Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(1), 2011, Pages 92–95
ISSN: 1223-6934

On an inequality due to Amrahov

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Abstract. We prove a Hadamard type inequality for the product of two convex functions in the framework of Orlicz spaces.

2000 *Mathematics Subject Classification.* Primary 26A51, 28A10; Secondary 46E30.
Key words and phrases. convex function, Hadamard's inequality, Hölder's inequality, Orlicz space, Lebesgue space.

According to the Hermite-Hadamard inequality (see [8], p. 50) every convex function $u : [a,b] \to \mathbb{R}$ satisfies an upper estimate of the form

$$\frac{1}{b-a}\int_a^b u(t)\,dt \le \frac{u(a)+u(b)}{2}\,. \tag{1}$$

The aim of the present paper is to prove an analogue of this result for a larger class of integrable functions. The starting point is the following simple remark due to Amrahov [3]: assuming $u,\, v : [a,b] \to \mathbb{R}$ are two nonnegative functions such that both u^2 and v^2 are convex, then

$$\frac{1}{b-a}\int_a^b u(t)v(t)\,dt \le \frac{\sqrt{u^2(a)+u^2(b)}\sqrt{v^2(a)+v^2(b)}}{2}\,. \tag{2}$$

This follows easily from the Cauchy-Schwarz inequality and the Hermite-Hadamard inequality (used in this order). What we need is the fact that the square of a nonnegative convex function is convex too. This is covered by the following straightforward result:

Lemma 0.1. *Assume $u : D \to \mathbb{R}$ is a convex function (defined on a convex subset of a linear space) and Φ is a convex and increasing function defined on an interval including $u(D)$. Then $\Phi \circ u$ is a convex function.*

Clearly, inequality (2) works also in the case where uv is a convex function. What makes Amrahov's remark interesting is the fact that the product of two convex functions is not necessary a convex function (e.g., the functions $u(t) = t^2$ and $v(t) = (1-t)^2$ are both convex for $t \in [0,1]$ but their product is not a convex function on $[0,1]$).

Hölder's inequality allows us to extend Amrahov's remark as follows:

Theorem 0.1. *Assume $p \in (1,\infty)$ is a given real number and $q = p/(p-1)$ is the conjugate exponent of p. Assume $u,\, v : [a,b] \to \mathbb{R}$ are two nonnegative functions such that both u^p and v^q are convex. Then*

$$\frac{1}{b-a}\int_a^b u(t)v(t)\,dt \le \left(\frac{u(a)^p+u(b)^p}{2}\right)^{1/p}\left(\frac{v(a)^q+v(b)^q}{2}\right)^{1/q}.$$

Received January 04, 2011. Revision received February 25, 2011.

The aim of the present paper is to extend Theorem 0.1 in the context of Orlicz spaces.

We start by recalling some basic facts about Orlicz spaces. For more details we refer to the books by D. R. Adams & L. L. Hedberg [2], R. Adams [1], J. Musielak [7] and M. M. Rao & Z. D. Ren [9] and the papers by Ph. Clément *et al.* [4], M. Garciá-Huidobro *et al.* [5] and J. P. Gossez [6].

Assume $\phi : [0,\infty) \to \mathbb{R}^+$ is an increasing and continuous function satisfying $\phi(0) = 0$. We associate to it the functions

$$\Phi(t) = \int_0^t \phi(s)\, ds, \quad \Phi^\star(t) = \int_0^t \phi^{-1}(s)\, ds\,.$$

We observe that Φ is a *Young function*, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, $\Phi(t) = 0$ if and only if $t = 0$ and

$$\lim_{x\to 0} \Phi(t)/t = 0, \qquad \lim_{x\to\infty} \Phi(t)/t = +\infty.$$

Thus Φ is actually an *N-function.* The function $\Phi^\star$, represents the *complementary* function of Φ, and satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s);\ s \geq 0\}, \qquad \text{for all } t \geq 0\,.$$

We also observe that $\Phi^\star$ is also a N-function and Young's inequality holds true

$$st \leq \Phi(s) + \Phi^\star(t), \qquad \text{for all } s, t \geq 0\,.$$

Example 0.1. *1) Let $p, q \in (1,\infty)$ be given real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\Phi(t) = \frac{|t|^p}{p}$ and $\Phi^\star(t) = \frac{|t|^q}{q}$ are complementary N–functions.*

2) The functions $\Phi(t) = e^t - t - 1$ and $\Phi^\star(t) = (1+t)\log(1+t) - t$ are complementary N-functions.

Assume a and b are two real numbers satisfying $0 \leq a < b < \infty$. The *Orlicz space* $L_\Phi(a,b)$ associated to the N-function Φ (see [2, 1]) is the space of measurable functions $u : (a,b) \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{\int_a^b uv\, dx;\ \int_a^b (\Phi)^\star(|g|)\, dx \leq 1\right\} < \infty\,.$$

Then $(L_\Phi(a,b), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the *Luxemburg norm*

$$\|u\|_\Phi := \inf\left\{k > 0;\ \int_a^b \Phi\left(\frac{u(x)}{k}\right)\, dx \leq 1\right\}.$$

For Orlicz spaces, Hölder's inequality reads as follows:

$$\int_a^b uvdx \leq 2\, \|u\|_{L_\Phi}\, \|v\|_{L_{\Phi^\star}} \qquad \text{for all } u \in L_\Phi(a,b) \text{ and } v \in L_{\Phi^\star}(a,b)\,,$$

or

$$\int_a^b uvdx \leq C\, \|u\|_\Phi\, \|v\|_{\Phi^\star} \qquad \text{for all } u \in L_\Phi(a,b) \text{ and } v \in L_{\Phi^\star}(a,b)\,, \tag{3}$$

where C is a positive constant. See [9, Inequality 4, p. 79].

For an easier manipulation of Orlicz-Sobolev spaces we define

$$\phi^- := \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} \ \text{ and } \ \phi^+ := \sup_{t>0} \frac{t\phi(t)}{\Phi(t)}\,.$$

Assume

$$1 < \phi^- \leq \phi^+ < \infty\,. \tag{4}$$

Then it is easy to show that

$$\|u\|_\Phi^{\phi^+} \leq \int_a^b \Phi(u(t))\,dt \leq \|u\|_\Phi^{\phi^-}, \quad \text{for all } u \in L_\Phi(a,b) \text{ with } \|u\|_\Phi \leq 1\,, \tag{5}$$

and

$$\|u\|_\Phi^{\phi^-} \leq \int_a^b \Phi(u(t))\,dt \leq \|u\|_\Phi^{\phi^+}, \quad \text{for all } u \in L_\Phi(a,b) \text{ with } \|u\|_\Phi \geq 1\,. \tag{6}$$

Example 0.2. *We point out certain examples of functions* $\phi : [0,\infty] \to \mathbb{R}^+$ *which are increasing, continuous and satisfy* $\phi(0) = 0$*. For more details the reader can consult* [4, Examples 1-3, p. 243].

1) Let $\phi(t) = t^{p-1}$ *with* $p \in (1,\infty)$*. It is easy to check that* $\Phi(t) = \frac{t^p}{p}$ *and in this case we have*

$$\phi^- = \phi^+ = p\,,$$

and

$$L_\Phi(a,b) = L^p(a,b)\,,$$

where $L^p(a,b)$ *stands for the classical Lebesgue space. Moreover, using inequalities (5) and (6) we find that*

$$\|u\|_\Phi^p = \int_a^b |u(t)|^p\,dt\,.$$

In this particular case we will denote $\|\cdot\|_\Phi$ *by* $\|\cdot\|_{L^p}$.

2) Consider

$$\phi(t) = \log(1+t^r)t^{p-1}, \quad \textit{for all } t \in [0,\infty)\,,$$

with p*,* $r > 1$*. In this case it can be proved that*

$$\phi^- = p, \quad \phi^+ = p + r\,.$$

3) Let

$$\phi(t) = \frac{t^{p-1}}{\log(1+t)}, \quad \textit{if } t > 0, \quad \phi(0) = 0\,,$$

with $p > 2$*. In this case we have*

$$\phi^- = p - 1, \quad \phi^+ = p\,.$$

We are in a position to state our main result:

Theorem 0.2. *Assume* $\phi : [0,\infty] \to \mathbb{R}^+$ *is an increasing continuous function such that* $\phi(0) = 0$ *and* Φ *and* $\Phi^\star$ *are complementary* N*-functions asociated to* ϕ*. Assume that inequalities (4) are fulfilled. Let* a*,* $b \in [0,\infty)$ *be two real numbers such that* $a < b$ *and let* u*,* $v : [a,b] \to \mathbb{R}^+$ *be two convex functions. If*

$$\Phi(u(a)) + \Phi(u(b)) \leq \frac{2}{b-a} \tag{7}$$

and

$$\Phi^\star(v(a)) + \Phi^\star(v(b)) \leq \frac{2}{b-a}, \tag{8}$$

then

$$\frac{1}{b-a}\int_a^b u(t)v(t)\,dt \le C(b-a)^{(1/\phi^+)+(1/(\phi^{-1})^+)-1}$$
$$\left(\frac{\Phi(u(a))+\Phi(u(b))}{2}\right)^{1/\phi^+}\left(\frac{\Phi^\star(v(a))+\Phi^\star(v(b))}{2}\right)^{1/(\phi^{-1})^+},$$

where C is the constant given in inequality 3.

Proof. Since Φ and $\Phi^\star$ are complementary N–functions it follows that they are increasing, continuous and convex on $[0,\infty)$. Thus, by Lemma 0.1 it follows that $\Phi\circ u$ and $\Phi^\star\circ v$ are convex functions. Then inequality (1) assures that

$$\frac{1}{b-a}\int_a^b(\Phi\circ u)(t)\,dt \le \frac{(\Phi\circ u)(a)+(\Phi\circ u)(b)}{2},$$

and

$$\frac{1}{b-a}\int_a^b(\Phi^\star\circ v)(t)\,dt \le \frac{(\Phi^\star\circ v)(a)+(\Phi^\star\circ v)(b)}{2}.$$

The above inequalities combined with assumptions (7) and (8) yield $\|u\|_\Phi<1$ and $\|v\|_{\Phi^\star}<1$. That fact and inequality 5 imply

$$\frac{1}{b-a}\|u\|_\Phi^{\phi^+} \le \frac{\Phi(u(a))+\Phi(u(b))}{2}, \tag{9}$$

and

$$\frac{1}{b-a}\|v\|_{\Phi^\star}^{(\phi^{-1})^+} \le \frac{\Phi^\star(v(a))+\Phi^\star(v(b))}{2}. \tag{10}$$

Relations (9) and (10) combined with inequality (3) lead to the conclusion of Theorem 0.2. □

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