Note on tense $SHn$–algebras

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Abstract. In this article, we continue the study of tense $SHn$–algebras [5]. These algebras constitute a generalization of tense Lukasiewicz–Moisil algebras [4]. In particular, we describe a discrete duality for tense $SHn$–algebras bearing in mind the results indicated by Orłowska and Rewitzky in [13], for $SHn$–algebras. In addition, we introduce a propositional calculus and prove this calculus has tense $SHn$–algebras as algebraic counterpart. Finally, the duality mentioned above allowed us to show the completeness theorem for this calculus.

2010 Mathematics Subject Classification. Primary 03G25; Secondary 03B44.

Key words and phrases. $SHn$–algebras, tense $SHn$–algebras, discrete duality.

1. Introduction

Classical tense logic is a logical system obtained from bivalent logic by adding the tense operators $G$ (it is always going to be the case that) and $H$ (it has always been the case that) (see [1, 10]). Starting with other logical systems and adding appropriate tense operators, we produce new tense logics (see [2, 3, 4]).

On the other hand, the propositional $SHn$–logics were introduced by Iturrioz in [7]. In [7], she gave a lattice–based semantics for these logics, by means of symmetrical Heyting algebras of order $n$, or for short $SHn$–algebras. There are two motivations for the study of $SHn$–logics and $SHn$–algebras. On the one hand, $SHn$–algebras are intended to provide an exact algebraic approach to the $n$–valued counterpart of the symmetrical modal propositional calculus introduced by Moisil in 1942. The symmetrical modal propositional calculus is obtained by the addition of one unary connective (a negation, characterized by the double negation law and the contraposition rule) to the alphabet of the intuitionistic propositional calculus. On the other hand, it has been shown that Post and Lukasiewicz–Moisil algebras are both Heyting algebras with operators. In both Lukasiewicz–Moisil and Post algebras a symmetry can be expressed in terms of the primitive operations [11, 12], cf. also [8]. This led to the study of more general algebras, called $SHn$–algebras [7, 8], which are Heyting algebras with a symmetry and which, in addition, have $n−1$ unary operations that satisfy certain properties, or, alternatively, can be seen as Lukasiewicz–Moisil algebras with a generalized negation [8]. In [9], Iturrioz and Orłowska give a completeness theorem for $SHn$–logics with respect to a Kripke–style semantics. Also a discrete duality for $SHn$–algebras is given by Orłowska and Rewitzky in [13].

This paper is devoted to the tense propositional $SHn$–calculus, a logical system obtained from the propositional $SHn$–logic by adding the tense operators $G$ and $H$. The algebraic basis of this logic consists of tense $SHn$–algebras, algebraic structures studied in our paper [5].
2. Preliminaries

$\text{SH}_n$–logics were introduced by Iturrioz [7] and further studied by other authors [9, 16]. The language of $\text{SH}_n$–logics is a propositional language, whose formulae are built from propositional variables taken from a set $V$, with operations $\lor$ (disjunction), $\land$ (conjunction), $\rightarrow$ (intuitionistic implication), $\sim, \neg$ (a De Morgan, resp. an intuitionistic negation), and a family $\{S_i : i = 1, \ldots, n - 1\}$ of unary operations (which, intuitively, represent degrees of truth). The following Hilbert style axiomatization of $\text{SH}_n$–logics is given in [7].

**Axioms:**
(A1) $\alpha \rightarrow (\beta \rightarrow \alpha)$
(A2) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$
(A3) $(\alpha \land \beta) \rightarrow \alpha$
(A4) $(\alpha \land \beta) \rightarrow \beta$
(A5) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \land \gamma)))$
(A6) $\alpha \rightarrow (\alpha \lor \beta)$
(A7) $\beta \rightarrow (\alpha \lor \beta)$
(A8) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \lor \beta) \rightarrow \gamma))$
(A9) $\sim \sim \alpha \leftrightarrow \alpha$
(A10) $S_i(\alpha \land \beta) \rightarrow S_i(\alpha) \land S_i(\beta)$
(A11) $S_i(\alpha \rightarrow \beta) \leftrightarrow \bigwedge_{k=1}^n S_k(\alpha) \rightarrow S_k(\beta)$
(A12) $S_i(S_j(\alpha)) \rightarrow S_j(S_i(\alpha))$, for every $i, j = 1, \ldots, n - 1$
(A13) $S_1(\alpha) \rightarrow \alpha$
(A14) $S_i(\sim \alpha) \leftrightarrow S_{n-i}(\alpha)$, for $i = 1, \ldots, n - 1$
(A15) $S_1(\alpha) \lor \sim S_i(\alpha)$

where $\neg \alpha = (\alpha \rightarrow \sim (\alpha \rightarrow \alpha)$ and $\alpha \rightarrow \beta$ is an abbreviation for $(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)$.

**Inference rules:**

(R1) $\frac{\alpha}{\beta} \quad \alpha \rightarrow \beta$
(R2) $\frac{\alpha \rightarrow \beta}{\sim \beta \rightarrow \sim \alpha}$
(R3) $\frac{\alpha \rightarrow \beta}{S_1 \alpha \rightarrow S_1 \beta}$

The $\text{SH}_n$–algebras constitute the algebraic counterpart of the $\text{SH}_n$–logics. The Lindembaum–Tarski algebra of the $\text{SH}_n$–logics is an $\text{SH}_n$–algebra (see [7], p. 300).

We shall recall the definition of $\text{SH}_n$–algebras.

**Definition 2.1.** A symmetrical Heyting algebra of order $n$ ($\text{SH}_n$–algebra for short) is an algebra $(W, \lor, \land, \sim, \rightarrow, \neg, \{S_i\}_{i=1}^{n-1}, 0, 1)$ where $(W, \lor, \land, \sim, 0, 1)$ is a Heyting algebra and the unary operations $\sim$ and $S_i$ (for $i = 1, \ldots, n - 1$) satisfy the following axioms:

(n1) $\sim (\sim x) = x$,
(n2) $\sim (x \lor y) = \sim x \land \sim y$,
(S1) $S_i(x \land y) = S_i(x) \land S_i(y)$,
(S2) $S_i(x \rightarrow y) = \bigwedge_{k=1}^n (S_k(x) \rightarrow S_k(y))$,
(S3) $S_i(S_j(x)) = S_j(x)$, for every $i, j = 1, \ldots, n - 1$,
(S4) $S_1(x) \lor \sim \sim x = x$,
(S5) $S_i(\sim x) = \sim S_{n-i}(x)$, for $i = 1, \ldots, n - 1$,
(S6) $S_1(x) \lor \sim S_1(x) = 1$, with $\neg x = x \rightarrow 0$.

In [13], Orłowska and Rewitzky introduced the notion of $\text{SH}_n$–frame as a structure $(X, \leq, N, \{s_i\}_{i=1}^{n-1})$, where $X$ is a non–empty set, $\leq$ is a quasi–order on $X$ and $N$ and $s_i$ (for $i = 1, \ldots, n - 1$) are functions on $X$ satisfying, for any $x, y \in X$,

(N1) $x \leq y$ implies $N(y) \leq N(x)$,
(N2) $N(N(x)) = x$,
(s1) \( N(s_i(x)) = s_{n-i}(N(x)), \) \( i = 1, \ldots, n-1, \)
(s2) \( s_j(s_i(x)) = s_j(x), \) \( i, j = 1, \ldots, n-1, \)
(s3) \( s_1(x) \leq x, \)
(s4) \( x \leq s_{n-1}(x), \)
(s5) \( s_i(x) \leq s_j(x) \) for \( i \leq j, \)
(s6) \( x \leq y \) implies \( s_i(x) \leq s_i(y) \) and \( s_i(y) \leq s_i(x), \) \( i = 1, \ldots, n-1, \)
(s7) \( s_i(y) \leq y \) and \( y \leq s_i(y) \) imply \( s_i(y) = y, \) \( i = 1, \ldots, n-1, \)
(s8) \( x \leq s_i(x) \) or \( s_{i+1}(x) \leq x, \) \( i = 1, \ldots, n-2, n \geq 3. \)

Let \( T \) be a binary relation on a set \( X \) and let \( A \) be a subset of \( X. \) In what follows we will denote by \([T]A\) the set \( \{x \in X : \forall y, xTy \implies y \in A\}. \)

Recall that, the complex algebra of an \( SHn \)-frame \( (X \leq, N, \{s_i\}_{i=1}^{n-1}) \) is

\[
(C(X), V^c, \wedge^c, \rightarrow^c, \sim^c, \{S_i\}_{i=1}^{n-1}, 0^c, 1^c)
\]

where \( C(X) = \{A \subseteq X : \leq(A = A), 0^c = \emptyset, 1^c = X, A \vee^c B = A \cup B, A \wedge^c B = A \cap B, A \rightarrow^c B = [\leq((X \setminus A) \cup B), \sim^c A = \{x \in X : N(x) \notin A\} \text{ and } S_i(A) = \{x \in X : s_i(x) \in A\} \text{ for all } A, B \in C(X). \}

On the other hand, the canonical frame of an \( SHn \)-algebra \((W, \vee, \wedge, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)\) is

\[
(X(W), \leq^c, N^c, \{s_i^c\}_{i=1}^{n-1})
\]

where \( X(W) \) is the set of all prime filters of \( W, \leq^c \) is \( \subseteq \) and for every \( F \in X(W), \)

\( N^c(F) = \{a \in W : a \notin F\}, s_i^c(F) = \{a \in W : S_i(a) \in F\}, i = 1, \ldots, n-1. \)

These results allowed them to obtain a discrete duality for \( SHn \)-algebras by defining the embeddings as follows:

(E1) \( h : W \rightarrow C(X(W)), \) defined by \( h(a) = \{F \in X(W) : a \in F\}, \) for any \( a \in W, \)
(E2) \( k : X \rightarrow X(C(X)), \) defined by \( k(x) = \{A \in C(X) : x \in A\}, \) for any \( x \in X. \)

3. Tense \( SHn \)-algebras

In this section we shall recall the definition and basic results on tense \( SHn \)-algebras from [5].

**Definition 3.1.** A tense \( SHn \)-algebra is an algebra \((W, \vee, \wedge, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)\), where the reduct \((W, \vee, \wedge, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)\) is an \( SHn \)-algebra and \( G, H \) are unary operators on \( W \) verifying the following conditions:

(T1) \( G(1) = 1, H(1) = 1, \)
(T2) \( G(x \wedge y) = G(x) \land G(y), H(x \wedge y) = H(x) \land H(y), \)
(T3) \( x \leq G(\sim H(\sim x)), \) \( x \leq H(\sim G(\sim x)), \)
(T4) \( S_i(G(x)) = G(S_i(x)), S_i(H(x)) = H(S_i(x)), \) for \( i = 1, \ldots, n-1. \)

**Remark 3.1.** (i) From (T2) it follows that \( G \) and \( H \) are increasing.
(ii) If \((W, \vee, \wedge, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)\) is a tense \( SHn \)-algebra in which satisfies the identity \((x \wedge \sim y) \sim (y \vee \sim y) = y \vee \sim y, \) then \((W, \vee, \wedge, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)\) is a tense Łukasiewicz–Moisil algebra.

**Lemma 3.1.** Let \( G, H \) be two unary operations on an \( SHn \)-algebra \( W \) such that \( G(1) = 1, H(1) = 1. \) Then condition (T2) is equivalent to the following one:

\[(T2)' \ G(x \rightarrow y) \leq G(x) \rightarrow G(y), \ (H(x \rightarrow y) \leq H(x) \rightarrow H(y). \)

**Proof.** We will only prove the equivalence between (T2) and \((T2)' \) in the case of \( G. \) From (T2) and (i) in Remark 3.1, we have that \( G(x) \land G(x \rightarrow y) = G(x \land (x \rightarrow y)) = G(x \land y) \leq G(y). \) Therefore, \( G(x \rightarrow y) \leq G(x) \rightarrow G(y). \) Conversely, let
Taking into account the results established in [13], we only have to prove (K4) that

\[ G(x) \leq G(y) \]

from which we get that \( G \) is increasing. This last assertion and (T2)' we infer that \( G(x) \leq G(y \rightarrow (x \wedge y)) \leq G(y) \rightarrow G(x \wedge y) \). Thus, \( G(x) \wedge G(y) \leq G(x \wedge y) \).

From this statement and taking into account that \( G \) is increasing we conclude that \( G(x) \wedge G(y) = G(x \wedge y) \). \( \square \)

Thus, if in Definition 3.1 we replace the axiom (T2) by (T2)', we obtain an equivalent definition of tense \( SHn \)-algebras.

4. A discrete duality for \( SHn \)-algebras

In this section, we describe a discrete duality for tense \( SHn \)-algebras bearing in mind the results indicated in Section 2 for \( SHn \)-algebras. To this end, we introduce the following

**Definition 4.1.** A tense \( SHn \)-frame is a structure \((X, \leq, N, \{s_i\}_{i=1}^{n-1}, R, Q)\) where \((X, \leq, N, \{s_i\}_{i=1}^{n-1})\) is an \( SHn \)-frame, \( R, Q \) are binary relations on \( X \) and the following conditions are satisfied:

(K1) \( (\leq \circ R \circ \leq) \subseteq R \),

(K2) \( (\leq \circ Q \circ \leq) \subseteq Q \),

(K3) \( x R N(y) \) if and only if \( y Q N(x) \),

(K4) \( x R_T y \) implies \( s_i(x) R_T s_i(y) \) for \( T = G \) and \( T = H \),

(K5) \( s_i(z) R_T y \) implies that there is \( x \in X \) such that \( z R_T x \) and \( s_i(x) \leq y \) for \( T = G \) and \( T = H \).

In what follows, tense \( SHn \)-frames will be denoted simply by \( X \) when no confusion may arise.

**Definition 4.2.** A canonical frame of a tense \( SHn \)-algebra \((W, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, G, H, 0, 1)\) is a structure \((A(W), \leq^c, N^c, \{S_i\}_{i=1}^{n-1}, R^c, Q^c)\), where \((A(W), \leq^c, N^c, \{S_i\}_{i=1}^{n-1})\) is the canonical frame of the reduct \((W, \vee, \wedge, \rightarrow, \sim, \{S_i\}_{i=1}^{n-1}, 0, 1)\) and the following conditions are verified for \( P, F \in A(W) \):

(F1) \( PR^c F \) if and only if \( G^{-1}(P) \subseteq F \),

(F2) \( PQ^c F \) if and only if \( H^{-1}(P) \subseteq F \).

**Lemma 4.1.** The canonical frame of a tense \( SHn \)-algebra is a tense \( SHn \)-frame.

**Proof.** Taking into account the results established in [13], we only have to prove

(K1) − (K5).

(K1): Let \((P, F) \in (\leq^c \circ R^c \circ \leq^c)\). Then there exist \( T, S \in A(W) \) such that \( P \subseteq T, T R^c S \) and \( S \subseteq F \). From the last two assertions we have that \( G^{-1}(T) \subseteq F \). Therefore, since \( P \subseteq T \) we infer that \( P R^c F \).

(K2): It is proved in a similar way to (K1).

(K3): Let \( F R^c N^c(P) \) and \( a \in H^{-1}(P) \). Suppose that \( \sim a \in F \). On the other hand, from (T3) we have that \( \sim a \leq G(\sim H(a)) \) and so, we get that \( G(\sim H(a)) \in F \). From this last assertion and the fact that \( G^{-1}(F) \subseteq N^c(P) \), we obtain \( \sim H(a) \in N^c(P) \). Hence, \( H(a) \not\in P \) which is a contradiction. Therefore, \( a \not\in N^c(F) \) from which we conclude that \( PQ^c N^c(F) \). The converse is proved similarly.

(K4): It is a direct consequence of (F1), (F2) and (T4).

(K5): Let \( G^{-1}(s_i(F)) \subseteq P \) and considering \( E = \{ z \in W : G(z) \in F \} \), then we have that \( \bigwedge I \nleq \bigvee J \), for all finite subsets \( I \subseteq E, J \subseteq S_i(W \setminus P) \). Indeed: Suppose that there is \( I \subseteq E, J \subseteq S_i(W \setminus P) \) finite subsets such that \( \bigwedge I \nleq \bigvee J \). From
this last assertion and (T2) we infer that \( G(\bigvee I) \in F \). Since, \( G \) is increasing we obtain that \( G(\bigvee J) \in F \). On the other hand, it is straightforward to prove that \( \bigvee J \in S(W \setminus P) \). From this last assertion, there is \( a \in W \setminus P \) such that \( S_i(a) = \bigvee J \). Hence, \( G(S_i(a)) \in F \). Then, from (T4) we have deduced that \( a \in P \), which is a contradiction. Therefore \( E \) is separated (see [6, p. 186]) from \( S_i(W \setminus P) \), then from [6, p. 186], there is \( Z \in \mathcal{X}(W) \) such that \( E \subseteq Z \) and \( Z \cap S_i(W \setminus P) = \emptyset \). This last assertion allows us to conclude that \( s_i(Z) \subseteq P \) and \( FR^cZ \). Similarly, it is proved (K5) for \( T = H \).

**Definition 4.3.** The complex algebra of a tense \( SHn \)-frame \((X, \leq, N, \{S_i\}_{i=1}^{n-1}, R, Q)\) is \((C(X), \bigvee, \bigwedge, \sim, \{S_i\}_{i=1}^{n-1}, G^c, H^c, 0^c, 1^c)\), where the reduct \((C(X), \bigvee, \bigwedge, \sim, \{S_i\}_{i=1}^{n-1})\) is the complex algebra of the \( SHn \)-frame \((X, \leq, N, \{S_i\}_{i=1}^{n-1})\), \( G^c(A) = [R]A \) and \( H^c(A) = [Q]A \), for all \( A \in C(X) \).

**Lemma 4.2.** The complex algebra of a tense \( SHn \)-frame is a tense \( SHn \)-algebra.

**Proof.** From [13], \( C(X) \) is closed under the lattice operations, \( \sim, \sim^c \) and \( \{S_i\}_{i=1}^{n-1} \). Now, we show that it is also closed under \( G^c \) i.e., \( G^cA = [\leq]G^cA \). From the reflexivity of \( \leq \), we have that \( [\leq]G^cA \subseteq G^cA \). Assume that \( x \in G^cA \). Let \( y \in X \) be such that \( x \leq y \) and take any \( z \in X \) verifying \( yRz \). Hence, from the reflexivity of \( \leq \) and (K1) we infer that \( xRz \). So, \( z \in A \) and therefore, \( x \in [\leq]G^cA \). Thus, \( G^cA \subseteq [\leq]G^cA \). Similarly, it is proved that \( H^cA \subseteq [\leq]H^cA \). On the other hand, clearly (T1) and (T2) are verified. Therefore, it only remains to prove (T3) and (T4).

(T3): Let \( x \in A \) and suppose that \( x \not\in G^c(\sim^c H^c(\sim^c A)) \). Then there is \( y \) such that \( xRy \) and \( y \not\in G^c(\sim^c H^c(\sim^c A)) \). From this last statement, \( y \in N(H^c(\sim^c A)) \) and so, \( y = N(z) \) for some \( z \in H^c(\sim^c A) \). Hence, \( xRN(z) \) and from (K3) we get that \( zQN(x) \). This assertion and the fact that \( z \in H^c(\sim^c A) \) enable us to infer that \( N(x) \notin N(A) \), which is a contradiction. So, \( A \subseteq G^c(\sim^c H^c(\sim^c A)) \). Analogously, it is proved that \( A \subseteq H^c(\sim^c G^c(\sim^c A)) \).

(T4): Let \( s_i(y) \in [R]A \) and \( yRx \). Then, from (K4) we have that \( s_i(y)R s_i(x) \). From this last assertion we infer that \( s_i(x) \in A \). Therefore, \( S_i([R]A) \subseteq [R]S_i^c(A) \). On the other hand, let \( z \in [R](S_i^c(A)) \) and \( s_i(z)Ry \). Then by virtue of (K5), there is \( x \in X \) such that \( zRx \) and \( s_i(x) \leq y \). This last assertion allows us to conclude that \( y \in A \). Similarly, it is proved that \( S_i^c([Q]A) = [Q]S_i^c(A) \).

**Theorem 4.1.** Every tense \( SHn \)-algebra is embeddable into the complex algebra of its canonical frame.

**Proof.** Let us consider the function \( h : W \to \mathcal{C}(\mathcal{X}(W)) \) defined by \( h(a) = \{P \in \mathcal{X}(W) : a \in P\} \), for all \( a \in W \). Let \( F \in h(G(a)) \); then \( G(a) \in F \). Suppose that \( P \in \mathcal{X}(W) \) verifies that \( FR^cP \). Then from (F1), \( G^c(F) \subseteq P \) and so, \( a \in P \). Therefore, \( F \in G^c(h(a)) \) from which we infer that \( h(G(a)) \subseteq G^c(h(a)) \). Conversely, assume that \( F \in G^c(h(a)) \). Then for every \( P \in \mathcal{X}(W) \), \( FR^cP \) implies that \( P \in h(a) \). Suppose that \( G(a) \notin F \). Then \( G^c(F) \) is a filter and \( a \notin G^c(F) \). Hence, there is \( T \in \mathcal{X}(W) \) such that \( a \notin T \) and \( G^c(F) \subseteq T \). This last assertion and (F1) allow us to conclude that \( FR^cT \). From this statement we have that \( T \in h(a) \) and so, \( a \in T \), which is a contradiction. Therefore, \( h(G(a)) = G^c(h(a)) \). Similarly, it is shown that \( h(H(a)) = H^c(h(a)) \). Thus, by virtue of the results established in [13] the proof is completed.

Lemma 4.3 will show that the order–embedding \( k : X \to \mathcal{X}(\mathcal{C}(X)) \) defined by \( k(x) = \{A \in \mathcal{C}(X) : x \in A\} \) for every \( x \in X \) preserves the relations \( R \) and \( Q \).
Lemma 4.3. Let $(X, \leq, N, [s_i]_i^{n-1}, R, Q)$ be a tense $SH_n$–frame and let $x, y \in X$. Then

(i) $xRy$ if and only if $k(x)R^c k(y)$,
(ii) $xQy$ if and only if $k(x)Q^c k(y)$.

Proof. We will only prove (i). Assume that $xRy$ and suppose that $A \in \mathcal{C}(X)$ verifies $G^c(A) \in k(x)$. Then it is easy to see that $y \in A$ and so, $k(x)R^c k(y)$. Conversely, let $x, y \in X$ be such that $k(x)R^c k(y)$. Then $G^c(k(x)) \subseteq k(y)$. On the other hand, note that $[\leq](X \setminus \{y\}) \in \mathcal{C}(X)$ and $y \notin [\leq](X \setminus \{y\})$. Thus, $[\leq](X \setminus \{y\}) \notin k(y)$ and so, $[\leq](X \setminus \{y\}) \notin G^{-1}(k(x))$. Therefore, $[R](\leq)(X \setminus \{y\}) \notin k(x)$ from which we infer that $x \notin [R](\leq)(X \setminus \{y\})$. Then there is $z$ such that $xRz$ and $z \notin [\leq](X \setminus \{y\})$. From this last assertion there is $w$ such that $z \leq w$ and $w \leq y$, which allow us to infer that $z \leq y$. Hence, by virtue of the reflexivity of $\leq$ and (K1), $xRy$ as required.

Lemma 4.3 and the results indicated in [13] enable us to conclude

Theorem 4.2. Every tense $SH_n$–frame is embeddable into the canonical frame of its complex algebra.

Theorems 4.1 and 4.2 enable us to obtain a discrete duality for tense $SH_n$–algebras.

5. A propositional calculus based on tense $SH_n$–algebras

In this section, we will describe a propositional calculus that has tense $SH_n$–algebras as the algebraic counterpart. The terminology and symbols used here coincide in general with those used in [14].

Let $\mathcal{L} = (A^0, For[V])$ be a formalized language of zero order, where in the alphabet $A^0 = (V, L_0, L_1, L_2, U)$ the set

- $V$ of propositional variables is enumerable,
- $L_0$ is empty,
- $L_1$ contains $n+2$ elements denoted by $\sim$, $S_i$ (for $i = 1, \cdots, n-1$), $G$ and $H$ called negation sign, modal operators signs and tense operators signs, respectively,
- $L_2$ contains three elements denoted by $\vee$, $\wedge$, $\rightarrow$, called disjunction sign, conjunction sign and implication sign, respectively,
- $U$ contains two elements denoted by $(.,.)$.

For any $\alpha, \beta$ in the set $For[V]$ of all formulas over $A^0$, instead of $\alpha \rightarrow (\alpha \rightarrow \alpha)$, $\sim \alpha \sim \alpha$ and $\sim \alpha \sim \alpha$ and $\sim H \sim \alpha$ we will write for brevity $\neg \alpha$, $\alpha \rightarrow \beta$, $F\alpha$ and $P\alpha$, respectively.

We assume that the set $\mathcal{A}_i$ of logical axioms consists of all formulas of the following form, where $\alpha, \beta, \gamma$ are any formulas in $For[V]$:

(M0) the axioms of the $SH_n$–logic, i.e., the axioms (A1)-(A15) indicated in Section 2,
(M1) $G(\alpha \rightarrow \beta) \rightarrow (G\alpha \rightarrow G\beta)$,
(M2) $\alpha \rightarrow GP\alpha$, $\alpha \rightarrow HF\alpha$,
(M3) $S_iG\alpha \leftrightarrow GS_i\alpha$, $S_iH\alpha \leftrightarrow HS_i\alpha$.

The consequence operation $C_\mathcal{L}$ in $\mathcal{L}$ is determined by $\mathcal{A}_i$ and by the following rules of inference:

(R1) $\alpha, \alpha \rightarrow \beta \quad (R2) \alpha \rightarrow \beta \quad (R3) \alpha \rightarrow \beta \quad (R4) \alpha \rightarrow \beta \quad (R5) \alpha \rightarrow \beta.$

The system $TSH_n = (\mathcal{L}, C_\mathcal{L})$ thus obtained will be called the tense propositional $SH_n$–calculus. We will denote by $T$ the set of all formulas derivable in $TSH_n$. If $\alpha$ belongs to $T$ we will write $\vdash \alpha$.

Let $\equiv$ be the binary relation on $For[V]$ defined by

\[ \alpha \equiv \beta \iff \vdash \alpha \iff \vdash \beta. \]
Then it is easy to check that \( \cong \) is a congruence relation on \((\text{For}[V], \lor, \land, \rightarrow, \sim, \{S_i\}^{n-1}_{i=1}, G, H)\) and \( T \) determines an equivalence class which we will denote by 1. Moreover, taking into account [7], p. 300 it is straightforward to prove

**Theorem 5.1.** \((\text{For}[V]/ \cong, \lor, \land, \rightarrow, \sim, \{S_i\}^{n-1}_{i=1}, G, H, 0, 1)\) is a tense \( SHn \)–algebra, being 0 \( \sim 1 \).

**Definition 5.1.** A tense \( SHn \)–model based on a tense \( SHn \)–frame \( K = (X, \leq, N, \{s_i\}^{n-1}_{i=1}, R, Q) \) is a system \( M = (K, m) \) such that \( m : V \rightarrow \mathcal{P}(X) \) is a meaning function that assigns subsets of states to propositional variables, i.e. satisfies the following condition:

\[ (\text{her}) \quad x \leq y \quad \text{and} \quad x \in m(p) \quad \text{imply} \quad y \in m(p). \]

**Definition 5.2.** A tense \( SHn \)–model \( M = ((X, \leq, N, \{s_i\}^{n-1}_{i=1}, R, Q); m) \) satisfies a formula \( \alpha \) at the state \( x \) and we write \( M \models_x \alpha \), if the following conditions are satisfied:

- \( M \models_x p \) if and only if \( x \in m(p) \) for \( p \in V \),
- \( M \models_x \alpha \lor \beta \) if and only if \( M \models_x \alpha \) or \( M \models_x \beta \),
- \( M \models_x \alpha \land \beta \) if and only if \( M \models_x \alpha \) and \( M \models_x \beta \),
- \( M \models_x \sim \alpha \) if and only if \( M \not\models_{N(x)} \alpha \),
- \( M \models_x \alpha \rightarrow \beta \) if and only if for all \( y \), if \( x \leq y \) and \( M \models_y \alpha \) then \( M \models_y \beta \),
- \( M \models_x \sim \alpha \) if and only if for all \( y \), if \( x \leq y \) then \( M \not\models_y \alpha \),
- \( M \models_x S_\alpha \) if and only if \( M \models_{s(x)} \alpha \),
- \( M \models_x G \alpha \) if and only if for all \( y \), if \( xRy \) then \( M \models_y \alpha \),
- \( M \models_x H \alpha \) if and only if for all \( y \), if \( xQy \) then \( M \models_y \alpha \).

A formula \( \alpha \) is true in a tense \( SHn \)–model \( M \) (denoted by \( M \models \alpha \)) if and only if for every \( x \in W \), \( M \models_x \alpha \). The formula \( \alpha \) is true in a tense \( SHn \)–frame \( K \) (denoted by \( K \models \alpha \)) if and only if it is true in every tense \( SHn \)–model based on \( K \). The formula \( \alpha \) is valid if and only if it is true in every tense \( SHn \)–frame.

**Proposition 5.1.** Given a tense \( SHn \)–model \( M = (K, m) \), the meaning function \( m \) can be extended to all formulae by \( m(\alpha) = \{ x \in X : M \models_x \alpha \} \). For every tense \( SHn \)–model \( M \) and for every formula \( \alpha \), this extension has the property

\[ (\text{her}) \quad x \leq y \quad \text{and} \quad x \in m(\alpha) \quad \text{then} \quad y \in m(\alpha). \]

**Proof.** The proof is by induction with respect to complexity of \( \alpha \). By way of an example we show (her) for formulas of the form \( G \alpha \). Let (1) \( x \leq y \) and (2) \( M \models_x G(\alpha) \). Suppose that \( yRz \), then by (1),(2) and (K1), we have \( M \models_z \alpha \). □

**Theorem 5.2. (Completeness Theorem)** Let \( \alpha \) be a formula in \( TSHn \). Then the following conditions are equivalent:

(i) \( \alpha \) is derivable in \( TSHn \),
(ii) \( \alpha \) is valid.

**Proof.** (i) \( \Rightarrow \) (ii): We proceed by induction on the complexity of the formula \( \alpha \). For example, we shall prove that the axioms (M2) and (M3) are valid. Let \( K = (X, \leq, N, \{s_i\}^{n-1}_{i=1}, R, Q) \) be a tense \( SHn \)–frame and \( M \) a tense \( SHn \)–model based on \( K \).

(M2) \( \alpha \rightarrow H \alpha \) is valid. Indeed:

(1) Let \( y \in X \) be such that \( x \leq y \), \[ \text{hip.} \]
(2) \( M \models_y \alpha \), \[ \text{hip.} \]
(3) Let \( z \in X \) be such that \( yQz \), \[ \text{hip.} \]
Suppose that

(4) \( M \models_{N(z)} G \sim \alpha \), \[ \text{hip.} \]
(5) \( N(z) R N(y) \), \[(3),(K3)\]
(6) \( M \models N(y) \sim \alpha \), \[(4),(6)\]
(7) \( M \not\models \alpha \). \[(6)\]

(7) contradicts (2). Then
(8) \( M \not\models N(z) G \sim \alpha \), \[(4),(7)\]
(9) \( M \models z \sim G \sim \alpha \), \[(8)\]
(10) \( M \models y H \sim G \sim \alpha \), \[(3),(9)\]
(11) \( M \models x \alpha \rightarrow H \sim G \sim \alpha \). \[(1),(2),(10)\]

In a similar way we can prove that \( \alpha \rightarrow GP \alpha \) is valid.

(M3) \( S_1 G \alpha \leftrightarrow G S_1 \alpha \) is valid. Indeed:
(1) Let \( y \in X \) be such that \( x \leq y \), \[\text{hip.}\]
(2) \( M \models y S_1 G \alpha \), \[\text{hip.}\]
(3) Let \( z \in X \) be such that \( y \sim R z \), \[\text{hip.}\]
(4) \( s_i(y) \sim R s_i(z) \), \[(3),(K4)\]
(5) \( M \models s_i(z) \alpha \), \[(2),(4)\]
(6) \( M \models z S_1 \alpha \), \[(5)\]
(7) \( M \models G S_1 \alpha \), \[(3),(6)\]
(8) \( M \models S_1 G \alpha \rightarrow G S_1 \alpha \), \[(2),(7)\]

On the other hand
(9) Let \( y \in X \) be such that \( x \leq y \), \[\text{hip.}\]
(10) \( M \models y G S_1 \alpha \), \[\text{hip.}\]
(11) Let \( z \in X \) be such that \( s_i(y) \sim R z \), \[\text{hip.}\]
(12) there is \( w \in X \) such that \( y \sim R w \) and \( s_i(w) \leq z \), \[(11),(K5)\]
(13) \( M \models s_i(w) \alpha \), \[(10),(12)\]
(14) \( M \models z \alpha \), \[(12),(13),(her)\]
(15) \( M \models y S_1 G \alpha \), \[(11),(14)\]
(16) \( M \models G S_1 \alpha \rightarrow S_1 G \alpha \). \[(9),(10),(15)\]

Therefore, \( S_1 G \alpha \leftrightarrow G S_1 \alpha \) is valid. Analogously we can prove that \( S_1 H \alpha \leftrightarrow H S_1 \alpha \) is valid.

(ii) \(\Rightarrow\) (i): Assume that \( \alpha \) is not derivable, i.e. \( [\alpha] \approx \neq 1 \). We apply Theorem 4.1 to the tense \( S H_n \)-algebra \( \text{For}[\mathcal{V}] \approx \), hence there exists a tense \( S H_n \)-frame \( \mathcal{X}(\text{For}[\mathcal{V}] \approx) \) and an injective morphism of tense \( S H_n \)-algebras \( h : \text{For}[\mathcal{V}] \approx \rightarrow \mathcal{C}(\mathcal{X}(\text{For}[\mathcal{V}] \approx)) \). Let us consider the function \( m : T S H_n \rightarrow \mathcal{C}(\mathcal{X}(\text{For}[\mathcal{V}] \approx)) \) defined by \( m(\alpha) = h([\alpha]_{\approx}) \) for all \( \alpha \in \text{For}[\mathcal{V}] \). It is straightforward to prove that \( m \) is an meaning function. Since \( h \) is injective, \( m(\alpha) = h([\alpha]_{\approx}) \neq \mathcal{X}(\text{For}[\mathcal{V}] \approx) \), i.e. \( \mathcal{X}(\text{For}[\mathcal{V}] \approx), m \) \( \not\models s_0 \alpha \) for some \( s_0 \in \mathcal{X}(\text{For}[\mathcal{V}] \approx) \). Thus \( \alpha \) is not valid. \(\square\)

References

& Control 5 (2010), no. 5, 642–653.

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