exercises upon the species s. In this line, (1) and (2) gives

$$\frac{dN_r}{dt} = \left(\epsilon_r + \sum_{s=1}^{n} A_{sr}N_s(\tau) \int_{0}^{t} N_s(\tau)F_{sr}(t-\tau)d\tau\right) N_r(t). \tag{3}$$

We may suppose that the historical actions may be prolonged indefinitely in the past, and then the equations (3) must be replaced by the following:

$$\frac{dN_r}{dt} = \left(\epsilon_r + \sum_{s=1}^{n} A_{sr}N_s(\tau) \int_{-\infty}^{t} N_s(\tau)F_{sr}(t-\tau)d\tau\right) N_r(t).$$

In this way we will have nonlinear Volterra integro-differential equations (VIDE's) with infinite delay,

$$y'(t) = f(t, y(t)) + \int_{-\infty}^{t} K(t, s, y(t), y(s))ds, \qquad t \in [0, T], \tag{4}$$

where on $(-\infty, 0]$ the solution y is agree with a given initial function φ:

$$y(t) = \varphi(t), \qquad\qquad t \leq 0. \tag{5}$$

There are many important applications of the Volterra's population growth models (4)-(5), for more details see [**?**, 13, 14].

We are interested in this work with kernel functions in (4) of the form

$$K(t, s, y, z) = a(t-s).y.z, \tag{6}$$

and

$$f(t, y) = y.(a_0 - a_1 y), \qquad \text{with } \ a_0, a_1 > 0. \tag{7}$$

Many models of population growth employ delay kernels reflecting the presence of some instantaneous effect on growth rate response, with delayed maximum effect, i.e.,

$$a(t) = -(\frac{\gamma_0}{b} + \frac{\gamma_1}{b^2}t)e^{-\frac{t}{b}}, \tag{8}$$

with $\gamma_0 + \gamma_1 = 1$, $\gamma_1 > \gamma_0 \geq 0$, $b > 0$. This function $a(t)$ attains its maximum at $t = b(\gamma_1 - \gamma_0)/\gamma_1$, and we have $\int_0^\infty |a(t)|\, dt = 1$. We cite the following result concerning Volterra's population equations (4)-(7).

Theorem 1.1 (Miller [15]). *Suppose that $a_0 > 0, a_1 > 0$, and let $a \in C[0, \infty) \cap L^1[0, \infty)$, with $a(t) \neq 0$, satisfy*

$$a_1 - \int_0^\infty |a(s)|\, ds > 0.$$

Then for any positive, continuos, bounded function $\varphi(t)$, $t \leq 0$, the problem (4)-(5), with K and f given by (6) and (7), respectively, has a unique solution $y \in C^1[0,\infty)$. This solution satisfies $y(t) > 0$ for all $t > 0$, and we have

$$y(\infty) = \lim_{t\to\infty} y(t) = a_0 \Big/ \left(a_1 - \int_0^\infty a(s)ds\right).$$

In this paper, an optimal homotopy analysis approach (HAM) [1, 2, 5, 6, 10, 11, 12, 13, 14] is applied to study the population growth model (4)-(5), with K and f given by (6)-(7). The results contained herein are new for such models.

2. Homotopy solution of the Volterra's population model

One can express $y(t)$ by such a set of base functions

$$\{t^m e^{-n\lambda t} \mid m, n = 0, 1, 2, \ldots, \lambda > 0\} \tag{9}$$

that

$$y(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha_{m,n} t^m e^{-n\lambda t}, \tag{10}$$

where $\alpha_{m,n}$ is a coefficient. This provides us with the *Rule of Solution Expression.*

Considering the *Rule of Solution Expression* described by (9) and taking into account the solution property at infinity given in Theorem 1, it is obvious that

$$y_0(t) = \kappa + (1-\kappa)e^{-\lambda t} \tag{11}$$

where

$$\kappa = a_0 \Big/ \left(a_1 - \int_0^\infty a(s)ds\right)$$

is a good initial guess of $y(t)$. One chooses such an auxiliary linear operator

$$\mathcal{L}[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t} + \lambda\phi(t;q) \tag{12}$$

that

$$\mathcal{L}[Ce^{-\lambda t}] = 0$$

where C is a coefficient.

Due to (4), one defines the non-linear operator

$$N[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t} - f(t, \phi(t;q)) - \int_{-\infty}^{t} K(t, s, \phi(t;q), \phi(s;q))ds. \tag{13}$$

Let $q \in [0,1]$ be the embedding parameter and $\hbar$ a non-zero auxiliary parameter. One constructs such a homotopy

$$H[\phi(t;q);\hbar, q] = (1-q)\mathcal{L}[\phi(t;q) - y_0(t)] - q\hbar N[\phi(t;q)]. \tag{14}$$

Setting $H[\phi(t;q);\hbar, q] = 0$, one has a family of equations

$$(1-q)\mathcal{L}[\phi(t;q) - y_0(t)] = q\hbar N[\phi(t;q)], \tag{15}$$

subject to the initial condition

$$\phi(t;q) = \varphi(t), \qquad t \leq 0.$$

Obviously, when $q = 0$, because of the property $\mathcal{L}(0) = 0$ of any linear operator $\mathcal{L}$, Eq. (15) has the solution

$$\phi(t;0) = y_0(t), \tag{16}$$

and when $q = 1$, since $\hbar \neq 0$, Eq. (15) is equivalent to the original one (4), provided

$$\phi(t;1) = y(t). \tag{17}$$

Thus, according to (16) and (17), as the embedding parameter q increases from 0 to 1, $\phi(t;q)$ varies continuously from the initial approximation $y_0(t)$ to the exact solution $y(t)$. This kind of deformation $\phi(t;q)$ is totally determined by the so-called zeroth-order deformation equation (15).

Assume that $\phi(t;q)$ is analytic in $q \in [0,1]$ so that $\phi(t;q)$ can be expanded in Maclaurin's series of q as follows

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{18}$$

where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{19}$$

If the auxiliary linear operator $\mathcal{L}$ and the nonzero auxiliary parameter $\hbar$ are properly chosen so that the power series (18) of $\phi(t;q)$ converges at $q = 1$. Then, we have under these assumptions the the so-called homotopy-series solution

$$y(t) = \sum_{m=0}^{\infty} y_m(t). \tag{20}$$

The solution at nth-order approximation is given by

$$y(t) \approx \sum_{m=0}^{n} y_m(t).$$

Differentiating the zero-order deformation equation (15) m times with respect to q and then dividing by $m!$ and finally setting $q = 0$, we have the so-called high-order deformation equation

$$\mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar \Re_m(t), \tag{21}$$

subject to the initial conditions

$$y_m(0) = 0, \tag{22}$$

where

$$\Re_m(t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}|_{q=0} \tag{23}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}.$$

In this line we have that,

$$\begin{aligned} \Re_m(t) \quad &= \quad y'_{m-1}(t) - a_0 y_{m-1}(t) + a_1 \sum_{k=0}^{m-1} y_k y_{m-k-1} - \\ & y_{m-1}(t) \int_{-\infty}^{0} a(t-s)\varphi(s)ds - \sum_{k=0}^{m-1} y_k \int_0^t a(t-s) y_{m-k-1}(s) ds. \end{aligned} \tag{24}$$

Notice that when $m \geq 1$, the IVPs (21)-(22) are always linear and due to the properties of $a(t)$ mentioned at the end of the previous section, the integrals in (24) can usually be calculated analytically. So, it is easy to gain the solution

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar e^{-\lambda t} \int_0^t e^{\lambda\zeta} \Re_m(\zeta) d\zeta + C e^{-\lambda t}, \tag{25}$$

where the integral coefficients are determined by initial conditions (22).

In this way, it is easily to obtain $y_m(t)$ one by one in the order $m = 1, 2, 3, ...$ and we have the n-th order approximation

$$y(t) = y_0(t) + \sum_{m=1}^{n} \sum_{k=2}^{2m+3} \alpha_{m,k} t^k e^{-k\lambda t}. \tag{26}$$

When $n \to \infty$, we get an accurate approximation of the original problem (4)-(7).

In our approach λ is an optimal convergence-control parameter that can be used to accelerate the convergence of the homotopy-series solution (26). At the m^{th} order of approximation, one can define the square residual error

$$\Delta_m = \int_0^{+\infty} \left(N \left[\sum_{i=0}^{m} y_i(\xi) \right] \right)^2 d\xi.$$

Note that Δ_m contains λ as an unknown parameter. At a given order of approximation m, the optimal value of λ is given by the minimum of Δ_m, corresponding to the nonlinear algebraic equation

$$\frac{d\Delta_m}{d\lambda} = 0.$$

For the sake of simplicity, we will find an optimal λ by minimizing the square residual error of the governing equation for the initial guess $y_0(t)$:

$$\lambda = \min \Delta_0 = \min \int_0^{+\infty} \left(N \left[y_0(\xi) \right] \right)^2 d\xi. \tag{27}$$

3. Result Analysis

In the following numerical illustration the values of the parameters were chosen as

$$\gamma_0 = 0.05, \quad \gamma_1 = 0.95, \quad b = 1, \quad a_0 = 14, \quad a_1 = 1.1;$$

the initial function is $\varphi(t) = e^{\gamma_2 t}$, with $\gamma_2 = 0.5$.

By (11) we can easily obtain

$$y_0(t) = 20/3 - 17/3\, e^{-\lambda t},$$

and using (27), we are able to estimate the optimal value of λ. The curve of Δ_0 versus λ is shown in Fig. 1, which indicates that the optimal value of λ is about 0.5.

Note that we have great freedom to choose the value of the auxiliary parameter $\hbar$. Mathematically the value of $y(t)$ at any finite order of approximation is dependent upon the auxiliary parameter $\hbar$, because the zeroth and high-order deformation equations contain $\hbar$. The best value of the parameter $\hbar$ can be studied by investigating the convergence of the series (20) at $t = 2$. This can be done by plotting the so-called $\hbar$-curve [10] which takes a line segment, nearly parallel to the horizontal axis, through the position of convergence. We plot the $\hbar$-curve of $y(2)$ as shown in Fig. 2.

According to this $\hbar$-curve, it is easy to conclude that $-1.7 \leq \hbar \leq -0.8$ is the valid region of $\hbar$.

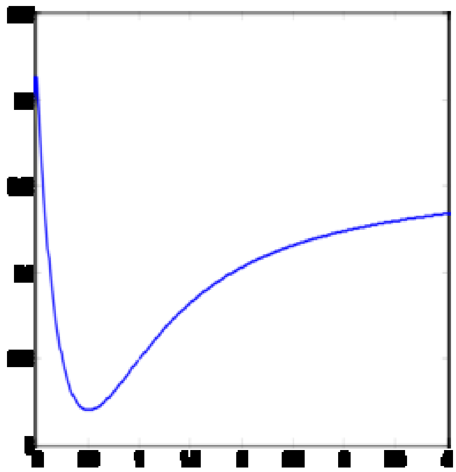

FIGURE 1. λ versus Δ_0

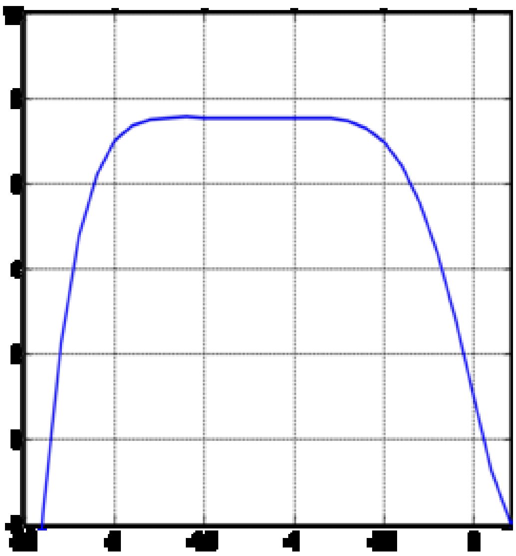

FIGURE 2. $y(2)$ versus $\hbar$ for 7th order of approximation

Some approximations of $y(2)$ are listed in Table 1, which show the convergence of the solution series (20) when $\hbar = -1.5$. The results in Table 1 agree well with that obtained by Brunner [6] using the collocation method.

A proper value of $\hbar = -1.5$ is taken and then the seven terms from the series solution expression by HAM is plotted in Fig. 3.

According to Fig. 3, a rise occurs along the solution curve, of the population growth model (4)-(7), from 1 at $t = 0$ to reach a peak about 11.67 near $t = 0.6$ and

TABLE 1. Approximation of $y(2)$ using 7th order of approximation and $\hbar = -1.5$

order of approximation	$y(2)$
2	7.4321
4	7.499
6	7.521
8	7.5681
10	7.5682
12	7.5682
14	7.5682

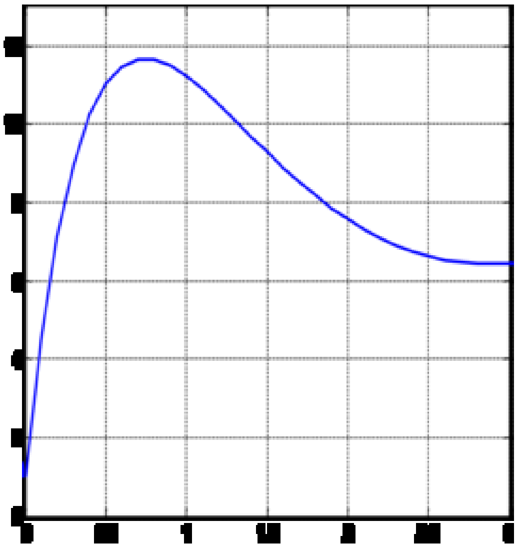

FIGURE 3. seven terms from the homotopy-series solution; $y_7(t)$

then tends to the asymptotic value $y(\infty) \approx 6.6$. This agree well with the theoretical results [15], and the numerical results obtained by the collocation method [8].

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