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Divisors of order k

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Abstract. The aim of this paper is to present the notion of *divisor of order k* and to study some properties about the arithmetical functions which use divisors of order k. We also investigate the maximal order and the minimal order of these arithmetical functions.

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1. Introduction

Many important relations involving arithmetic functions can be developed by introducing new classes of divisors. We start by enumerating several types of divisors found in some papers on the number theory.

The notion of *block-factor* was used for the first time by R. Vaidyanathaswamy in [22]. He introduced this notion in the following way: a divisor d of n is a block-factor when $\left(d, \frac{n}{d}\right) = 1$. Several years later, E. Cohen [2] introduced the current terminology for a block-factor, namely, *the unitary divisor*. In 1966, M. V. Subbarao and L. J. Warren [16] introduced the *unitary perfect numbers* satisfying $\sigma^*(n) = 2n$, where $\sigma^*(n)$ denotes the sum of the unitary divisors on n. Let $\tau^*(n)$ denote the number of unitary divisors of n, which is, in fact, the number of the squarefree divisors of n. Several characterization of these arithmetical function are given below.

The following relation was introduced by F. Mertens, in [5]:

$$\sum_{n \le x} \tau^*(n) = \frac{x}{\zeta(2)} \left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)} \right) + S_2(x), \tag{1}$$

where $S_2(x) = O\left(x^{\frac{1}{2}} \log x\right)$, ζ is *the zeta function of Riemann* and γ is *Euler's constant.*

But in [3] A. A. Gioia and A. M. Vaidya showed that $S_2(x) = O\left(x^{\frac{1}{2}}\right)$.

In 1973, R. Sitaramachandrarao and D. Suryanarayana [14] found the following result:

$$\sum_{n \le x} \sigma^*(n) = \frac{\pi^2 x^2}{12\zeta(3)} + O\left(x \log^{\frac{5}{3}} x\right), \tag{2}$$

where $\zeta(3)$ is *Apéry's constant.*

The notion of *exponential divisor* was introduced by M. V. Subbarao in [15] in the following way: d is said to be an *exponential divisor* (or *e-divisor*) of $n = p_1^{a_1}...p_r^{a_r} > 1$, if $d = p_1^{b_1}...p_r^{b_r}$, where $b_i | a_i$ for any $1 \le i \le r$. A series of results related to the exponential divisors are given in many sources, such as: [4,8,13,18].

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N. Minculete and L. Tóth in [8] presented some properties of the arithmetical functions which use *exponential unitary divisors* or *e-unitary divisors*. A divisor d of $n = p_1^{a_1}...p_r^{a_r} > 1$ is called e-unitary divisor if $d = p_1^{b_1}\dots p_r^{b_r}$, where b_i is an unitary divisor of a_i, so $\left(b_i, \dfrac{a_i}{b_i}\right) = 1$, for any $1 \le i \le r$.

In [6] N. Minculete introduced a new class of divisors, namely, a divisor d of n, so that $\gamma(d) = \gamma(n)$ and $\left(\dfrac{d}{\gamma(n)}, \dfrac{n}{d}\right) = 1$. This divisor was called an *exponential semiproper divisor* or an *e-semiproper divisor* of n, where $\gamma(n) = p_1p_2\dots p_r$, for $n = p_1^{a_1}\dots p_r^{a_r} > 1$ and $\gamma(1) = 1$.

2. Main results

We generalize the class of the unitary divisors and the class of the exponential semiproper divisors as in [7].

Let n be a positive integer and $k \ge 0$ another integer. If

$$n = p_1^{a_1}p_2^{a_2}\dots p_u^{a_u}p_{u+1}^{a_{u+1}}\dots p_r^{a_r} > 1,$$

where $a_1, a_2, \dots a_u < k+1$, and $a_{u+1}, a_{u+2}, \dots, a_r \ge k+1$, then we define the arithmetical function $\gamma_k : \mathbb{N}^* \to \mathbb{C}$ such that $\gamma_k(1) = 1$ and

$$\gamma_k(n) = p_1^{a_1}p_2^{a_2}\dots p_u^{a_u}(p_{u+1}p_{u+2}\dots p_r)^k,$$

where $\mathbb{N}^* = \{1, 2, 3, ...\}$ and $\mathbb{C}$ is the set of the complex numbers. It is easy to see that the arithmetical function γ_k is a multiplicative function.

A divisor d of n, so that $\gamma_k(d) = \gamma_k(n)$ and $\left(\dfrac{d}{\gamma_k(n)}, \dfrac{n}{d}\right) = 1$, will be called a *divisor of order* k of n.

For example, we consider the number $n = 2^6 \cdot 3^4$; as $\gamma_2(n) = 2^2 \cdot 3^2$, then the divisors of order 2 of n are the following:

$$2^2 \cdot 3^2,\ 2^2 \cdot 3^4,\ 2^6 \cdot 3^2,\ 2^6 \cdot 3^4.$$

Let $\tau^{(k)}(n)$ denote the number of the divisors of order k of n, and $\sigma^{(k)}(n)$ denote the sum of the divisors of order k of n. We observe that 1 is a divisors of order k of itself, so that $\sigma^{(k)}(1) = \tau^{(k)}(1) = 1$. For $n > 1$ and $k \ge 1$, the smallest divisor of order k of n is $\gamma_k(n)$ and the greatest divisor of order k of n is n. In the above example, the divisors of order 2 of $n = 2^6 \cdot 3^4$ are the following: $\gamma_2(n)\cdot 1,\ \gamma_2(n)\cdot 3^2,\ \gamma_2(n)\cdot 2^4$ and $\gamma_2(n)\cdot 2^4 \cdot 3^2$. This suggest the following: any divisor of order k of n is written as $d = \gamma_k(n)\cdot d'$, where d' is a unitary divisor of $\dfrac{n}{\gamma_k(n)}$. Therefore, the number of the divisors of order k of n is $\tau^*\left(\dfrac{n}{\gamma_k(n)}\right)$ and the sum of the divisors of order k of n is $\gamma_k(n)\cdot\sigma^*\left(\dfrac{n}{\gamma_k(n)}\right)$, so we have the following relations:

$$\tau^{(k)}(n) = \tau^*\left(\frac{n}{\gamma_k(n)}\right),\ \sigma^{(k)}(n) = \gamma_k(n)\cdot\sigma^*\left(\frac{n}{\gamma_k(n)}\right). \tag{3}$$

We observe that if the integer $d = p_1^{b_1}\dots p_r^{b_r}$ is a divisor of order k of $n = p_1^{a_1}\dots p_r^{a_r} > 1$, then $b_i \in \{k, a_i\}$, for any $1 \le i \le r$.

According to the previous statements, we have

$$\tau^{(k)}(p^a) = \begin{cases} 1, & for \;\; a < k+1 \\ 2, & for \;\; a \geq k+1, \end{cases} \tag{4}$$

so, p^a is the only divisor of order k of p^a, when $a \leq k$, and the divisors of order k of p^a $(a \geq k+1)$ are p^k and p^a, which means that

$$\sigma^{(k)}(p^a) = \begin{cases} p^a, & for \;\; a < k+1 \\ p^a + p^k, & for \;\; a \geq k+1. \end{cases} \tag{5}$$

Note that for $k = 0$ the notion of the *divisor of order 0* is identical with the notion of the unitary divisor, and for $k = 1$ the notion of *the divisor of order 1* is identical with the notion of the exponential semiproper divisor. Similar to the unitary analogue of Euler's totient (see e.g. [8], [12]), we define the multiplicative function $\varphi^{(k)} : \mathbb{N}^* \to \mathbb{C}$, so that $\varphi^{(k)}(1) = 1$ and

$$\varphi^{(k)}(p^a) = \begin{cases} p^a, & for \;\; a < k+1 \\ p^a - p^k, & for \;\; a \geq k+1. \end{cases} \tag{6}$$

We observe that $\varphi^{(0)}(n) = \varphi^*(n)$, where φ^* is the unitary analogue of Euler's arithmetical function, and $\varphi^{(1)}(n) = \varphi^{(e)s}(n)$, where the multiplicative function $\varphi^{(e)s} : \mathbb{N}^* \to \mathbb{C}$, is defined as $\varphi^{(e)s}(1) = 1$ and

$$\varphi^{(e)s}(p^a) = \begin{cases} p, & for \;\; a = 1 \\ p^a - p, & for \;\; a \geq 2, \end{cases} \tag{7}$$

which refers to the exponential semiproper divisors, see [6].

It is easy to see that the arithmetical functions $\tau^{(k)}$ and $\sigma^{(k)}$ are multiplicative and we have

$$\tau^{(k)}(n) = 2^t, \;\; \sigma^{(k)}(n) = p_1^{a_1} \ldots p_u^{a_u} \prod_{i=u+1}^{r} (p_i^{a_i} + p_i^k), \tag{8}$$

where $n = p_1^{a_1} \ldots p_u^{a_u} p_{u+1}^{a_{u+1}} \ldots p_r^{a_r}$, with $a_i \leq k$ for any $i \in \{1, \ldots, u\}$ and $a_i \geq k+1$ for any $i \in \{u+1, \ldots r\}$, and $t = r - u$, so t is the number of the exponents in the prime factorization of n which are $\geq k+1$.

If n is squarefree and $k \geq 1$, then $\tau^{(k)}(n) = 1$ and $\sigma^{(k)}(n) = n$.

Similar to the exponential unitary convolution and to the e-semiproper convolution, we introduce the *convolution of order* k of two arithmetical functions $f, g : \mathbb{N} \to \mathbb{C}$, as the arithmetical function $f *_{(k)} g$, which is defined by $(f *_{(k)} g)(1) = 1$ and

$$\begin{aligned} (f *_{(k)} g)(n) = f(p_1^{a_1} \ldots p_u^{a_u}) g(p_1^{a_1} \ldots p_u^{a_u}) & \sum_{\substack{b_{u+1} \neq c_{u+1} \\ b_{u+1}, c_{u+1} \in \{k, a_{u+1}\}}} \ldots \\ \ldots \sum_{\substack{b_r \neq c_r \\ b_r, c_r \in \{k, a_r\}}} & f(p_{u+1}^{b_{u+1}} \ldots p_r^{b_r}) g(p_{u+1}^{c_{u+1}} \ldots p_r^{c_r}), \end{aligned} \tag{9}$$

if $n = p_1^{a_1} \ldots p_u^{a_u} p_{u+1}^{a_{u+1}} \ldots p_r^{a_r}$, with $a_i \leq k$ for any $i \in \{1, \ldots, u\}$ and $a_i \geq k+1$ for any $i \in \{u+1, \ldots r\}$.

The convolution of order k is commutative, associative and has the identity element $\overline{\mu}^{(k)}$, where $\overline{\mu}^{(k)}(1) = 1$ and

$$\overline{\mu}^{(k)}(p^a) = \begin{cases} 1, & for \;\; a < k+1 \\ 0, & for \;\; a \geq k+1. \end{cases} \tag{10}$$

We observe that

$$\overline{\mu}^{(k)}(n) = \begin{cases} 1, & for \;\; n \in \mathbb{Q}_{k+1} \\ 0, & otherwise, \end{cases} \tag{11}$$

where $\mathbb{Q}_k$ denotes the set of k-free integers (positive integers whose prime factors are all of multiplicity $\leq k$), so $\overline{\mu}^{(k)}$ is the characteristic function of $\mathbb{Q}_{k+1}$.

In [1], T.M. Apostol resumed the Gegenbauer's result, which proved that the number of k- free integers $\leq x$ is given by the asymptotic estimation

$$\sum_{n\leq x} \overline{\mu}^{(k)}(n) = \frac{x}{\zeta(k+1)} + O\left(x^{\frac{1}{k+1}}\right), \quad \text{for any } k \geq 1. \tag{12}$$

Remark 2.1. *In [1], T. M. Apostol defined an arithmetical function μ_k, the Möbius function of order k, as follows:*
$\mu_k(1) = 1,$
$\mu_k(n) = 0$ *if* $p^{k+1}|n$ *for some prime* p,
$\mu_k(n) = (-1)^u$ *if* $n = p_1^k \dots p_u^k \prod_{i>u} p_i^{a_i}, \quad 0 \leq a_i < k,$
$\mu_k(n) = 1$ *otherwise.*
It is easy to see that $\overline{\mu}^{(k)}(n) = |\mu_k(n)|$.

Furthermore, a function f has an inverse with respect to the convolution of order k iff $f(1) \neq 0$ and $f(p_1^{a_1} \dots p_u^{a_u}(p_{u+1}^{a_{u+1}} \dots p_r^{a_r})^k) \neq 0$, for any distinct primes $p_1, \dots, p_r$.

The inverse with respect to the convolution of order k of the constant 1 function is denoted by $\mu^{(k)}$ $(1 *_{(k)} \mu^{(k)} = \overline{\mu}^{(k)})$. This multiplicative arithmetic function is given by $\mu^{(k)}(1) = 1$ and for a prime number p and $a \geq 1$, we have

$$\mu^{(k)}(p^a) = \begin{cases} 1, & for \;\; a < k+1 \\ -1, & for \;\; a \geq k+1. \end{cases} \tag{13}$$

Hence, we obtain the identity

$$\mu^{(k)} *_{(k)} \mu^{(k)} = \mu^{(k)} \cdot \tau^{(k)}. \tag{14}$$

Therefore, the arithmetical function $\mu^{(k)}$ is another Möbius type function. If we have the arithmetical functions F and f such that $F = f *_{(k)} 1$, then $f = F *_{(k)} \mu^{(k)}$.

An asymptotic formula for $\mu^{(k)}$ can be obtained from the the following general result of L. Tóth given by the following:

Theorem 2.1. ([19], p.2). *Let f be a complex valued multiplicative function such that $|f(n)| \leq 1$, for every $n \geq 1$, and $f(p) = 1$, for every prime p. Then*

$$\sum_{n\leq x} f(n) = m(f)x + O\left(x^{\frac{1}{2}} \log x\right).$$

where

$$m(f) = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right)$$

is the mean value of f i.e. $m(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{1\leq x\leq n} f(n)$.

Applying this theorem for the multiplicative function $f = \mu^{(k)}$, we deduced the following:

Theorem 2.2. ([7], Theorem 2.1). *For $k \geq 1$, we have*

$$\sum_{n\leq x} \mu^{(k)}(n) = Ax + O\left(x^{\frac{1}{2}} \log x\right), \tag{15}$$

where

$$A = \prod_p \left(1 - \frac{2}{p^{k+1}}\right) \tag{16}$$

is the mean value of $\mu^{(k)}$.

In [11], we meet the regular convolutions of Narkiewicz-type, namely: denote by $\mathbb{A}$ the set of arithmetical functions $f : \mathbb{N} \to \mathbb{C}$; let $A(n)$ be a subset of the set $D(n)$ of positive divisors of n for each natural number n. The A-convolution of the functions $f, g \in \mathbb{A}$ is given by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right).$$

An A-convolution is called *regular* if

(a) $\mathbb{A}$ is a commutative ring with unity δ (where $\delta(1) = 1$ and $\delta(n) = 0$ for all $n > 1$) with respect to ordinary addition and to $*_A$,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function I, defined by $I(n) = 1$ for all natural numbers n, has an inverse μ_A with respect to $*_A$ and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power $p^a (a \geq 1)$.

We observe that the convolution of order k is a special case of these only for $k = 0$. In [9], [10] and [17] we found several elementary methods in number theory which will suggest some further results.
We present the following result of L. Tóth and E. Wirsing:

Theorem 2.3. ([21], p.3). *Let f be a nonnegative real-valued multiplicative function. Suppose that for all primes p we have $\rho(p) := \sup_{a \geq 0} f(p^a) \leq \dfrac{1}{1 - \frac{1}{p}}$ and that for all primes p there is an exponent $e_p = p^{o(1)}$ such that $f(p^{e_p}) \geq 1 + \dfrac{1}{p}$. Then*

$$\lim_{n \to \infty} \sup \frac{f(n)}{\log \log n} = e^{\gamma} \prod_p \left(1 - \frac{1}{p}\right) \rho(p).$$

For the maximal order of the function $\sigma^{(k)}$, we have

Theorem 2.4.

$$\lim_{n \to \infty} \sup \frac{\sigma^{(k)}(n)}{n \log \log n} = \frac{6}{\pi^2} e^{\gamma}, \tag{17}$$

where γ is Euler's constant.

Proof. In Theorem 2.3 we choose $f(n) = \dfrac{\sigma^{(k)}(n)}{n}$, which is a multiplicative function, and for $e_p = k + 1$, we have

$$\frac{\sigma^{(k)}(p^{k+1})}{p^{k+1}} = 1 + \frac{1}{p}.$$

But

$$\sup_{a \geq 0} \frac{\sigma^{(k)}(p^a)}{p^a} = \sup_{a \geq 0} \frac{p^k + p^a}{p^a} < 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots = \frac{1}{1 - \frac{1}{p}}, \quad \text{so} \quad \rho(\mathrm{p}) \leq \frac{1}{1 - \frac{1}{\mathrm{p}}}.$$

Consequently, relation (17) holds. $\square$

So, the maximal order of $\dfrac{\sigma^{(k)}(n)}{n}$ is $\dfrac{6}{\pi^2}e^{\gamma}\log\log n$.

L. Tóth in ([20], p. 2) proved the following general result:

Theorem 2.5. *Let f be a complex valued multiplicative arithmetic function, such that*

a) $f(p) = f(p^2) = \ldots = f(p^{l-1}) = 1$, $f(p^l) = f(p^{l+1}) = s$, *for every prime p, where* $l, s \geq 2$ *are fixed integers, and*

b) there exist constants $C, m > 0$, *such that* $|f(p^a)| \leq Ca^m$ *for every prime p and every* $a \geq l+2$.

Then, for $t \in \mathbb{C}$,

i)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^t} = \zeta(t)\cdot\zeta^{s-1}(lt)\cdot V(t), \quad for \quad \mathrm{Re}\, t > 1,$$

where the Dirichlet series $V(t) = \sum_{n=1}^{\infty} \frac{v(n)}{n^t}$ *is absolutely convergent for* $\mathrm{Re}\, t > \dfrac{1}{l+2}$, *and*

$$v(p) = v(p^2) = \ldots = v(p^{l+1}) = 0 \quad \text{and}$$

$$v(p^a) = \sum_{j\geq 0}(-1)^j \binom{s-1}{j} \left(f(p^{a-jl}) - f(p^{a-jl-1})\right)$$

for $a \geq l+2$,

ii)

$$\sum_{n\leq x} f(n) = C_f x + x^{\frac{1}{l}} P_{f,s-2}(\log x) + O(x^{u_{s,l}+\epsilon}),$$

for every $\epsilon > 0$, *where* $P_{f,s-2}$ *is a polynomial of degree* $s-2$, $u_{s,l} = \dfrac{2s-1}{3+(2s-1)l}$ *and*

$$C_f := \prod_p \left(1 + \sum_{a=l}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right).$$

Theorem 2.6.

$$\sum_{n\leq x} \tau^{(k)}(n) = \frac{\zeta(k+1)}{\zeta(2k+2)}x + Ax^{\frac{1}{k+1}} + O\left(x^{\frac{1}{k+2}+\epsilon}\right), \tag{18}$$

for every $\epsilon > 0$, *where A is a constant, and the Dirichlet series of* $\tau^{(k)}(n)$ *is*

$$\sum_{n=1}^{\infty} \frac{\tau^{(k)}(n)}{n^t} = \frac{\zeta(t)\zeta(t(k+1))}{\zeta(2t(k+1))}, \quad for \quad \mathrm{Re}\ t > 1. \tag{19}$$

Proof. In Theorem 2.5, for the arithmetic function $f(n) = \tau^k(n)$, take $l = k+1$ and $s = 2$, because $\tau^{(k)}(p) = \ldots \tau^{(k)}(p^k) = 1, \tau^{(k)}(p^{k+1}) = \tau^{(k)}(p^{k+2}) = 2$, and for every $a \geq k+3$, we have

$$|\tau^{(k)}(p^a)| = 2 \leq Ca^m,$$

where C and m are two constants. Therefore, the conditions from Tóth's theorem are satisfied, so it follows the relation

$$\sum_{n\leq x} \tau^{(k)}(n) = C_f x + x^{\frac{1}{k+1}} P_{f,0}(\log x) + O(x^{u_{2,k+1}+\epsilon}).$$

But $C_f := \prod_p \left(1 + \sum_{a=l}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right)$, so

$$C_f = \prod_p \left(1 + \sum_{a=k+1}^{\infty} \frac{\tau^{(k)}(p^a) - \tau^{(k)}(p^{a-1})}{p^a}\right) = \prod_p \left(1 + \frac{1}{p^{k+1}}\right) = \frac{\zeta(k+1)}{\zeta(2k+2)}.$$

By several calculations, we obtain that $u_{2,k+1} = \frac{1}{k+2}$, and $P_{f,0}$ is a constant, which is denoted by A. Therefore, the proof of relation (18) is complete.

As in Theorem 2.5, let $v(p) = \ldots = v(p^{k+2}) = 0$ and, for $a \geq k+3$,

$$v(p^a) = \sum_{j \geq 0} (-1)^j \begin{pmatrix} 1 \\ j \end{pmatrix} (\tau^{(k)}(p^{a-jl}) - \tau^{(k)}(p^{a-jl-1})) = \tau^{(k)}(p^a) - \tau^{(k)}(p^{a-1})$$
$$- \tau^{(k)}(p^{a-k-1}) + \tau^{(k)}(p^{a-k-2}).$$

Using relation (4), we obtain $v(p^a) = 0$, for $k+3 \leq a \leq 2k+1$, $v(p^{2k+2}) = -1$ and $v(p^a) = 0$, for $a \geq 2k+3$.

Therefore, we obtain $v(p^{2k+2}) = -1$, and $v(p^a) = 0$ for any $a \neq 2k+2$.

But the Dirichlet series $V(t) = \sum_{n=1}^{\infty} \frac{v(n)}{n^t}$ is absolutely convergent for $\operatorname{Re} t > \frac{1}{k+3}$ and is equal to $\prod_{p\ prime} \left(1 - \frac{1}{p^{2t(k+1)}}\right) = \frac{1}{\zeta(2t(k+1))}$, so $V(t) = \frac{1}{\zeta(2t(k+1))}$, thus, relation (19) is true. □

We mention that a number n is a *perfect number of order k* if we have

$$\sigma^{(k)}(n) = 2n.$$

If m is a squarefree number and n is a perfect number of order k, so that $(m,n) = 1$, then mn is a perfect number of order k, because

$$\sigma^{(k)}(m \cdot n) = \sigma^{(k)}(m) \cdot \sigma^{(k)}(n) = m \cdot 2n = 2mn.$$

An example of a perfect number of order k is the number $n = 2^{k+1} \cdot 3^{k+1}$.
There is an infinity of perfect numbers of order k.

Remark 2.2. *The number n is a perfect number of order k if and only if $\frac{n}{\gamma_k(n)}$ is unitary perfect number.*

In [12] is given the following result:

Theorem 2.7. *Let g be an arithmetical function. Assume that*
(i) g is integral valued and $g(n) \geq 1$ for every $n \geq 1$,
(ii) $g(n) \geq n$ for every sufficiently large $n(n \geq n_0)$,
(iii) either $g(p) = p+1$ for every sufficiently large prime $p(p \geq p_0)$, or g is multiplicative and $g(p) = p$ for every sufficiently large prime $p(p \geq p_0)$.
Then

$$\lim_{n\to\infty} \inf \frac{\varphi(g(n)) \log\log n}{n} = \lim_{n\to\infty} \inf \frac{\varphi(g(n)) \log\log g(n)}{g(n)} = e^{-\gamma}.$$

Theorem 2.8.

$$\lim_{n\to\infty} \inf \frac{\varphi(\sigma^{(k)}(n)) \log\log n}{n} = e^{-\gamma}, \tag{20}$$

where γ is Euler's constant and $\varphi(n)$ is Euler's totient.

Proof. Since $n \leq \sigma^{(k)}(n)$ for any $n \geq 1$, $\sigma^{(k)}$ is multiplicative and $\sigma^{(0)}(p) = p + 1$ or $\sigma^{(k)}(p) = p$, when $k \geq 1$, we apply Theorem 2.7 and we deduce the statement. $\square$

In [12] is given another result, namely:

Theorem 2.9. *Let $h(n)$ be an arithmetical function such that $n \leq h(n) \leq \sigma(n)$ for every sufficiently large $n(n \geq n_0)$. Then*

$$\lim_{n\to\infty} \inf \frac{h(\sigma(n))}{n} = 1.$$

Theorem 2.10.

$$\lim_{n\to\infty} \inf \frac{\sigma^{(k)}(\sigma(n))}{n} = 1 \tag{21}$$

where $\sigma(n)$ is the sum of the divisors of n.

Proof. Since $n \leq \sigma^{(k)}(n) \leq \sigma(n)$ for any $n \geq 1$, we apply Theorem 2.9 and we deduce the statement. $\square$

Theorem 2.11. *For every $n \geq 1$ and $k \geq 1$ the following inequality holds:*

$$\tau(n) \leq \sqrt{n\gamma_k(n)} \leq \frac{\sigma^{(k)}(n)}{\tau^{(k)}(n)},\ n \neq 4. \tag{22}$$

Proof. For $n = 1$ we have $\tau(1) = 1 = \sqrt{1\gamma_k(1)} = 1 = \frac{\sigma^{(k)}(1)}{\tau^{(k)}(1)}$.
For

$$n = p_1^{a_1} p_2^{a_2} \dots p_u^{a_u} p_{u+1}^{a_{u+1}} \dots p_r^{a_r} > 1,$$

where $a_1, a_2, \dots, a_u < k+1$ and $a_{u+1}, a_{u+2}, \dots, a_r \geq k+1$, we deduce the inequality

$$p_1^{a_1} p_2^{a_2} \dots p_u^{a_u} p_{u+1}^{\frac{a_{u+1}+k}{2}} \dots p_r^{\frac{a_r+k}{2}} \leq p_1^{a_1} p_2^{a_2} \dots p_u^{a_u} \prod_{j=u+1}^{r} \left(\frac{p_j^{a_j} + p_j^k}{2} \right) =$$

$$= \frac{1}{2^{r-u}} p_1^{a_1} p_2^{a_2} \dots p_u^{a_u} \prod_{j=u+1}^{r} (p_j^{a_j} + p_j^k) = \frac{\sigma^{(k)}(n)}{\tau^{(k)}(n)}.$$

But, we have the equality $p_1^{a_1} p_2^{a_2} \dots p_u^{a_u} p_{u+1}^{\frac{a_{u+1}+k}{2}} \dots p_r^{\frac{a_r+k}{2}} = \sqrt{n\gamma_k(n)}$. Therefore, we obtain the inequality

$$\sqrt{n\gamma_k(n)} \leq \frac{\sigma^{(k)}(n)}{\tau^{(k)}(n)},$$

for any $n \geq 1$. The left side of the inequality (22) should be treated separately, because for $n = 4$ the inequality is not true.

If $n = p^a \neq 4$, then first show that

$$\sqrt{p^a \gamma_k(p^a)} \geq \tau(p^a).$$

For $a \geq k+1$, we have $p^{\frac{a+k}{2}} \geq a+1$, which is true, because $p^{\frac{a+k}{2}} \geq 2^{\frac{a+1}{2}} \geq a+1$.
For $a < k+1$ and $k \geq 1$, we have $p^a \geq a+1$, which is true, because $p^a \geq 2^a \geq a+1$ for every $a \geq 1$. For $a \leq k$, we have $p^a \geq a+1$ and inequality is true. We remark that we need to check separately inequality (22) for natural numbers of type $n = 4p^a$. If we have $k = 1$ and $a = 1$, then implies the inequality $\tau(4p) = 6 \leq \sqrt{4p \cdot 2p} = 2\sqrt{2}p$, which is true, because $p \geq 3$. In the case $k \geq 2$ and $a \leq k$ or $a \geq k+1$, we obtain $\tau(4p^a) = 3(a+1) \leq \sqrt{4p^a \cdot 4p^k} = 4p^{\frac{a+k}{2}}$. This inequality is true, because $p \geq 3$ and accordance with the above.

Using the fact that the arithmetic functions τ and γ_k are multiplicative, it follows that

$$\sqrt{n\gamma_k(n)} \geq \tau(n) \text{ for any } n \geq 1,\ n \neq 4 \text{ and } k \geq 1.$$

Thus, the proof is completed. $\square$

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References

[1] T. M. Apostol, *Möbius function of order k*, Pacific Journal of Mathematics **32** (1970), no. 1, 21-27.

[2] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Z. **74** (1960), 66-80.

[3] A. A. Gioia and A. M. Vaidya, *The number of squarefree divisors of an integer*, Duke Math. J. **33** (1966), 797-799.

[4] J. Fabrykowski, J. and M. V. Subbarao, *The maximal order and the average order of multiplicative function $\sigma^{(e)}(n)$*, Théorie des Nombres (Quebec, PQ, 1987), de Gruyter, Berlin-New York (1989), 201-206.

[5] F. Mertens, *Über einige asymptotische Gesetze der Zahlentheorie*, Crelle's Journal **77** (1874), 289-338.

[6] N. Minculete, *A New Class of Divisors: the Exponential Semiproper Divisors* (submitted for publication).

[7] N. Minculete and C. Pozna, *On some properties of divisors of order k*, Int. J. Res. Rev. Appl. Sci. **8** (2011), no. 3, 283-287.

[8] Minculete, N. and Tóth, L., *Exponential unitary divisors*, Annales Univ. Sci. Budapest, Sect. Comp. **35** (2011), 205-216.

[9] H. Montgomery and R. Vaughan, *Multiplicative Number Theory: I. Classical Theory*, Cambridge University Press, New York, 2006.

[10] M. Nathanson, *Elementary Methods in Number Theory*, Springer, New York, 2006.

[11] W. Narkiewicz, *On a class of arithmetical convolutions*, Colloq. Math. **10** (1963), 81-94.

[12] J. Sándor and L. Tóth, *Extremal orders of compositions of certain arithmetical functions*, Electronic Journal of Combinatorial Number Theory **8** (2008).

[13] E. G. Straus and M. V. Subbarao, *On exponential divisors*, Duke Math. J. **41** (1974), 465-471.

[14] R. Sitaramachandrarao and D. Suryanarayana, *On $\sum_{n\leq x}\sigma^*(n)$ and $\sum_{n\leq x}\varphi^*(n)$*, Proc. Amer. Math. Soc. **41** (1973), 61-66.

[15] M. V. Subbarao, *On some arithmetic convolutions in The Theory of Arithmetic Functions*, Lecture Notes in Mathematics, New York, Springer-Verlag, 1972.

[16] M. V. Subbarao and L. J. Warren, *Unitary perfect numbers*, Canad. Math. Bull. **9** (1966), 147-153.

[17] D. Suryanarayana, R. Sita Rama Chandra Rao, *On the true maximum order of a class of arithmetical functions*, Math. J. Okayama Univ. **17** (1975), 95-101.

[18] L. Tóth, *On Certain Arithmetic Functions Involving Exponential Divisors*, Annales Univ. Sci. Budapest., Sect. Comp. **24** (2004), 285-294.

[19] L. Tóth, *On Certain Arithmetic Functions Involving Exponential Divisors, II*, Annales Univ. Sci. Budapest., Sect. Comp. **27** (2007), 155-166.

[20] L. Tóth, *An Order Result for the Exponential Divisors Function*, Publ. Math. Debrecen **71** (2007), no. 1-2, 165-171.

[21] L. Tóth and E. Wirsing, *The maximal order of a class of multiplicative arithmetical functions*, Annales Univ. Sci. Budapest., Sect. Comp. **22** (2003), 353-364.

[22] R. Vaidyanathaswamy, *The theory of multiplicative arithmetic functions*, Trans. Amer. Math. Soc. **33** (1931), 579-662.

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