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Some Inequalities About Certain Arithmetic Functions

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Abstract. Let $\sigma_k^{(e)}(n)$ denote the sum of kth powers of the exponential divisors of $n, \tau^{(e)}(n)$ denote the number of the exponential divisors of $n, \sigma_k^{(e)*}(n)$ denote the sum of kth powers of the $e-$ unitary divisors of n and $\tau^{(e)*}(n)$ denote the number of the $e-$ unitary divisors of n. The purpose of this paper is to present several inequalities about the arithmetic functions $\sigma_k^{(e)}$, $\tau^{(e)}, \sigma_k^{(e)*}$, $\tau^{(e)*}$ and other well-known arithmetic functions. Among these, we have the following: $\sigma_k^{(e)}(n) \geq \gamma^k(n)\left[1^k + 2^k + ... + \left(\tau^{(e)}(n)\right)^k\right]$ and $\sigma_k(n) + n^k \geq \sigma_k^{(e)}(n) + \sigma_k^*(n)$, for any $n \geq 1$ and $k \geq 0$.

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1. Introduction

An interesting part of number theory is related to multiplicative arithmetic functions. Many inequalities between some of the functions are developed in many papers and several inequalities can be found in the papers [4], [5], [8] and [14].

Some functions use an important type of divisor, namely, *the exponential divisor* that was introduced by M. V. Subbarao in [13], thus: if $n > 1$ is an integer of canonical form $n = p_1^{a_1}p_2^{a_2}...p_r^{a_r}$, then the integer $d = \prod_{i=1}^{r} p_i^{b_i}$ is called an *exponential divisor* (or *$e-$divisor*) *of* $n = \prod_{i=1}^{r} p_i^{a_i} > 1$, if $b_i \geq 1$ and $b_i \mid a_i$ for every $i = \overline{1,r}$. We write $d \mid_{(e)} n$. We note with $\sigma_k^{(e)}(n)$ the sum of kth powers of the exponential divisors of n, so, $\sigma_k^{(e)}(n) = \sum_{d|_{(e)}n} d^k$, whence we obtain the following equalities: $\sigma_1^{(e)}(n) = \sigma^{(e)}(n)$ and $\sigma_0^{(e)}(n) = \tau^{(e)}(n) -$ the number of the exponential divisors of n.

A particular case of exponential divisor is "the exponential unitary divisors"or "e-unitary divisors"given in [15] by L. Tóth and N. Minculete, thus: an integer $d = \prod_{i=1}^{r} p_i^{b_i}$ is called a *$e-$unitary divisor* of $n = \prod_{i=1}^{r} p_i^{a_i} > 1$ if b_i is a unitary divisor of a_i, so $\left(b_i, \frac{a_i}{b_i}\right) = 1$, for every $i = \overline{1,r}$. Let $\sigma_k^{(e)*}(n)$ denote the sum of kth powers of the $e-$unitary divisors of n, and $\tau^{(e)*}(n)$ denote the number of the $e-$unitary divisors of n.

By convention, 1 is an exponential divisor of itself, so that $\sigma_k^{(e)*}(1) = \tau^{(e)*}(1) = 1$.

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We notice that 1 is not a $e-$unitary divisor of $n>1$, the smallest $e-$unitary divisor of $n=p_1^{a_1}p_2^{a_2}...p_r^{a_r}>1$ is $p_1p_2...p_r$, where $p_1p_2...p_r=\gamma(n)$ is called the "core" of n.

J. Fabrykowski and M. V. Subbarao in [1] study the maximal order and the average order of the multiplicative function $\sigma^{(e)}(n)$. E.G. Straus and M. V. Subbarao in [12] obtained several results concerning $e-$perfect numbers (n is an $e-$perfect number if $\sigma^{(e)}(n)=2n$).

In [6], it is shown that

$$\tau(n)+1\geq\tau^{(e)}(n)+\tau^{*}(n), \tag{1.1}$$

for all integers $n\geq 1$.

In [9], J. Sándor and L. Tóth proved the inequalities

$$\frac{n^k+1}{2}\geq\frac{\sigma_k^{*}(n)}{\tau^{*}(n)}\geq\sqrt{n^k}, \tag{1.2}$$

and

$$\frac{\sigma_{k+m}^{*}(n)}{\sigma_m^{*}(n)}\geq\sqrt{n^k}, \tag{1.3}$$

for all $n\geq 1$ and $k,m\geq 0$, real numbers, where $\tau^{*}(n)$ is the number of the unitary divisors of $n,\sigma_k^{*}(n)$ is the sum of kth powers of the unitary divisors of n.

We remark the following inequalities:

$$\tau^{(e)*}(n)\leq\tau^{(e)}(n)\leq\tau(n) \tag{1.4}$$

and

$$\sigma_k^{(e)*}(n)\leq\sigma_k^{(e)}(n)\leq\sigma_k(n), \tag{1.5}$$

for every $n\geq 1$.

Using the same proof from inequality (7) of [5], we deduce the inequality

$$\sigma_k^{(e)}(n)\leq n^k\prod_{i=1}^{r}(1+\frac{1}{p_i^k})\leq\sigma_k(n), \tag{1.6}$$

for all integers $n\geq 1$ and $k\geq 1$.

An important function in number theory is the Euler totient $\varphi(n)$. This is defined to be the number of positive integers not exceeding n, which are relatively prime to n, thus, we have

$$\varphi(n)=n\prod_{p/n}\left(1-\frac{1}{p}\right) \tag{1.7}$$

for all $n\geq 1$.

In [2], C. Jordan has introduced the function $J_k(n)$ defined as the number of ordered sets of k elements from a complete residue system (mod n) such that the greatest common divisor of each set is prime to n. It will be called *Jordan's totient* which is a generalization of Euler's totient and can be expressed as

$$J_k(n)=n^k\prod_{p/n}\left(1-\frac{1}{p^k}\right), \tag{1.8}$$

for all $n\geq 1$ and $k\geq 1$.

A. Makowski, in [3], found the inequality

$$n^k\tau(n)\geq J_k(n)+\sigma_k(n)\geq 2n^k \tag{1.9}$$

for all $n \geq 1$ and $k \geq 1$.

In [10], J. Sándor gave some inequalities related to the function J_k, for example:

$$J_k(n)\tau(n) \geq n^k \tag{1.10}$$

and

$$\sigma_k(n) \leq J_k(n)\tau^2(n), \tag{1.11}$$

for all $n \geq 1$ and $k \geq 1$.

In [7], K. Nageswara Rao has introduced the unitary analogue $J_k^*(n)$ of Jordan's totient which can be expressed as:

$$J_k^*(n) = n^k \prod_{p/n} \left(1 - \frac{1}{p^{ak}}\right). \tag{1.12}$$

J. Sándor and L. Tóth established in [9]] several interesting inequalities for function J_k^*.

Among these, we remark the following:

$$J_k^*(n) + \tau^*(n) \leq \sigma_k^*(n), \tag{1.13}$$

$$J_k^*(n) + \sigma_k^*(n) \leq n^k \tau^*(n), \tag{1.14}$$

and

$$n^k \leq J_k^*(n) \cdot \tau^*(n),$$

for all $n \geq 1$ and $k \geq 1$. Next, the principal aim of the paper is to illustrate several inequalities between the above mentioned arithmetic functions.

2. Inequalities for the functions $\tau^{(e)}, \sigma_k^{(e)}, \tau^{(e)*}$ and $\sigma_k^{(e)*}$

Theorem 2.1. *For all $n \geq 1$ and for all integers $k \geq 0$, there are the following inequalities:*

$$\sigma_k^{(e)}(n) \geq \gamma^k(n) \left[1^k + 2^k + ... + \left(\tau^{(e)}(n)\right)^k\right] \tag{2.1}$$

and

$$\sigma_k^{(e)*}(n) \geq \gamma^k(n) \left[1^k + 2^k + ... + \left(\tau^{(e)*}(n)\right)^k\right]. \tag{2.2}$$

Proof. For $n = 1$, we have equality in relations (2.1) and (2.2).

If $n > 1$, then we take the divisors in increasing order. The smallest exponential divisor of $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$ is $p_1 p_2 ... p_r$, where $p_1 p_2 ... p_r = \gamma(n)$. The second divisor is at least $2p_1 p_2 ... p_r = 2\gamma(n)$. If $d_1, d_2, ..., d_s$ are the exponential divisors of n, then it is easy to see that $d_i \geq \gamma(n) \cdot i$, for any $i = \overline{1, s}$. The last inequality is in fact the inequality $n \geq \gamma(n) \cdot \tau^{(e)}(n)$, which is true, for all $n \geq 1$. Hence

$$\begin{aligned} \sigma_k^{(e)}(n) &= \sum_{d|_{(e)}n} d^k \geq \gamma^k(n) + (2 \cdot \gamma(n))^k + (3 \cdot \gamma(n))^k + ... + (s \cdot \gamma(n))^k = \\ &= \gamma^k(n) \left(1^k + 2^k + ... + s^k\right), \end{aligned}$$

where $s = \tau^{(e)}(n)$. In an analogous way, we deduce the second inequality by replacing the exponential divisors of n with the unitary $e-$divisors of n. $\square$

Remark 2.1. *In Theorem 2.1, the equality in relations* (2.1) *and* (2.2) *holds, when we have* $n = \gamma(n) \cdot \tau^{(e)}(n)$, *so, for* $n = 1, n = p_1 p_2 ... p_r$ *and* $n = 4p_2 ... p_r$ $(p_i \neq 2)$, *where* p_i *is a prime number, for all* $1 \leq i \leq r$.

Corollary 2.1. *For all* $n \geq 1$ *and* $k \geq 2$, *there are the following inequalities:*

$$\sigma_k^{(e)}(n) > \frac{\left[\tau^{(e)}(n)\right]^2 \cdot \gamma(n)}{\zeta(k)} \tag{2.3}$$

and

$$\sigma_k^{(e)*}(n) > \frac{\left[\tau^{(e)*}(n)\right]^2 \cdot \gamma(n)}{\zeta(k)}, \tag{2.4}$$

where ζ *is the Riemann-Zeta function.*

Proof. We apply Cauchy's inequality, thus:

$$\left(\frac{1}{1^k} + \frac{1}{2^k} + ... + \frac{1}{s^k}\right)\left(1^k + 2^k + ... + s^k\right) \geq s^2,$$

where $s = \tau^{(e)}(n)$. But

$$\zeta(k) = \frac{1}{1^k} + \frac{1}{2^k} + ... + \frac{1}{s^k} + ... > \frac{1}{1^k} + \frac{1}{2^k} + ... + \frac{1}{s^k}.$$

Therefore, we obtain the inequality $1^k + 2^k + ... + s^k \geq \frac{s^2}{\zeta(k)} = \frac{\left[\tau^{(e)}(n)\right]^2}{\zeta(k)}$.

Using Theorem 2.1 and the above inequality, we deduce inequality (2.3). Similarly, we obtain inequality (2.4). □

Corollary 2.2. *For all* $n \geq 1$, *there are the following inequalities:*

$$\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \gamma(n) \cdot \frac{\tau^{(e)}(n) + 1}{2} \geq \gamma(n) \tag{2.5}$$

and

$$\frac{\sigma^{(e)*}(n)}{\tau^{(e)*}(n)} \geq \gamma(n) \cdot \frac{\tau^{(e)*}(n) + 1}{2} \geq \gamma(n). \tag{2.6}$$

Proof. For $k = 1$, in Theorem 2.1, we obtain

$$\sigma^{(e)}(n) \geq \gamma(n)(1 + 2 + ... + s) = \gamma(n) \cdot \frac{s(s+1)}{2} = \gamma(n) \cdot \frac{\tau^{(e)}(n)\left(\tau^{(e)}(n) + 1\right)}{2},$$

so

$$\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \gamma(n) \cdot \frac{\tau^{(e)}(n) + 1}{2}.$$

But $\tau^{(e)}(n) \geq 1$, which means that we have

$$\frac{\sigma^{(e)}(n)}{\tau^{(e)}(n)} \geq \gamma(n) \cdot \frac{\tau^{(e)}(n) + 1}{2} \geq \gamma(n).$$

In an analogous way, we deduce the second inequality, thus, the proof is complete. □

Lemma 2.1. *For any $x_i > 0$ with $i \in \{1, 2, ..., n\}$, there is the following inequality:*

$$\prod_{i=1}^{n} \left(1 + x_i + x_i^2\right) + \prod_{i=1}^{n} x_i^2 \geq \prod_{i=1}^{n} \left(x_i + x_i^2\right) + \prod_{i=1}^{n} \left(1 + x_i^2\right). \tag{2.7}$$

Proof. We consider

$$p(n) : \left\{ \prod_{i=1}^{n} \left(1 + x_i + x_i^2\right) + \prod_{i=1}^{n} x_i^2 \geq \prod_{i=1}^{n} \left(x_i + x_i^2\right) + \prod_{i=1}^{n} \left(1 + x_i^2\right) \right\}, \text{ for any } n \geq 1.$$

We check that $p(1)$ is true, so,

$$1 + x_i + x_i^2 + x_i^2 \geq x_i + x_i^2 + 1 + x_i^2,$$

and we suppose that $p(k)$ is true, so

$$\prod_{i=1}^{k} \left(1 + x_i + x_i^2\right) + \prod_{i=1}^{k} x_i^2 \geq \prod_{i=1}^{k} \left(x_i + x_i^2\right) + \prod_{i=1}^{k} \left(1 + x_i^2\right).$$

We prove that $p(k+1)$ is true, so

$$\prod_{i=1}^{k+1} \left(1 + x_i + x_i^2\right) + \prod_{i=1}^{k+1} x_i^2 \geq \prod_{i=1}^{k+1} \left(x_i + x_i^2\right) + \prod_{i=1}^{k+1} \left(1 + x_i^2\right),$$

which is equivalent to the inequality

$$x_{k+1}^2 \left(\prod_{i=1}^{k} \left(1 + x_i + x_i^2\right) + \prod_{i=1}^{k} x_i^2 - \prod_{i=1}^{k} \left(x_i + x_i^2\right) - \prod_{i=1}^{k} \left(1 + x_i^2\right) \right) +$$
$$+ x_{k+1} \left(\prod_{i=1}^{k} \left(1 + x_i + x_i^2\right) - \prod_{i=1}^{k} \left(x_i + x_i^2\right) \right) + \prod_{i=1}^{k} \left(1 + x_i + x_i^2\right) - \prod_{i=1}^{k} \left(1 + x_i^2\right) \geq 0.$$

According to the principle of mathematical induction, $p(n)$ is true for any $n \geq 1$. □

Theorem 2.2. *For any $n \geq 1$ and $k \geq 0$, the following inequality:*

$$\sigma_k(n) + n^k \geq \sigma_k^{(e)}(n) + \sigma_k^*(n) \tag{2.8}$$

holds.

Proof. For $k = 0$, we deduce the inequality

$$\tau(n) + 1 \geq \tau^{(e)}(n) + \tau^*(n),$$

for all integers $n \geq 1$, which is in fact inequality (1.1). If $n = 1$ and $k \geq 1$, then we obtain $\sigma_k(1) + 1 = 2 = \sigma_k^{(e)}(1) + \sigma_k^*(1)$.

We consider $n > 1$ and $k \geq 1$. To prove the above inequality, we will have to study several cases, namely:

Case I. If $n = p_1^2 p_2^2 ... p_r^2$, then we deduce the equalities $\sigma_k(n) = \prod_{i=1}^{r}\left(1 + p_i^k + p_i^{2k}\right)$, $\sigma_k^{(e)}(n) = \prod_{i=1}^{r}\left(p_i^k + p_i^{2k}\right)$ and $\sigma_k^*(n) = \prod_{i=1}^{r}\left(1 + p_i^{2k}\right)$,which means that inequality (2.8) implies the inequality

$$\prod_{i=1}^{r}\left(1 + p_i^k + p_i^{2k}\right) + \prod_{i=1}^{r} p_i^{2k} \geq \prod_{i=1}^{r}\left(p_i^k + p_i^{2k}\right) + \prod_{i=1}^{r}\left(1 + p_i^{2k}\right),$$

which is true, because we use inequality (2.7), for $n = r$ and $x_i = p_i^k$, for all $i = \overline{1, r}$.

Case II. If $a_k \neq 2$, for all $k = \overline{1, r}$, then the numbers

$$\frac{n}{p_1}, \frac{n}{p_2}, ..., \frac{n}{p_r}, \frac{n}{p_1 p_2}, ..., \frac{n}{p_i p_j}, ..., \frac{n}{p_i p_j p_k}, ..., \frac{n}{p_1 p_2 ... p_r}$$

are not exponential divisors of n, so they are in a total number of $2^r - 1$, and their sum is $n^k \prod_{i=1}^{r}(1 + \frac{1}{p_i^k}) - n^k$, such as we have the inequality

$$\sigma_k(n) = \sum_{d \nmid_{(e)} n} d^k + \sum_{d \nmid_{(e)} n} d^k = \sigma_k^{(e)}(n) + \sum_{d \nmid_{(e)} n} d^k \geq \sigma_k^{(e)}(n) + n^k \prod_{i=1}^{r}(1 + \frac{1}{p_i^k}) - n^k.$$

Since we have the inequality $\sigma_k^*(n) = n^k \prod_{i=1}^{r}(1 + \frac{1}{p_i^{a_i k}}) \leq n^k \prod_{i=1}^{r}(1 + \frac{1}{p_i^k})$, it follows that

$$\sigma_k(n) + n^k \geq \sigma_k^{(e)}(n) + \sigma_k^*(n).$$

Case III. If there is at least one $a_k \neq 2$, and at least one $a_j = 2$, where $j, k \in \{1, 2, ..., r\}$, then without decreasing the generality, we renumber the prime factors from the factorization of n and we obtain

$$n = p_1^2 p_2^2 ... p_s^2 p_{s+1}^{a_{s+1}} ... p_r^r, \text{ with } a_{s+1}, a_{s+2}, ..., a_r \neq 2.$$

Hence, we will write $n = n_1 \cdot n_2$, where $n_1 = p_1^2 p_2^2 ... p_s^2$ and $n_2 = p_{s+1}^{a_{s+1}} ... p_r^r$, which means that $(n_1, n_2) = 1$, and by simple calculations, it is easy to see that

$$\begin{gathered}
\sigma_k(n) = \sigma_k(n_1 \cdot n_2) = \sigma_k(n_1) \cdot \sigma_k(n_2) \geq \\
\left(\sigma_k^{(e)}(n_1) + \sigma_k^*(n_1) - n_1^k\right)\left(\sigma_k^{(e)}(n_2) + \sigma_k^*(n_2) - n_2^k\right) = \\
= \sigma_k^{(e)}(n_1)\sigma_k^{(e)}(n_2) + \sigma_k^{(e)}(n_1)\left(\sigma_k^*(n_2) - n_2^k\right) + \sigma_k^*(n_1)\left(\sigma_k^{(e)}(n_2) - n_2^k\right) + \\
+ \sigma_k^*(n) - n_1^k \sigma_k^{(e)}(n_2) - n_1^k \sigma_k^*(n_2) + n_1^k n_2^k \geq \\
\geq \sigma_k^{(e)}(n) + n_1^k\left(\sigma_k^*(n_2) - n_2^k\right) + n_1^k\left(\sigma_k^{(e)}(n_2) - n_2^k\right) + \\
+ \sigma_k^*(n) - n_1^k \sigma_k^{(e)}(n_2) - n_1^k \sigma_k^*(n_2) + n_1^k n_2^k = \sigma_k^{(e)}(n) + \sigma_k^*(n) - n^k.
\end{gathered}$$

We used the inequalities $\sigma_k^{(e)}(n_1) \geq n_1^k$ and $\sigma_k^*(n_1) \geq n_1^k$ and we took into account the fact that the functions $\sigma_k^{(e)}(n)$,$\sigma_k^*(n)$ and $\sigma_k(n)$ are multiplicative. Thus, the demonstration is complete. $\square$

Remark 2.2. *Another interesting relationship between the above functions can be achieved if we make the same proof as in Theorem 2.2 of* [4], *as follows:*

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \geq \frac{\sigma_k^{(e)}(n)}{\sigma_k^{(e)*}(n)}. \tag{2.9}$$

for all $n \geq 1$ and $k \geq 0$.

Theorem 2.3. *For any $n \geq 1$, $n \neq 2,4,6$, there is the following inequality:*

$$\varphi(n) + 1 \geq \tau^{(e)}(n) + \tau^*(n), \tag{2.10}$$

with equality for $n = 1,3,10$ and 12.

Proof. Combining inequality (1.1) and the inequality

$$\varphi(n) > \tau(n) \tag{2.11}$$

for all $n > 30$, from [11], we deduce

$$\varphi(n) + 1 \geq \tau^{(e)}(n) + \tau^*(n)$$

for every $n > 30$. By simple calculations for $n \leq 30$, we find cases where the inequality is true. □

Lemma 2.2. *For every $n \geq 1$ and $k \geq 1$ the inequality*

$$J_{k+1}(n) \geq \sigma_k(n), \tag{2.12}$$

holds.

Proof. If $n = 1$, the lemma is obvious. Assume $n > 1$.

For $n = p$, where p is a prime number, relation (2.12) becomes

$$p^{k+1} - 1 \geq p^k + 1$$

which is immediate because $p^{k+1} \geq 2p^k \geq p^k + 2$.

For $n = p^a$ with $a \geq 2$ and p is a prime number we have

$$p^{a(k+1)} - 1 \geq \frac{p^{k(a+1)} - 1}{p^k - 1},$$

which is equivalent to inequality

$$p^{ak+a+k} + 2 \geq p^{ak+k} + p^{ak+a} + p^k. \tag{2.13}$$

But

$$\begin{aligned} p^{ak+a+k} &\geq 2p^{ak+k+a-1} \geq p^{ak+k+a-1} + p^{ak+k+a-2} + p^{ak+k+a-2} \\ &\geq p^{ak+a} + p^{ak+k} + p^k, \end{aligned}$$

for all $k \geq 1$ and $a \geq 2$.

Therefore, inequality (2.14) is true. Taking into account that the arithmetic functions J_{k+1} and σ_k are multiplicative and the canonical representation of n is $n = \prod_{p/n} p^a$, we obtain the inequality of the statement. □

Theorem 2.4. *For any $n \geq 1$ and $k \geq 1$ the inequality*

$$J_{k+1}(n) + n^k \geq \sigma_k^{(e)}(n) + \sigma_k^*(n), \tag{2.14}$$

holds.

Proof. According to inequalities (2.8) and (2.12), we deduce relation (2.14). □

Remark 2.3. *Another proof can be given by mathematical induction after k using the inequality*

$$J_{k+1}(n) \geq nJ_k(n), \tag{2.15}$$

for any $n \geq 1$ and $k \geq 1$.

From relation 1.8, we obtain the relation

$$\frac{J_{k+1}(n)}{J_k(n)} = n \prod_{p/n} \frac{p^{k+1}-1}{p^{k+1}-p} \geq n$$

which implies inequality (2.15).

Similarly for the unitary analogue J_k^*, we find the following relations:

$$J_{k+1}^*(n) \geq \sigma_k^*(n) \tag{2.16}$$

and

$$J_{k+1}^*(n) \geq nJ_k^*(n). \tag{2.17}$$

But, taking into account to relations (1.13), (1.14), (2.16) and (2.17), we deduce the following inequalities:

$$J_{k+1}^*(n) - J_k^*(n) \geq \tau^*(n), \tag{2.18}$$

$$n^k J_{k+1}^* \geq (n^k+1)J_k^* + \sigma_k^*(n), \tag{2.19}$$

$$J_{k+1}(n) + n^k \geq J_k^*(n) + \sigma_k^{(e)} + \tau^*(n) \tag{2.20}$$

and

$$J_{k+l}^*(n) \geq n^{l-1}\sigma_k^*(n) \tag{2.21}$$

for all $n \geq 1$, $k \geq 1$ and $l \geq 1$.

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