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Solving Fractional Oscillators Using Laplace Homotopy Analysis Method

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Abstract. In this paper, we present an algorithm of the Laplace homotopy analysis method (LHAM) to obtain approximate solutions for linear and nonlinear oscillator fractional differential equations. The proposed algorithm presents a procedure of constructing the set of base functions and gives the high-order deformation equation in a simple form. The method provides the solution in the form of a rapidly convergent series. Numerical examples are used to illustrate the preciseness and effectiveness of the proposed method.

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1. Introduction

Fractional order ordinary differential equations, as generalizations of classical integer order ordinary differential equations, are increasingly used to model problems fluid flow, mechanics, viscoelasticity, biology, physics, engineering and other applications [9, 5, 6]. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [9]. The solution of fractional differential equations is much involved. In general, there exists no method that yields exact solutions for fractional differential equations. Only approximate solutions can be found using linearization or perturbation method. In recent years, much research has been focused on the numerical solution of fractional differential equations. Some numerical methods have been developed, such as differential transform method [1, 20, 16], Laplace transform method [15, 10], Pade approximation method [7], homotopy perturbation method [12, 13], Adomain decomposition method [17, 19] and variation iteration method [14]. In this paper, we will consider the dynamics of the so-called driven fractional oscillator. This fractional oscillator is obtained by replacing the second time derivative term in the corresponding harmonic oscillator by a fractional derivative of order α with $1 < \alpha \leq 2$. The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 2$ the fractional system of oscillators reduces to the standard system of simple harmonic oscillators. Some aspects of such a system have been studied previously by other researchers [11, 2, 3]. Liao [8] employed the basic ideas of the homotopy in topology to propose a general analytic method for linear and nonlinear problems, namely homotopy analysis method. This method has been successfully applied to solve many types of nonlinear problems [21, 4, 18, 22]. In this paper, we further apply the homotopy analysis method to solve fractional oscillator differential equations.

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The objective of the present paper is to modify the homotopy analysis method to provide symbolic approximate solutions for linear and nonlinear oscillator fractional initial value problems. The LHAM is a combination of HAM and Laplace transforms. This method is characterized by choosing the identity auxiliary linear operator. The organization of this paper is as follows: Brief definitions of the fractional calculus in are given in Section 2. The LHAM is presented in Section 3. In Section 4, four numerical examples are solved to illustrated the applicability of the considered method. Conclusions are presented in Section 5.

2. Fractional Calculus

Some basic definitions and properties of the fractional calculus theory which are used in this paper.

Definition 2.1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$,where $f_1(x) \in C[0,\infty)$. Clearly $C_\mu \subset C_\beta$ if $\beta \leq \mu$.*

Definition 2.2. *A function $f(x)$, $x > 0$,is said to be in the space C_μ^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.*

Definition 2.3. *The left sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as*

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \quad x > 0, \\ J^0 f(x) &= f(x), \end{aligned} \tag{1}$$

Definition 2.4. *Let $f \in C_{-1}^m$, $m \in N \cup \{0\}$ then the Caputo fractional derivative of $f(x)$ is defined as*

$$D_*^\alpha f(x) = \begin{cases} J^{m-\alpha} f^{(m)}(x), & m-1 < \alpha < m, \quad m \in N \\ \frac{d^m f(x)}{dx^m}, & \alpha = m. \end{cases} \tag{2}$$

Hence, we have the following properties

$$\begin{aligned} 1.\ J^\alpha J^v f(t) &= J^{\alpha+v} f(t), \quad \alpha, v \geq 0. \\ 2.\qquad J^\alpha t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \quad \alpha > 0,\ \gamma > -1, \quad t > 0. \\ 3. J^\alpha D_*^\alpha f(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0, \quad m-1 < \alpha \leq m. \end{aligned} \tag{3}$$

Lemma 2.1. *If $m-1 < \alpha \leq m$, $m \in N$, then the Laplace transform of the fractional derivative $D_*^\alpha f(t)$ is*

$$£(D_*^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \quad t > 0, \tag{4}$$

where $F(s)$ be the Laplace transform of $f(t)$.

Proof. The convolution integral with two functions and is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

If $F(s)$and $G(s)$are the Laplace transforms of $f(t)$ and $g(t)$, respectively then

$$\pounds(\int_0^t f(t-\tau)g(\tau)d\tau) = F(s)\ G(s),$$

and from Definition 2

$$\pounds(J^\alpha f(t)) = \frac{1}{\Gamma(\alpha)}\pounds(\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau),$$

so

$$\pounds(J^\alpha f(t)) = \frac{F(s)}{\alpha^s}.$$

Take the Laplace transform of both sides of the property (3), we have

$$\frac{\pounds(D_*^\alpha f(t))}{s^\alpha} = F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{-(k+1)},$$

and so

$$\pounds(D_*^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)s^{\alpha-k-1}, \quad m-1 < \alpha \leq m.$$

□

3. Laplace Homotopy analysis method

The homotopy analysis method which provides an analytical approximate solution is applied to various nonlinear problems. They use the auxiliary linear operator to be D_*^α. In this section, we present a modification of the HAM. This modification is based on the Laplace transform of the fractional derivative $D_*^\alpha f(t)$. To illustrate the basic idea, let us consider the following fractional differential equation

$$D_*^\alpha u(t) = g(t, u(t), u'(t)), \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{5}$$

subject to the initial conditions

$$u(0) = a \ \text{ and } u'(0) = b \tag{6}$$

Applying the Laplace transform to both sides of Equation (5) and by using linearity of Laplace transforms we get

$$\pounds(D_*^\alpha u(t)) = \pounds(g(t, u(t), u'(t))).$$

Using (4), then we have

$$s^\alpha \widetilde{u}(s) - s^{\alpha-1}a - s^{\alpha-2}b = \pounds(g(t, u(t), u'(t))),$$

and

$$\widetilde{u}(s) = \frac{a}{s} + \frac{b}{s^2} + \frac{1}{s^\alpha}\pounds(g(t, u(t), u'(t))), \tag{7}$$

where $\pounds(u(t)) = \widetilde{u}(s)$.

The so-called zero-order deformation equations of the Laplace Eq. (7) has the form

$$(1-q)[\widetilde{\phi}(s;q) - \widetilde{u}_0(s)] = qh[\widetilde{\phi}(s;q) - \frac{a}{s} - \frac{b}{s^2} - \frac{1}{s^\alpha}\pounds(g(t,\phi(t;q),\frac{d}{dt}\phi(t,q)))], \quad (8)$$

where $q \in [0,1]$ is an embedding parameter, when $q = 0$ and $q = 1$, we have $\widetilde{\phi}(s;0) = \widetilde{u}_0(s)$ and $\widetilde{\phi}(s;1) = \widetilde{u}(s)$ respectively. Thus, as q increasing from 0 to 1, $\widetilde{\phi}(s;q)$ varies from $\widetilde{u}_0(s)$ to $\widetilde{u}(s)$. Expanding $\widetilde{\phi}(s;q)$ in Taylor series with respect to q, one has

$$\widetilde{\phi}(s;q) = \widetilde{u}_0(s) + \sum_{m=1}^{\infty} \widetilde{u}_m(s) q^m, \quad (9)$$

where

$$\widetilde{u}_m(s) = \frac{1}{m!}\frac{\partial^m \widetilde{\phi}(s;q)}{\partial q^m}|_{q=0}. \quad (10)$$

If the auxiliary parameter h and the initial guesses $\widetilde{u}_0(s)$ are so properly chosen, then the Series (9) is converge at $q = 1$ and one has

$$\widetilde{u}(s) = \widetilde{u}_0(s) + \sum_{m=1}^{\infty} \widetilde{u}_m(s). \quad (11)$$

Define the vectors

$$\vec{\widetilde{u}}_m = \{\widetilde{u}_0(s), \widetilde{u}_1(s), \widetilde{u}_2(s), \ldots, \widetilde{u}_m(s)\}. \quad (12)$$

Differentiating Equation (8) m times with respect to the embedding parameter q, and then setting $q = 0$, $h = -1$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\widetilde{u}_m(s) = \chi_m \widetilde{u}_{(m-1)}(s) - R_m(\vec{\widetilde{u}}_{m-1}(s)), \quad (13)$$

where

$$\begin{aligned} R_m(\vec{\widetilde{u}}_{m-1}(s)) &= \widetilde{u}_{(m-1)}(s) - \frac{1}{s^\alpha}(\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}[\pounds(g(t,\phi(t;q),\frac{d}{dt}\phi(t,q)))]|_{q=0}) \\ &\quad -(\frac{a}{s} + \frac{b}{s^2})(1-\chi_m), \end{aligned} \quad (14)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}. \quad (15)$$

Applying the inverse Laplace transforms of (13), then we have a power series solution $y(t) = \sum_{i=0}^{\infty} y_i(t)$ of (5).

4. Numerical Results

In order to assess the accuracy of the Laplace homotopy analysis method presented in this paper for fractional oscillator differential equations, we applied it to the following problems [1].

Example 4.1. *Consider the following simple harmonic fractional oscillator*

$$D_*^\alpha u(t) + \omega^\alpha u(t) = 0, \quad \omega > 0, \quad t \ge 0, \; 1 < \alpha \le 2, \quad (16)$$

subject to the initial conditions

$$u(0) = 1 \quad \text{and } u'(0) = 0. \quad (17)$$

Take the Laplace transform of both sides of (16) and by using (17), we have

$$\widetilde{u}(s) + (\frac{\omega}{s})^{\alpha}\ \widetilde{u}(s) - \frac{1}{s} = 0. \tag{18}$$

In view of (13) and (14) is follows

$$\widetilde{u}_m(s) = \chi_m \widetilde{u}_{(m-1)}(s) - R_m(\overrightarrow{\widetilde{u}}_{m-1}(s)), \tag{19}$$

where

$$R_m(\overrightarrow{\widetilde{u}}_{m-1}(s)) = \widetilde{u}_{m-1}(s) + (\frac{\omega}{s})^{\alpha}\widetilde{u}_{m-1}(s) - \frac{1}{s}(1-\chi_m), \tag{20}$$

so

$$\widetilde{u}(s) = \frac{1}{s} - \frac{\omega^{\alpha}}{s^{\alpha+1}} + \frac{\omega^{2\alpha}}{s^{2\alpha+1}} - \frac{\omega^{3\alpha}}{s^{3\alpha+1}} + \frac{\omega^{4\alpha}}{s^{4\alpha+1}} - .... \tag{21}$$

The inverse Laplace transform of (21) has the form

$$u(t) = 1 - \frac{(\omega t)^{\alpha}}{\Gamma(1+\alpha)} + \frac{(\omega t)^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{(\omega t)^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{(\omega t)^{4\alpha}}{\Gamma(1+4\alpha)} - .... \tag{22}$$

In order to improve the accuracy of the homotopy analysis solution of the simple harmonic fractional equation, we calculate is obtained for different values of α. For $\alpha = 2$, we obtain the same solution found in Al-rabtah et al.[1] using the differential transform method

$$u(t) = 1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} - \frac{(\omega t)^6}{6!} + \frac{(\omega t)^8}{8!} - .... \tag{23}$$

Equation (23) is the solution of a simple harmonic oscillator and given by $u(t) = \cos(\omega t)$. Now, taking $\alpha = 1.9$, $u(t)$ is obtained as follows

$$u(t) = 1 - \frac{(\omega t)^{\frac{19}{10}}}{\Gamma(\frac{29}{10})} + \frac{(\omega t)^{\frac{19}{5}}}{\Gamma(\frac{24}{5})} - \frac{(\omega t)^{\frac{57}{10}}}{\Gamma(\frac{67}{10})} + \frac{(\omega t)^{\frac{38}{5}}}{\Gamma(\frac{43}{5})} - ....$$

Finally, the following series solution is obtained for $\alpha = 1.8$,

$$u(t) = 1 - \frac{(\omega t)^{\frac{9}{5}}}{\Gamma(\frac{14}{5})} + \frac{(\omega t)^{\frac{18}{5}}}{\Gamma(\frac{23}{5})} - \frac{(\omega t)^{\frac{27}{5}}}{\Gamma(\frac{32}{5})} + \frac{(\omega t)^{\frac{36}{5}}}{\Gamma(\frac{41}{5})} - ....$$

Fig. 1 shows the series solution (22) exhibit the periodic behavior which is the characteristic of the simple harmonic Equations (16) and (17) obtained for $\alpha = 2$, 1.9 and 1.8. Comparison between these results shows how the displacement of the fractional oscillator varies as a function of time and how this time variation depends on the parameter α. It can be seen that the behavior of the driven fractional oscillator is very similar to the behavior of the damped harmonic oscillator, where the motion is still oscillatory, but the total energy decrease and the phase plane diagram is no longer a closed curve, but a logarithmic spiral.

Example 4.2. *Consider the following fractional oscillator equation*

$$D_*^{\alpha} u(t) + \omega^{\alpha} u(t) = 1, \quad \omega > 0, \quad t \geq 0, \ 1 < \alpha \leq 2, \tag{24}$$

subject to the initial conditions

$$u(0) = 0 \quad \textit{and } u'(0) = 0. \tag{25}$$

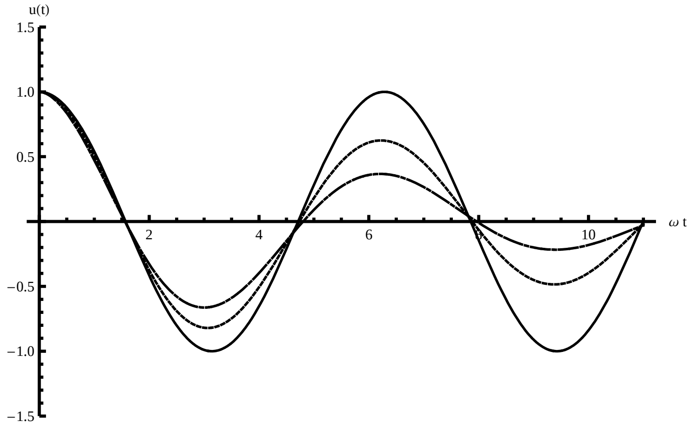

FIGURE 1. $u(t)$ function of Equation (16) for different values of α.

The Laplace transform of both sides of (24) has the form

$$\widetilde{u}(s) + (\frac{\omega}{s})^{\alpha}\ \widetilde{u}(s) = \frac{1}{s^{\alpha+1}}, \tag{26}$$

and the mth-order deformation equations has the form

$$\widetilde{u}_m(s) = \chi_m \widetilde{u}_{(m-1)}(s) - R_m(\overrightarrow{\widetilde{u}}_{m-1}(s)), \tag{27}$$

where

$$R_m(\overrightarrow{\widetilde{u}}_{m-1}(s)) = \widetilde{u}_{m-1}(s) + (\frac{\omega}{s})^{\alpha}\widetilde{u}_{m-1}(s) - \frac{1}{s^{\alpha+1}}(1-\chi_m), \tag{28}$$

and by solving the above linear system of equations, the first components of the Laplace homotopy analysis solution are derived as follows

$$\widetilde{u}(s) = \frac{1}{s^{\alpha+1}} - \frac{\omega^{\alpha}}{s^{2\alpha+1}} + (2 - \frac{\pi \csc(\pi\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha)})\frac{\omega^{2\alpha}}{s^{3\alpha+1}} - \frac{\pi \csc(\pi\alpha)\omega^{3\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)s^{4\alpha+1}} + ... \tag{29}$$

The inverse Laplace transform of (29) has the form

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(1+\alpha)}t^{\alpha} - \frac{\omega^{\alpha}}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{\omega^{2\alpha}}{\Gamma(1+3\alpha)}(2 - \frac{\pi\csc(\pi\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha)})t^{3\alpha} \\ &\quad - \frac{\pi\csc(\pi\alpha)\omega^{3\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(1+4\alpha)})t^{4\alpha} + ... \end{aligned} \tag{30}$$

For $\alpha = 2$, the first terms of the series solution are given by

$$u(t) = \frac{t^2}{2!} - \frac{\omega^2 t^4}{4!} + \frac{\omega^4 t^6}{6!} - \frac{\omega^6 t^8}{8!} + ...$$

Now taking $\alpha = 1.9$, $u(t)$ is obtained as follows:

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\frac{29}{10})}t^{\frac{19}{10}} - \frac{\omega^{\frac{19}{10}}}{\Gamma(\frac{24}{5})}t^{\frac{19}{5}} + \frac{\omega^{\frac{19}{5}}}{\Gamma(\frac{67}{10})}(2 - \frac{(-1-\sqrt{5})\pi}{\Gamma(\frac{-9}{10})\Gamma(\frac{19}{10})})t^{\frac{57}{10}} \\ &\quad - \frac{(-1-\sqrt{5})\pi\omega^{\frac{57}{10}}}{\Gamma(\frac{-9}{10})\Gamma(\frac{19}{10})\Gamma(\frac{43}{5})})t^{\frac{38}{5}} + ..., \end{aligned}$$

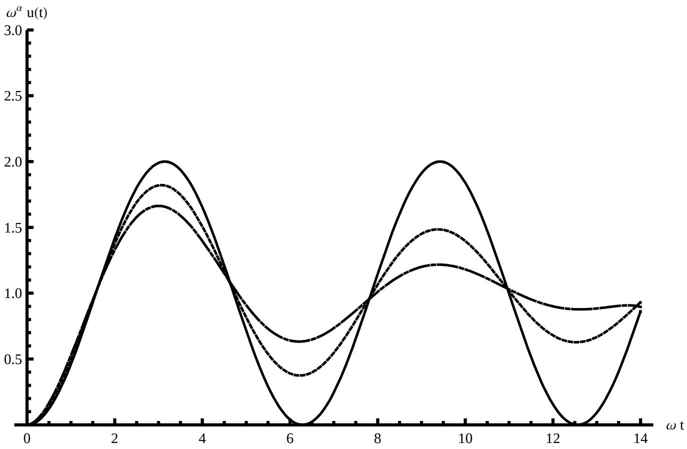

FIGURE 2. $\omega^\alpha u(t)$ function of Equation (24) for different values of α.

and for $\alpha = 1.8$, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\frac{14}{5})} t^{\frac{9}{5}} - \frac{\omega^{\frac{9}{5}}}{\Gamma(\frac{23}{5})} t^{\frac{18}{5}} + \frac{\omega^{\frac{18}{5}}}{\Gamma(\frac{32}{5})} (2 + \frac{\pi}{\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} \Gamma(\frac{-4}{5}) \Gamma(\frac{9}{5})}) t^{\frac{27}{5}} \\ &+ \frac{\pi \omega^{\frac{27}{5}}}{\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} \Gamma(\frac{-4}{10}) \Gamma(\frac{9}{5}) \Gamma(\frac{41}{5})} t^{\frac{36}{5}} + ... \end{aligned}$$

Fig. 2 shows the response function for $\alpha = 2$, 1.9 and 1.8. The results here are very similar to the results of the previous example. The behavior of the driven fractional oscillator for the step function is similar to the behavior of the damped oscillator.

Example 4.3. *Consider the following fractional oscillator equation*

$$D_*^\alpha u(t) + \omega^\alpha u(t) = \sin(\omega t), \quad \omega > 0, \quad t \geq 0, \ 1 < \alpha \leq 2, \tag{31}$$

subject to the initial conditions

$$u(0) = 0 \quad \textit{and} \ u'(0) = 0. \tag{32}$$

If we take the first terms of the power series of $\sin(\omega t)$, and by using the construct of the homotopy (13) and (14), then we have

$$\widetilde{u}(s) = \frac{\omega}{s^{\alpha+2}} - \frac{\omega^3}{s^{\alpha+4}} + \frac{\omega^5}{s^{\alpha+6}} - \frac{\omega^{\alpha+1}}{s^{2\alpha+2}} + \frac{\omega^{\alpha+3}}{s^{2\alpha+4}} - \frac{\omega^{\alpha+5}}{s^{2\alpha+6}} + ... \tag{33}$$

The inverse Laplace transform of (33) has the form

$$\begin{aligned} u(t) &= \frac{\omega}{\Gamma(2+\alpha)} t^{\alpha+1} - \frac{\omega^3}{\Gamma(4+\alpha)} t^{\alpha+3} + \frac{\omega^5}{\Gamma(6+\alpha)} t^{\alpha+5} - \frac{\omega^{\alpha+1}}{\Gamma(2+2\alpha)} t^{2\alpha+1} \\ &+ \frac{\omega^{\alpha+3}}{\Gamma(4+2\alpha)} t^{2\alpha+3} - \frac{\omega^{\alpha+5}}{\Gamma(6+2\alpha)} t^{2\alpha+5} + ... \end{aligned} \tag{34}$$

For the value of $\alpha = 2$, $u(t)$ is obtained as follows

$$u(t) = \frac{\omega}{6} t^3 - \frac{\omega^3}{60} t^5 + \frac{\omega^5}{2520} t^7 - \frac{\omega^7}{362880} t^9 + ...$$

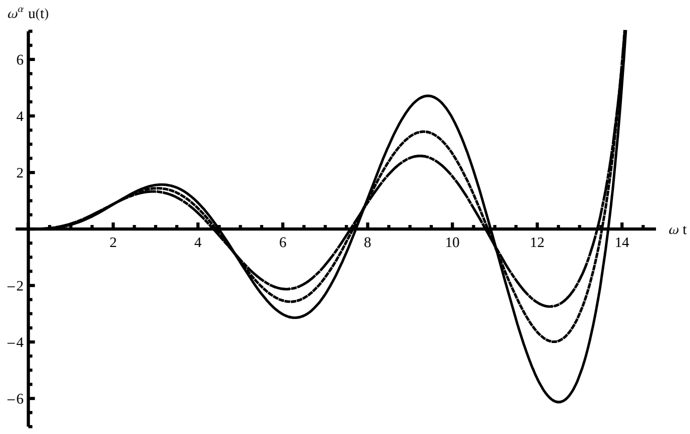

FIGURE 3. $\omega^\alpha u(t)$ function of Equation (31) for different values of α.

Now, substitute $\alpha = 1.9$, in equation (34), then $u(t)$ has the form:

$$u(t) = \frac{\omega}{\Gamma(\frac{39}{10})} t^{\frac{29}{10}} - \frac{\omega^3}{\Gamma(\frac{59}{10})} t^{\frac{49}{10}} + \frac{\omega^5}{\Gamma(\frac{79}{10})} t^{\frac{69}{10}} - \frac{\omega^{\frac{29}{10}}}{\Gamma(\frac{29}{5})} t^{\frac{24}{5}} + \frac{\omega^{\frac{49}{10}}}{\Gamma(\frac{39}{5})} t^{\frac{34}{5}} - \frac{\omega^{\frac{69}{10}}}{\Gamma(\frac{49}{5})} t^{\frac{44}{5}} + ...$$

and for $\alpha = 1.8$, we have

$$u(t) = \frac{\omega}{\Gamma(\frac{19}{5})} t^{\frac{14}{5}} - \frac{\omega^3}{\Gamma(\frac{29}{5})} t^{\frac{24}{5}} + \frac{\omega^5}{\Gamma(\frac{39}{5})} t^{\frac{34}{5}} - \frac{\omega^{\frac{14}{5}}}{\Gamma(\frac{28}{5})} t^{\frac{23}{5}} + \frac{\omega^{\frac{24}{5}}}{\Gamma(\frac{38}{5})} t^{\frac{33}{5}} - \frac{\omega^{\frac{34}{5}}}{\Gamma(\frac{48}{5})} t^{\frac{43}{5}} + ...$$

Fig. 3 shows the LHAM approximate solutions for various values of α which have the same trajectories.

Example 4.4. *Consider the nonlinear fractional Van Der Pol oscillator equation of the form*

$$D_*^\alpha u(t) + \frac{21}{20}(u^2(t) - 1)u'(t) + u(t) = \frac{6}{5}\sin(\omega t), \quad \omega > 0, \quad t \geq 0, \ 1 < \alpha \leq 2, \quad (35)$$

subject to the initial conditions

$$u(0) = 0 \quad \textit{and } u'(0) = 0. \quad (36)$$

If we take the first terms of the power series of $\sin(\omega t)$, then the Laplace transform of Equation (35) has the form

$$\widetilde{u}(s) + \frac{21}{20s^\alpha}\pounds(u^2(t)u'(t)) - \frac{21}{20s^{\alpha-1}}\widetilde{u}(s) + \frac{1}{s^\alpha}\widetilde{u}(s) = \frac{6}{5}(\frac{\omega}{s^{\alpha+2}} - \frac{\omega^3}{s^{\alpha+4}} + \frac{\omega^5}{s^{\alpha+6}} - ...),$$

and the *mth*-order deformation equations has the form

$$\widetilde{u}_m(s) = \chi_m \widetilde{u}_{(m-1)}(s) - R_m(\overrightarrow{\widetilde{u}}_{m-1}(s)),$$

where

$$\begin{aligned}R_m(\overrightarrow{\widetilde{u}}_{m-1}(s)) &= \widetilde{u}_{m-1}(s)+\frac{21}{20s^{\alpha}}\pounds(\sum_{i=0}^{m-1}u'_{m-1-i}(t)\sum_{j=0}^{i}u_j(t)u_{i-j}(t))-\\&-\frac{21}{20s^{\alpha-1}}\widetilde{u}_{m-1}(s)+\frac{1}{s^{\alpha}}\widetilde{u}_{m-1}(s)-\\&-\frac{6}{5}(\frac{\omega}{s^{\alpha+2}}-\frac{\omega^3}{s^{\alpha+4}}+\frac{\omega^5}{s^{\alpha+6}}-...)(1-\chi_m).\end{aligned}$$

By solving the above linear system of equations, and taking the inverse Laplace transform of $\widetilde{u}(s)$, then we have the first components of $u(t)$ as

$$\begin{aligned}u(t) &= \frac{6\omega}{5\Gamma(2+\alpha)}(2+\frac{\pi(\alpha-1)\csc(\pi\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha)})t^{\alpha+1}-\frac{6\omega^3}{5\Gamma(4+\alpha)}(2+\frac{\pi(\alpha-1)\csc(\pi\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha)})t^{\alpha+3}\\&+\frac{6\omega^5}{5\Gamma(6+\alpha)}(2+\frac{\pi(\alpha-1)\csc(\pi\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha)}t^{\alpha+5}+\frac{189\omega^2\Gamma(2+2\alpha)}{128(1+\alpha)\Gamma^2(1+\alpha)\Gamma(2+3\alpha)}t^{3\alpha+1}\\&-\frac{378(2+\alpha)\omega^4\Gamma(4+2\alpha)}{125(2+\alpha)\Gamma(4+\alpha)\Gamma(4+3\alpha)}t^{3\alpha+3}\\&+\frac{189\omega^6\Gamma(6+2\alpha)}{125\Gamma(6+3\alpha)}(\frac{5\alpha+6}{\Gamma(6+\alpha)\Gamma(2+\alpha)}+\frac{\alpha(\alpha+4)}{(1+\alpha)\Gamma^2(4+\alpha)})t^{3\alpha+5}-....\end{aligned} \tag{37}$$

For $\alpha=2$, in Equation (37), we have

$$u(t)=\frac{\omega}{6}t^3-\frac{\omega^3}{100}t^5+(\frac{3\omega^2}{1000}+\frac{\omega^5}{4200})t^7-\frac{7\omega^4}{30000}t^9+\frac{41\omega^6}{4400000}t^{11}-....$$

By taking $\alpha=1.9$, $u(t)$ is obtained as follows

$$\begin{aligned}u(t) &= \frac{6\omega}{5\Gamma(\frac{39}{10})}(2-\frac{(1+\sqrt{5})\pi}{\Gamma(\frac{1}{10})\Gamma(\frac{9}{10})})t^{\frac{29}{10}}-\frac{6\omega^3}{5\Gamma(\frac{59}{10})}(2-\frac{(1+\sqrt{5})\pi}{\Gamma(\frac{1}{10})\Gamma(\frac{9}{10})})t^{\frac{49}{10}}\\&+\frac{6\omega^5}{5\Gamma(\frac{79}{10})}(2-\frac{(1+\sqrt{5})\pi}{\Gamma(\frac{1}{10})\Gamma(\frac{9}{10})})t^{\frac{69}{10}}+\frac{189\omega^2\Gamma(\frac{29}{5})}{125\Gamma(\frac{29}{10})\Gamma(\frac{39}{10})\Gamma(\frac{77}{10})}t^{\frac{67}{10}}\\&-\frac{189\omega^4\Gamma(\frac{44}{5})}{125\Gamma(\frac{39}{10})\Gamma(\frac{59}{10})\Gamma(\frac{97}{10})}t^{\frac{87}{10}}+\frac{189\omega^6}{125\Gamma(\frac{117}{10})}(\frac{15.5}{\Gamma(\frac{39}{10})\Gamma(\frac{79}{10})}+\frac{1121}{290\Gamma^2(\frac{59}{10})})t^{\frac{107}{10}}-...,\end{aligned}$$

Finally, for $\alpha=1.8$, the following series solution is obtained

$$\begin{aligned}u(t) &= \frac{6\omega}{5\Gamma(\frac{19}{5})}(2-\frac{\pi}{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}})t^{\frac{14}{5}}-\frac{6\omega^3}{5\Gamma(\frac{29}{5})}(2-\frac{\pi}{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}})t^{\frac{24}{5}}\\&+\frac{6\omega^5}{5\Gamma(\frac{39}{5})}(2-\frac{\pi}{\Gamma(\frac{1}{5})\Gamma(\frac{4}{5})\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}})t^{\frac{34}{5}}+\frac{189\omega^2\Gamma(\frac{28}{5})}{125\Gamma(\frac{14}{5})\Gamma(\frac{19}{5})\Gamma(\frac{37}{5})}t^{\frac{32}{5}}\\&-\frac{189\omega^4\Gamma(\frac{43}{5})}{125\Gamma(\frac{19}{5})\Gamma(\frac{29}{5})\Gamma(\frac{47}{5})}t^{\frac{42}{5}}+\frac{189\omega^6\Gamma(\frac{48}{5})}{125\Gamma(\frac{57}{5})}(\frac{15}{\Gamma(\frac{19}{5})\Gamma(\frac{39}{5})}+\frac{261}{70\Gamma^2(\frac{29}{5})})t^{\frac{52}{5}}-...,\end{aligned}$$

Fig. 4 shows the homotopy analysis approximate solutions for various values of α, $\alpha=2$, 1.9 and 1.8. The parameter is taken as $\omega=0.5$.

5. Conclusion

The purpose of this paper is to construct the Laplace homotopy analysis method to linear and nonlinear oscillatory equations of fractional order. The Laplace homotopy

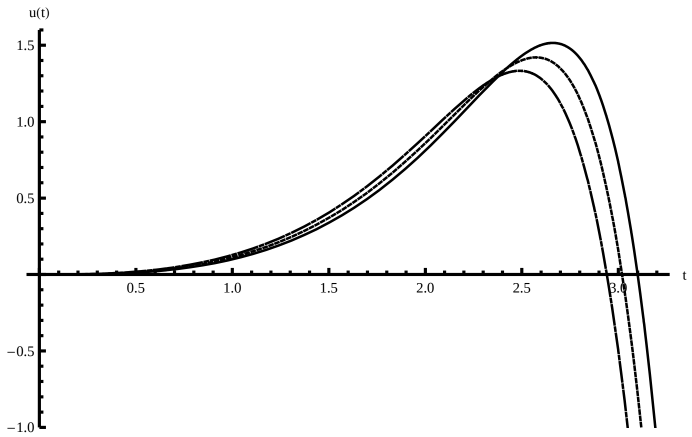

FIGURE 4. $u(t)$ function of Equation (35) for different values of α and $\omega = 0.5$.

analysis method is used to construct an approximate analytical solution for the linear harmonic fractional equation $D_*^\alpha u(t) + \omega^\alpha u(t) = f(t)$. The response function $u(t)$ of different force functions ($f(t) = 0$, 1 and $\sin(\omega t)$) are obtained for different values of α $(1 < \alpha \leq 2)$. The numerical results showed that the behavior of the fractional oscillator is similar to the behavior of the damped harmonic oscillator. Also, the Laplace homotopy analysis method is used to construct an approximate analytical solution for the nonlinear fractional Van Der Pol oscillator. However, the proposed approach can be further implemented to solve other nonlinear problems in the fractional calculus field.

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