Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(1), 2011, Pages 72–82
ISSN: 1223-6934

Slices and extensions of ω-trees

Nicolae Ţăndăreanu and Cristina Zamfir

Abstract. In [15] we defined the structure named ω-labeled tree as a binary, ordered and labeled tree with several features concerning the labels and order between the direct descendants of a node. In this paper we introduce two operators, which enable us to compare between them the structures or parts thereof. These operators work in opposite directions: one of them obtains some part of the structure and the other operator extends the structure. Both the first and the second operator preserves the basic features of an ω-tree from the point of view of the comparison binary relations and the equivalence relations introduced in [15] and [16].

2010 Mathematics Subject Classification. Primary 05C20; Secondary 18A10.
Key words and phrases. directed ordered graph, tree, ordered tree, partial order, embedding mapping.

1. Introduction

There are today two major implications of the mathematical results into the domain of computer science. The first direction is given by the theory of universal algebras. The second is the domain of graph theory. The Peano algebras and graph theory were applied successfully in knowledge representation and various applications were presented: risk management ([2]), conceptual graphs ([3], [9]), coherence graphs ([7]), labeled stratified graphs ([10], [11], [12]), semantic schemas ([14]), master-slave systems of semantic schemas ([13]). Several properties of pseudo-BCK algebras show their connection with fuzzy structures and the class of pseudo-BCK algebras with pseudo-double negation generalizes some particular structures with applications in mathematical logic ([4]). A Tree Algebra for XML, named TAX, was developed as a natural extension of relational algebra for manipulating XML data, modeled as forests of labeled ordered trees([6]). An algebra for manipulating collections with ordering specifications was developed in [8]. The partial algebra of conditional binary relations was used to model a dialogue system ([17]). The Peano Count Tree (P-tree) gives a tree representation of spatial data. The algebra and properties of P-tree structure as well as fast algorithms for P-tree generation and P-tree operations are treated in [5].

The concept of ω-tree was introduced in [15]. This structure is a binary tree whose nodes are labeled by means of a mapping ω that specifies the labeling process. There are two kinds of labels: terminal and non-terminal labels. Only the nodes labeled by non-terminal labels may contain direct descendants. On the set $OBT(\omega)$ of ω-trees we introduced a binary relation, which is not a partial order. An equivalence relation $\simeq$ on the set $OBT(\omega)$ was introduced in [16] and a partial order on the factor set $OBT(\omega)/\simeq$ was defined and studied.

In this paper we develop the idea presented in [15] and [16]. Two operators on the set $OBT(\omega)$ of all ω-trees are introduced: from a given structure the *slicing*

Received January 03, 2011. Revision received February 20, 2011.

operator obtains a substructure and the *immediate extension operator* transforms a given structure into an extended structure. Each of them is not the inverse operator of the other. These operators are studied from the point of view of the comparison and/or equivalence relations from $OBT(\omega)$.

This paper is organized as follows: in Section 2 we recall the basic notions and the main results used in the subsequent sections; in Section 3 we define the slicing operator and several useful properties are presented; in Section 4 we define the immediate extension operator on the set of ω-trees; the last section includes the conclusions of the study and the future work.

2. Basic notions and notations

A *directed ordered graph* ([1]) is a pair $G = (A, D)$, where A is a finite set of elements called *nodes*, D is a finite set of elements of the form $[(i, i_1), \ldots, (i, i_n)]$, where $i, i_1, \ldots, i_n \in A$ and D satisfies the following condition:

$$[(i, i_1), \ldots, (i, i_n)] \in D,\ [(j, j_1), \ldots, (j, j_s)] \in D \implies i \neq j$$

If $G = (A, D)$ is a directed ordered graph then we can associate to G a *directed graph* $G' = (A, D')$, where

$$D' = \{(i, j) \mid \exists [(i, i_1), \ldots, (i, i_n)] \in D, \exists r \in \{1, \ldots, n\} : j = i_r\}$$

An *ordered tree* is a directed ordered graph $G = (A, D)$ such that G' is a tree and the following property is satisfied:

$$[(i, i_1), \ldots, (i, i_n)] \in D, j, r \in \{1, \ldots, n\}, j \neq r \Rightarrow i_j \neq i_r \quad (1)$$

A path in G is defined as usual and the length of a path p is denoted by $length(p)$.

We consider a nonempty set L and a decomposition $L = L_N \cup L_T$, where $L_N \cap L_T = \emptyset$. The elements of L_N are called *nonterminal labels* and those of L_T are called *terminal labels*. The elements of L are called *labels*.

Let $L = L_N \cup L_T$ be a set of labels. A **split mapping** on L ([15]) is a function $\omega : L_N \longrightarrow L \times L$. An ω**-tree** ([15]) is a tuple $t = (A, D, h)$, where

- (A, D) is an ordered tree and every element of D is of the form $[(i, i_1), (i, i_2)]$;
- $h : A \longrightarrow L$ is a mapping such that if $[(i, i_1), (i, i_2)] \in D$ then

$$\begin{cases} h(i) \in L_N \\ \omega(h(i)) = (h(i_1), h(i_2)) \end{cases}$$

For each $i \in A$ the element $h(i)$ is called the **label** of the node i. The mapping h is named the **labeling mapping** of t. By $OBT(\omega)$ we denote the set of all ω-trees. If $t = (A, D, h) \in OBT(\omega)$ then $Path(t)$ denotes the set of all paths of (A, D).

Let $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$ be two elements of $OBT(\omega)$ and an arbitrary mapping $\alpha : A_1 \longrightarrow A_2$. For every $i, i_1, i_2 \in A_1$ if $u = [(i, i_1), (i, i_2)]$ then we denote

$$\overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))]$$

If $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ then we write $t_1 \preceq t_2$ ([15]) if there is a mapping $\alpha : A_1 \longrightarrow A_2$ such that:

$$u \in D_1 \Longrightarrow \overline{\alpha}(u) \in D_2$$

$$h_1(root(t_1)) = h_2(\alpha(root(t_1)))$$

Such a mapping α is an **embedding mapping** of t_1 into t_2 ([15]).

An useful result obtained in [15] states that if $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$, $t_1 \preceq t_2$ and α is an embedding mapping of t_1 into t_2 then $h_1(i) = h_2(\alpha(i))$ for every $i \in A_1$.

If $t = (A, D, h) \in OBT(\omega)$ and $\beta : A \longrightarrow B$ is an injective mapping then we consider the tuple $t^\beta = (A^\beta, D^\beta, h^\beta)$, where the components of t^β are defined as follows ([15]):

$A^\beta = \beta(A)$

$D^\beta = \{ \ [(\beta(i), \beta(i_1)), (\beta(i), \beta(i_2))] \ \mid \ [(i, i_1), (i, i_2)] \in D\}$

$h^\beta : A^\beta \longrightarrow L$, $h^\beta(\beta(i)) = h(i)$

The relation $\preceq$ on the set $OBT(\omega)$ is not a partial order. If $t_1 \preceq t_2$ and $t_2 \preceq t_1$ then we write $t_1 \simeq t_2$. The relation $\simeq$ is an equivalence relation on the set $OBT(\omega)$.

3. The slicing operator

In this section we define an operator which transforms an ω-tree in a "shortest" ω-tree such that several features of this structure are preserved in vision of the equivalence relation between ω-trees. This operator obtains a "slice" of an ω-tree and for this reason it is named the "slicing operator".

Definition 3.1. *For every $k \geq 1$ we define the operator $T_k : OBT(\omega) \longrightarrow OBT(\omega)$ as follows: $T_k(t)$ is obtained from t by deleting all nodes which are reachable from $root(t)$ by a path of length greater than k. T_k is named the **slicing operator**.*

Consider $t = (A, D, h) \in OBT(\omega)$ and $k \geq 1$. For every $j \in A \setminus \{root(t)\}$ there is a path and only one from $root(t)$ to j. We denote by $path(j)$ this path. We consider the set

$$A_k = \{root(t)\} \cup \{j \in A \mid j \neq root(t),\ length(path(j)) \leq k\} \tag{2}$$

Consider an element $[(j, j_1), (j, j_2)] \in D$. If $j_1 \in A_k$ and $j_2 \in A_k$ then $j \in A_k$. Based on this idea we conclude that $T_k(t) = (A_k, D_k, h_k)$, where

$$D_k = \{[(j, j_1), (j, j_2)] \in D \mid j_1, j_2 \in A_k\} \tag{3}$$

$$h_k : A_k \longrightarrow L, \quad h_k(j) = h(j), \quad j \in A_k \tag{4}$$

Proposition 3.1. *The operator T_k is well defined. In other words, if $t \in OBT(\omega)$ then $T_k(t) \in OBT(\omega)$. In addition we have $root(t) = root(T_k(t))$.*

Proof. Consider $t = (A, D, h) \in OBT(\omega)$. We have $T_k(t) = (A_k, D_k, h_k)$, where A_k, D_k and h_k are defined respectively by (2), (3) and (4). The pair $G_k = (A_k, D_k)$ is a directed ordered graph because $A_k \subseteq A$, $D_k \subseteq D$ and $G = (A, D)$ is a directed ordered graph. The associated graph G'_k is a tree because $(p_1, \ldots, p_{r+1})$ is a path of G'_k if and only if $(p_1, \ldots, p_{r+1})$ is a path of G and $r \leq k$.

The condition (1) is satisfied because $D_k \subseteq D$ and D satisfies this condition. In addition we have

$$[(i, i_1), (i, i_2)] \in D_k \Rightarrow h_k(i) \in L_N \quad \& \quad \omega(h_k(i)) = (h_k(i_1), h_k(i_2))$$

because if $[(i, i_1), (i, i_2)] \in D_k$ then $[(i, i_1), (i, i_2)] \in D$, $h_k(i) = h(i) \in L_N$ and $\omega(h_k(i)) = \omega(h(i)) = (h(i_1), h(i_2)) = (h_k(i_1), h_k(i_2))$. It follows that $T_k(t) \in OBT(\omega)$. Obviously $root(T_k(t)) = root(t)$. □

If $t = (A, D, h) \in OBT(\omega)$ and $i \in A$ has direct descendants then we denote by $t_{(i)}$ the structure obtained from t by preserving the node i and all nodes of t which are reachable from i. In other words, $t_{(i)} = (A_{(i)}, D_{(i)}, h_{(i)})$, where

$$A_{(i)} = \{i\} \cup \{j \in A \mid \exists (i, i_1, \ldots, i_k, j) \in Path(t), k \geq 0\} \tag{5}$$

$$D_{(i)} = \{[(j, j_1), (j, j_2)] \in D \mid j \in A_{(i)}\} \tag{6}$$

$$h_{(i)}(j) = h(j) \ for \ j \in A_{(i)} \tag{7}$$

Obviously if $t = (A, D, h) \in OBT(\omega)$ and $i \in A$ has direct descendants then we have $t_{(i)} = (A_{(i)}, D_{(i)}, h_{(i)}) \in OBT(\omega)$ and $root(t_{(i)}) = i$.

An element $t_{(i)} = (A_{(i)}, D_{(i)}, h_{(i)})$ such that $D_i = \emptyset$ is named a *degenerate element* of $OBT(\omega)$. Obviously, a degenerate element contains only one node.

Proposition 3.2. *Suppose that $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$, $[(root(t_1), i_1), (root(t_1), i_2)] \in D_1$, $[(root(t_2), j_1), (root(t_2), j_2)] \in D_2$ and $t_1 \simeq t_2$. Then $t_{1,(i_1)} \simeq t_{2,(j_1)}$, $t_{1,(i_2)} \simeq t_{2,(j_2)}$ and the corresponding embedding mappings are the restrictions of the embedding mapping of t_1 into t_2.*

Proof. Denote by α the embedding mapping of t_1 into t_2. This is a bijective mapping such that ([16])

$$\alpha(root(t_1)) = root(t_2) \tag{8}$$

$$\forall i, i_1, i_2 \in A_1 : [(i, i_1), (i, i_2)] \in D_1 \iff [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2 \tag{9}$$

$$h_1(root(t_1)) = h_2(root(t_2)) \tag{10}$$

Applying the implication from left to right of (9) for $i = root(t_1)$ and using (8) we deduce that $[(root(t_2), \alpha(i_1)), (root(t_2), \alpha(i_2))] \in D_2$. But $[(root(t_2), j_1), (root(t_2), j_2)] \in D_2$, therefore

$$\alpha(i_1) = j_1 \tag{11}$$

$$\alpha(i_2) = j_2 \tag{12}$$

We denote $t_{1,(i_1)} = (X_1, E_1, v_1)$ and $t_{2,(j_1)} = (Y_1, F_1, w_1)$. From (5), (6) and (7) we obtain

$$X_1 = \{i_1\} \cup \{j \in A_1 \mid \exists (i_1, r_1, \ldots, r_k, j) \in Path(t_1), k \geq 0\}$$
$$E_1 = \{[(m, m_1), (m, m_2)] \in D_1 \mid m \in X_1\}$$
$$v_1(j) = h_1(j) \ for \ j \in X_1$$
$$Y_1 = \{j_1\} \cup \{j \in A_2 \mid \exists (j_1, p_1, \ldots, p_k, j) \in Path(t_2), k \geq 0\}$$
$$F_1 = \{[(m, m_1), (m, m_2)] \in D_2 \mid m \in Y_1\}$$
$$w_1(j) = h_2(j) \ for \ j \in Y_1$$

We prove now that

$$\alpha(X_1 \setminus \{i_1\}) = Y_1 \setminus \{j_1\} \tag{13}$$

We have the following chain of equivalent sentences:
$p \in \alpha(X_1 \setminus \{i_1\}) \iff$ there is $j \in X_1 \setminus \{i_1\}$ such that $p = \alpha(j) \iff$
there is $(i_1, r_1, \ldots, r_k, j) \in Path(t_1)$ such that $p = \alpha(j) \iff$
there is $(\alpha(i_1), \alpha(r_1), \ldots, \alpha(r_k), \alpha(j)) \in Path(t_2)$ such that $p = \alpha(j) \iff$
there is $(j_1, \alpha(r_1), \ldots, \alpha(r_k), \alpha(j)) \in Path(t_2)$ such that $p = \alpha(j) \iff$
$\alpha(j) \in Y_1 \setminus \{j_1\}$ and $p = \alpha(j) \iff p \in Y_1 \setminus \{j_1\}$
therefore (13) is proved.

Based on (9) and (13) the following sentences are equivalent:

- $[(m, m_1), (m, m_2)] \in E_1$
- $[(m, m_1), (m, m_2)] \in D_1$ and $m \in X_1$
- $[(\alpha(m), \alpha(m_1)), (\alpha(m), \alpha(m_2))] \in D_2$ and $\alpha(m) \in Y_1$
- $[(\alpha(m), \alpha(m_1)), (\alpha(m), \alpha(m_2))] \in F_1$

therefore

$$\forall m, m_1, m_2 \in X_1 : [(m, m_1), (m, m_2)] \in E_1 \Leftrightarrow [(\alpha(m), \alpha(m_1)), (\alpha(m), \alpha(m_2))] \in F_1 \tag{14}$$

As we mentioned in Section 2 we have $h_1(i) = h_2(\alpha(i))$ for every $i \in A_1$. It follows that $v_1(i_1) = h_1(i_1) = h_2(\alpha(i_1)) = h_2(j_1) = w_1(j_1)$. But $i_1 = root(t_{1,(i_1)})$ and $j_1 = root(t_{2,(j_1)})$. This shows that (11) can be written as follows:

$$\alpha(root(t_{1,(i_1)})) = root(t_{2,(j_1)}) \tag{15}$$

The relation $v(i_1) = w(j_1)$ can be written as

$$v_1(root(t_{1,(i_1)})) = root(t_{2,(j_1)}) \tag{16}$$

From (15), (16) and (14) we deduce that that $t_{1,(i_1)} \simeq t_{2,(j_1)}$ and the corresponding embedding mapping is the restriction of the embedding mapping of t_1 into t_2. Similar we prove that $t_{1,(i_2)} \simeq t_{2,(j_2)}$. □

Remark 3.1. *For every $t \in OBT(\omega)$ we denote*

$$depth(t) = max\{length(p) \mid p \in Path(t)\}$$

Obviously, if $t_1 \simeq t_2$ then $depth(t_1) = depth(t_2)$. We observe that for $k \geq depth(t)$ we have $T_k(t) = t$. For $k \leq dept(t)$ we have $depth(T_k(t)) = k$.

Definition 3.2. *We define recursively the mapping $F : OBT(\omega) \longrightarrow L^*$ as follows:*

- *If $t = (\{i\}, \emptyset, h)$ is a degenerate element of $OBT(\omega)$ then $F(t) = h(i)$.*
- *If $t = (A, D, h) \in OBT(\omega)$ and $[(root(t), i), (root(t), j)] \in D$ then by means of the concatenation operation on L^* we define*

$$F(t) = F(t_{(i)})F(t_{(j)})$$

Proposition 3.3.

(1) *If $t_1 \simeq t_2$ then for every $k \geq 1$ we have*

$$T_k(t_1) \simeq T_k(t_2) \tag{17}$$

and the embedding mapping of $T_k(t_1)$ into $T_k(t_2)$ is the restriction of the embedding mapping of t_1 into t_2.

(2) *If $t_1 \simeq t_2$ then*

$$F(t_1) = F(t_2) \tag{18}$$

Proof. For $k \geq depth(t_1) = depth(t_2)$ the first part of the proposition is obvious because $T_k(t_1) = t_1$, $T_k(t_2) = t_2$ and $t_1 \simeq t_2$. It remains to consider the case $k < depth(t_1)$. Suppose that $t_1 \simeq t_2$, $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. Let us denote by α the embedding mapping of t_1 into t_2. This is a bijective mapping such that the conditions (8), (9) and (10) are satisfied.

For $r \in \{1, 2\}$ we denote $T_k(t_r) = (A_k^r, D_k^r, h_k^r)$. From Proposition 3.1 we obtain

$$\begin{cases} root(t_1) = root(T_k(t_1)) \\ root(t_2) = root(T_k(t_2)) \end{cases} \tag{19}$$

therefore (8) becomes

$$\alpha(root(T_k(t_1))) = root(T_k(t_2)) \tag{20}$$

Obviously d is a path in (A_k^1, D_k^1) if and only if $\alpha(d)$ is a path in (A_k^2, D_k^2), therefore $\alpha(A_k^1) = A_k^2$. Based on these results we observe that for every $i, i_1, i_2 \in A_k^1$, if $u = [(i, i_1), (i, i_2)]$ then

$$u \in D_k^1 \Longleftrightarrow \overline{\alpha}(u) \in D_k^2 \tag{21}$$

Taking into account (19) the relation (10) becomes

$$h_1(root(T_k(t_1))) = h_2(root(T_k(t_2))) \tag{22}$$

From (20), (21), (22) we obtain (17). Moreover, the embedding mapping of $T_k(t_1)$ into $T_k(t_2)$ is the embedding mapping α of t_1 into t_2.

The relation (18) is proved by induction on $depth(t_1)$. Suppose that $depth(t_1) = depth(t_2) = 1$. It follows that $t_1 = (A_1, D_1, h_1)$, $t_2 = (A_2, D_2, h_2)$, where
$A_1 = \{i, i_1, i_2\}$, $A_2 = \{j, j_1, j_2\}$, $root(t_1) = i$, $root(t_2) = j$;
$D_1 = \{[(i, i_1), (i, i_2)]\}$, $D_2 = \{[(j, j_1), (j, j_2)]\}$
From $t_1 \simeq t_2$ we deduce that $h_1(i) = h_2(j)$, $h_1(i_1) = h_2(j_1)$ and $h_1(i_2) = h_2(j_2)$. It follows that $F(t_1) = h_1(i_1)h_1(i_2) = h_2(j_1)h_2(j_2) = F(t_2)$.

Suppose that (18) is true for every $t_1, t_2 \in OBT(\omega)$ such that $depth(t_1) = depth(t_2) \leq n$. Consider the elements $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ such that $t_1 \simeq t_2$, $depth(t_1) = depth(t_2) = n + 1$. We denote $root(t_1) = i$, $root(t_2) = j$, $[(i, i_1), (i, i_2)] \in D_1$, $[(j, j_1), (j, j_2)] \in D_2$. From Proposition 3.2 we know that $t_{1,(i_1)} \simeq t_{2,(j_1)}$, $t_{1,(i_2)} \simeq t_{2,(j_2)}$ and the corresponding embedding mappings are the restrictions of the embedding mapping of t_1 into t_2. But $depth(t_{1,(i_1)}) \leq n$, $depth(t_{1,(i_2)}) \leq n$, $depth(t_{2,(j_1)}) \leq n$ and $depth(t_{2,(j_2)}) \leq n$. By the inductive assumption we have $F(t_{1,(i_1)}) = F(t_{2,(j_1)})$ and $F(t_{1,(i_2)}) = F(t_{2,(j_2)})$. But $F(t_1) = F(t_{1,(i_1)})F(t_{1,(i_2)})$ and therefore $F(t_1) = F(t_2)$. □

Corollary 3.1. *If $t_1 \simeq t_2$ then for every $k \geq 1$ we have $F(T_k(t_1)) = F(T_k(t_2))$.*

Proof. Really, we have $T_k(t_1) \simeq T_k(t_2)$. □

4. Extensions based on non-terminal labels

In this section we define another operator. In comparison with the slicing operator, this is an operator which extends the initial tree. In other words, from a given ω-tree another ω-tree with a greater depth is obtained.

Definition 4.1. *Consider an element $t = (A, D, h) \in OBT(\omega)$ such that $F(t) \notin L_T^*$. More precisely, suppose that*

$$F(t) = w_1 h(i_1) w_2 \dots w_n h(i_n) w_{n+1}$$

*where $w_1, \dots, w_{n+1} \in L_T^*$ and $h(i_1), \dots, h(i_n) \in L_N$. An **immediate extension** of t is an element $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ such that*

$$A_1 = A \cup \bigcup_{i \in \{i_1, \dots, i_n\}} \{j_{i,1}, j_{i,2}\} \tag{23}$$

$$D_1 = D \cup \bigcup_{i \in \{i_1, \dots, i_n\}} \{[(i, j_{i,1}), (i, j_{i,2})]\} \tag{24}$$

$$h_1(x) = h(x) \ for \ x \in A \tag{25}$$

We denote by $E(t)$ the set of all immediate extensions of t. If $t \in OBT(\omega)$ and $F(t) \in L_T^*$ then we take $E(t) = [t]$.

Remark 4.1. *The values $h_1(x)$ for $x \in A_1 \setminus A$ are obtained by means of the mapping ω. More precisely, if $[(i, j_{i,1}), (i, j_{i,2})] \in D_1 \setminus D$ then $\omega(h(i)) = (h_1(i_1), h_1(i_2))$.*

Proposition 4.1. *If $t = (A, D, h) \in OBT(\omega)$ and $F(t) \notin L_T^*$ then*
(1) *$t \prec t_1$ for all $t_1 \in E(t)$*
(2) *if $t_1 \in E(t)$ and $t_2 \in E(t)$ then $t_1 \simeq t_2$*

Proof. The first part of the proposition is immediately obtained. Really, if $t_1 \in E(t)$ then (23), (24) and (25) are fulfilled. It follows that the identity mapping is an embedding mapping of t into t_1, which is not surjective. Thus we have $t \prec t_1$.

Consider now $t_1 = (A_1, D_1, h_1) \in E(t)$ and $t_2 = (A_2, D_2, h_2) \in E(t)$. We define the mapping $\alpha : A_1 \longrightarrow A_2$ as follows:

- $\alpha(i) = i$ for $i \in A$
- If $[(i, i_1), (i, i_2)] \in D_1 \setminus D$ and $[(i, j_1), (i, j_2)] \in D_2 \setminus D$ then $\alpha(i_1) = j_1$ and $\alpha(i_2) = j_2$.

The mapping α is well defined because if $[(i, j_1), (i, j_2)] \in D_2 \setminus D$ and $[(s, r_1), (s, r_2)] \in D_2 \setminus D$ then $i \neq s$, therefore j_1 and j_2 are uniquely determined.

We verify now that α is an injective mapping. Consider $p \in A_1 \setminus A$ and $q \in A_1 \setminus A$ such that $p \neq q$. We have the following cases:

(1) There is $i \in A$ such that $[(i, p), (i, q)] \in D_1 \setminus D$. Then $[(i, \alpha(p)), (i, \alpha(q))] \in D_2 \setminus D_1$. From (1) we obtain $\alpha(p) \neq \alpha(q)$.
(2) There is $i \in A$, $j \in A$ such that $[(i, p), (i, p_1)] \in D_1 \setminus D$ and $[(j, q), (j, q_1)] \in D_1 \setminus D$. In this case $[(i, \alpha(p)), (i, \alpha(p_1))] \in D_2 \setminus D$ and $[(j, \alpha(q)), (j, \alpha(q_1))] \in D_2 \setminus D$, therefore $\alpha(p) \neq \alpha(q)$ because (A_2, D_2) is a tree.
(3) There is $i \in A$, $j \in A$ such that $[(i, p_1), (i, p)] \in D_1 \setminus D$ and $[(j, q), (j, q_1)] \in D_1 \setminus D$. A similar prove as the previous case can be obtained immediately.

In conclusion α is an injective mapping. Let us prove that α is a surjective mapping. Take an element $q \in A_2 \setminus A$. There is an element $[(j, q), (j, q_2)] \in D_2 \setminus D$ or an element $[(j, q_1), (j, q)] \in D_2 \setminus D$. If the first case is encountered then there is an element $[(j, p_1), (j, p_2)] \in D_1 \setminus D$. From the definition of α we have $\alpha(p_1) = q$. For the second case we have $\alpha(p_2) = q$. Thus α is a surjective mapping.

The mapping α satisfies obviously the condition (8), (9) and (10) and therefore α is the embedding mapping of t_1 into t_2, which is bijective. It follows that $t_1 \simeq t_2$. □

Remark 4.2. *If $F(t) \in L_T^*$ then the second sentence of the previous proposition is true because in this case we have $E(t) = [t]$.*

Corollary 4.1. *The following sentences are equivalent:*
(i) $t \in E(t)$
(ii) $F(t) \in L_T^*$

Proof. Suppose that $t \in E(t)$. If we suppose that $F(t) \notin L_T^*$ then by Proposition 4.1 we have $t \prec t$. But the relation $\prec$ is a strict order on $OBT(\omega)$, therefore it is irreflexive. Thus the assumption $F(t) \notin L_T^*$ is false. Therefore (ii) is true. Conversely, suppose that (ii) is true. In this case we have $E(t) = [t]$, therefore (i) is true because $t \in [t]$. □

Proposition 4.2. *If $t \in OBT(\omega)$ then*

$$\bigcup_{t_0 \in [t]} E(t_0) \in OBT(\omega)/_{\simeq} \tag{26}$$

Moreover, if $t_1 \in E(t)$ then $\bigcup_{t_0 \in [t]} E(t_0) = [t_1]$.

Proof. In order to prove (26) we take into consideration two cases: the case $F(t) \in L_T^*$ and the case $F(t) \notin L_T^*$. We begin with the first case because this is very simple. If $F(t) \in L_T^*$ then by Corollary 4.1 we have $E(t) = [t]$. If $t_0 \in [t]$ then by Proposition 3.3 we have $F(t_0) = F(t)$, therefore $F(t_0) \in L_T^*$. It follows that $E(t_0) = [t_0]$. The relation (26) is true because

$$\bigcup_{t_0 \in [t]} E(t_0) = [t_0] = [t]$$

and $[t] \in OBT(\omega)/_{\simeq}$.

We consider now the second case, $F(t) \notin L_T^*$. In order to prove (26) we have to verify the following two properties:

$$t_1, t_2 \in \bigcup_{t_0 \in [t]} E(t_0) \Longrightarrow t_1 \simeq t_2 \tag{27}$$

$$t_1 \in \bigcup_{t_0 \in [t]} E(t_0),\ t_1 \simeq t_2 \Longrightarrow t_2 \in \bigcup_{t_0 \in [t]} E(t_0) \tag{28}$$

First we prove (27). Take the elements $t_1, t_2 \in \bigcup_{t_0 \in [t]} E(t_0)$. There are $t_0^1 \in [t]$ and $t_0^2 \in [t]$ such that $t_1 \in E(t_0^1)$ and $t_2 \in E(t_0^2)$. Denote $t_0^1 = (A_0^1, D_0^1, h_0^1)$ and $t_0^2 = (A_0^2, D_0^2, h_0^2)$. Because $t_0^1 \simeq t_0^2$ we deduce that there is the bijective mapping $\alpha : A_0^1 \longrightarrow A_0^2$ such that

$$\alpha(root(t_0^1)) = root(t_0^2) \tag{29}$$

$$\forall i, i_1, i_2 \in A_0^1 : [(i, i_1), (i, i_2)] \in D_0^1 \Leftrightarrow [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_0^2 \tag{30}$$

$$h_1(root(t_0^1)) = h_2(root(t_0^2)) \tag{31}$$

Denote $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. Following Definition 4.1 we obtain:

$$A_1 = A_0^1 \cup \bigcup_{i \in \{i_1, \ldots, i_n\}} \{p_{i,1}, p_{i,2}\}$$

$$D_1 = D_0^1 \cup \bigcup_{i \in \{i_1, \ldots, i_n\}} \{[(i, p_{i,1}), (i, p_{i,2})]\}$$

$$h_1(x) = h_0^1(x) \ for \ x \in A_0^1$$

$$A_2 = A_0^2 \cup \bigcup_{i \in \{\alpha(i_1), \ldots, \alpha(i_n)\}} \{q_{i,1}, q_{i,2}\} \tag{32}$$

$$D_2 = D_0^2 \cup \bigcup_{i \in \{\alpha(i_1), \ldots, \alpha(i_n)\}} \{[(i, q_{i,1}), (i, q_{i,2})]\} \tag{33}$$

$$h_2(x) = h_0^2(x) \ for \ x \in A_0^2$$

As we mentioned in Section 2 we have $h_0^1(x) = h_0^2(\alpha(x))$ for every $x \in A_0^1$. This explains the relations (32) and (33) because $\{i_1, \ldots, i_n\}$ are all leaves of t_0^1 which are labeled by non-terminal labels if and only if $\{\alpha(i_1), \ldots, \alpha(i_n)\}$ is the set of all leaves of t_0^2 labeled the non-terminal labels.

The mapping $\alpha : A_0^1 \longrightarrow A_0^2$ can be extended to a bijective mapping $\alpha : A_1 \longrightarrow A_2$ such that

$$\forall i, i_1, i_2 \in A_1 : [(i, i_1), (i, i_2)] \in D_1 \Leftrightarrow [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2 \tag{34}$$

As we noted above we have $[(i, p_{i,1}), (i, p_{i,2})] \in D_1 \setminus D_0^1$ for $i \in \{i_1, \ldots, i_n\}$ if and only if $[(i, q_{i,1}), (i, q_{i,2})] \in D_2 \setminus D_0^2$ for $i \in \{\alpha(i_1), \ldots, \alpha(i_n)\}$. This property allows to define for $i \in \{i_1, \ldots, i_n\}$:

$$\alpha(p_{i,1}) = q_{\alpha(i),1},\ \alpha(p_{i,2}) = q_{\alpha(i),2}$$

and thus (34) is satisfied.

We have $root(t_0^1) = root(t_1)$ and $root(t_0^2) = root(t_2)$, therefore from (29) and (31) we obtain

$$\alpha(root(t_1)) = root(t_2) \tag{35}$$

$$h_1(root(t_1)) = h_2(root(t_2)) \tag{36}$$

Now from (34), (35) and (36) we deduce that $t_1 \simeq t_2$. Thus (27) is proved.

Let us prove now (28). Consider an element $t_1 \in E(t_0)$, where $t_0 \in [t]$. We can suppose that:

- $t = (A, D, h)$, $t_0 = (A_0, D_0, h_0)$ and $\alpha : A \longrightarrow A_0$ is the embedding mapping of t into t_0.
- $t_1 \simeq t_2$, $t_1 = (A_1, D_1, h_1) \in E(t_0)$, $t_2 = (A_2, D_2, h_2)$; we have

$$A_1 = A_0 \cup \bigcup_{i \in \{i_1,\ldots,i_n\}} \{j_{i,1}, j_{i,2}\} \tag{37}$$

$$D_1 = D_0 \cup \bigcup_{i \in \{i_1,\ldots,i_n\}} \{[(i, j_{i,1}), (i, j_{i,2})]\} \tag{38}$$

$$h_1(x) = h_0(x) \ for \ x \in A_0 \tag{39}$$

- $\beta : A_1 \longrightarrow A_2$ is the embedding mapping of t_1 into t_2;

From (37), (38) and (39) we obtain

$$A_1^\beta = A_0^\beta \cup \bigcup_{i \in \{i_1,\ldots,i_n\}} \{\beta(j_{i,1}), \beta(j_{i,2})\} \tag{40}$$

$$D_1^\beta = D_0^\beta \cup \bigcup_{i \in \{i_1,\ldots,i_n\}} \{[(\beta(i), \beta(j_{i,1})), (\beta(i), \beta(j_{i,2}))]\} \tag{41}$$

$$h_1^\beta(\beta(x)) = h_0(x) \ for \ x \in A_0 \tag{42}$$

and $t_1^\beta = (A_1^\beta, D_1^\beta, h_1^\beta)$, $t_0^\beta = (A_0^\beta, D_0^\beta, h_0^\beta)$. From (40), (41) and (42) we obtain

$$t_1^\beta \in E(t_0^\beta) \tag{43}$$

But $t_0 \in [t]$, α is the embedding mapping of t into t_0, therefore

$$t^\alpha = t_0 \tag{44}$$

We observe that $(t^\alpha)^\beta = t^{\alpha\circ\beta}$, where $\alpha \circ \beta(x) = \beta(\alpha(x))$. From (43) and (44) we obtain

$$t_1^\beta \in E(t^{\alpha\circ\beta}) \tag{45}$$

But $t_1^\beta = t_2$, therefore from (45) we obtain $t_2 \in E(t^{\alpha\circ\beta})$. The mapping $\alpha \circ \beta$ is injective, therefore we have $t^{\alpha\circ\beta} \in [t]$. It follows that $t_2 \in \bigcup_{t_0 \in [t]} E(t_0)$. Thus (28) is proved.

The last part of the proposition is obtained now immediately. Consider $t_1 \in E(t)$. The inclusion

$$[t_1] \subseteq \bigcup_{t_0 \in [t]} E(t_0) \tag{46}$$

is obtained from (28).

Consider $t_2 \in \bigcup_{t_0 \in [t]} E(t_0)$. But $t_1 \in E(t)$ and $E(t) \subseteq \bigcup_{t_0 \in [t]} E(t_0)$. From (27) we obtain $t_2 \simeq t_1$. In other words we have $t_2 \in [t_1]$. Thus we have the inclusion

$$\bigcup_{t_0 \in [t]} E(t_0) \subseteq [t_1] \tag{47}$$

From (46) and (47) we obtain

$$\bigcup_{t_0 \in [t]} E(t_0) = [t_1]$$

and the proposition is proved. □

5. Conclusions

In this paper we develop the results presented in [15] and [16]. Two operators on the set $OBT(\omega)$ of all ω-trees are introduced and studied. These operators preserve the basic features of the structures. The initial structure and the final structure obtained by applying these operators are compared from the point of view of the comparison relation and the equivalence relation introduced in [15] and [16]. The results presented in this paper allow to introduce the concept of ω-templates by means of which we can characterize the formal computations in a master-slave system based on semantic schemas. This concept is treated in a forthcoming paper.

References

[1] **A.V. Aho, J.D. Ullman**, *The Theory of Parsing, Translation, and Compiling*, Prentice-Hall, Vol. I, 1972.

[2] **S. Bhattacharya, S. K. Ghosh**, An Artificial Intelligence Based Approach for Risk Management Using Attack Graph, *Proceedings of the 2007 International Conference on Computational Intelligence and Security* (2007), 794–798.

[3] **M. Chein, M.L. Mugnier**, Graph-based Knowledge Representation- *Computational Foundations of Conceptual Graphs, Series: Advanced Information and Knowledge Processing* (2008), Springer.

[4] **L.C.Ciungu**, On pseudo-BCK algebras with pseudo-double negation, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **37** (2010), No.1, 19–26.

[5] **Qin Ding, Maleq Khan, Amalendu Roy, William Perrizo**, The P-tree algebra, *Proceedings of the 2002 ACM symposium on Applied computing* (2002), 426–431.

[6] **H. V. Jagadish, Laks V. S. Lakshmanan, Divesh Srivastava, Keith Thompson**, TAX: A Tree Algebra for XML, Database Programming Languages, *Lecture Notes in Computer Science, Springer Berlin / Heidelberg* (2002), Volume 2397, 149-164.

[7] **E. Miranda, M. Zaffalon**, Coherence graphs, *Artificial Intelligence*, **173** (2009), Issue 1, 104–144

[8] **S. Paparizos, H. V. Jagadish**, Pattern tree algebras: sets or sequences?, *Proceedings of the 31st VLDB Conference* (2005), Trondheim, Norway, 349–360.

[9] **J.F. Sowa**, Conceptual graphs for a database interface, *IBM Journal of Research and Development*, **20** (1976), no. 4, 336–357.

[10] **N. Ţăndăreanu**, Collaborations between distinguished representatives for labelled stratified graphs, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **30**(2) (2003), 184–192.

[11] **N. Ţăndăreanu**, Distinguished Representatives for Equivalent Labelled Stratified Graphs and Applications, *Discrete Applied Mathematics*, **144** (2004), no.1-2, 183–208.

[12] **N. Ţăndăreanu**, Knowledge representation by labeled stratified graphs, *Proceedings of the 8th World Multi-Conference on Systemics, Cybernetics and Informatics* (2004), Vol. 5, 345–350.

[13] **N. Ţăndăreanu**, Master-Slave Systems of Semantic Schemas and Applications, *The 10th IASTED International Conference on Intelligent Systems and Control (ISC 2007)* (2007), November 19–21, Cambridge, Massachusetts, USA, 150–155.

[14] **N. Ţăndăreanu**, An Extension of Semantic Schemas to Represent Multiple Meanings, *International Multi-Conference on Engineering and Technological Innovation* (IMETI 2008) (2008), June 29th- July 2nd 2008, USA, Vol.I, 143–148.

[15] **N. Ţăndăreanu, C.Zamfir**, Algebraic properties of ω-trees (I), *Annals of the University of Craiova, Mathematics and Computer Science Series*, **37**(1) (2010), 80–89.

[16] **N. Ţăndăreanu, C.Zamfir**, Algebraic properties of ω-trees (II), *Annals of the University of Craiova, Mathematics and Computer Science Series*, **37**(2), (2010), p.7-17

[17] **C. Zamfir**, DiaSys - A Dialogue System based on Conditional Knowledge, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **37**(3), (2010), p.78-91

(Nicolae Ţăndăreanu, Cristina Zamfir) Department of Informatics, University of Craiova, Al.I. Cuza Street, No. 13, Craiova RO-200585, Romania
E-mail address: ntand@rdslink.ro, cristina.zamfir@star-storage.ro