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Upper Bounds on the Order of Nearly Regular Induced Subgraphs in Random Graphs

Yilun Shang

Abstract. Finding the order of a largest induced regular subgraph in every graph on n vertices was posed long ago by Erdős, Fajtlowicz and Staton. Motivated by this problem and recent investigation in random graphs, we consider the order of nearly regular induced (bipartite) subgraphs in Erdős-Rényi random graph $G(n, 1/2)$ and random bipartite graph $G(n, m, 1/2)$. We obtain comparable upper bounds by using combinatorial and probabilistic techniques.

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1. Introduction

An old problem of Erdős, Fajtlowicz and Staton (c.f. [4] and [3, p. 85]) asks for the order of a largest induced regular subgraph that can be found in every graph on n vertices. They conjecture that the quantity in question is $\omega(\ln n)$, which has not been completely settled so far. By the known estimates for graph Ramsey numbers [7], every graph on n vertices contains either a clique or an independent set of order $\Omega(\ln n)$, providing a trivial lower bound for the problem. An upper bound of order $O(n^{1/2}\ln^{3/4} n)$ has recently been obtained in [1].

Given the notorious difficulty of the problem, it is natural to study the asymptotic behavior of this graph theoretic parameter for a typical graph on n vertices, that is, in the Erdős-Rényi random graph model $G(n, p)$ with edge density $p = 1/2$ (see e.g. [8]). It is shown that the random graph $G(n, 1/2)$ almost surely contains no induced regular subgraphs on $cn/\ln n$ vertices for some constant $c > 0$. An improvement has recently been made in the work [9], where the authors take the upper bound down to $2n^{2/3}$.

In this paper, to extend the above asymptotic upper bounds, we analyze the order of nearly regular induced subgraphs in random graph $G(n, 1/2)$ and another classical random graph model $G(n, m, 1/2)$, the bipartite counterpart. For some recent progresses of random bipartite graphs, we mention the works [5, 11, 13]. Without loss of generality, we suppose that $m \le n$ throughout the paper. We define the nearly regular graph as follows (a different but less technical definition can be found in [1])

Definition 1.1. *A graph $G = (V, E)$ with $|V| = n$ is (c, ε)-nearly regular if there is some constant $c \ge 0$, a set $V_1 \subseteq V$ with $|V_1| = O(n^\varepsilon)$, $0 < \varepsilon < 1$, and $r = r(n) \ge 0$ such that*

$$d_i = r \ for\ v_i \in V\backslash V_1 \quad and \quad |d_i - r| \le c\ for\ v_i \in V_1,$$

where d_i is the degree of vertex v_i.

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Observe that V_1 serves as an exceptional set and that a $(0, \varepsilon)$-nearly regular graph is clearly a strictly r-regular graph.

The rest of the paper is organized as follows. We present our main results in Section 2. In Section 3, we give the proofs. Finally, we conclude the paper in Section 4.

2. Upper Bounds in Random (Bipartite) Graphs

The following result provides an upper bound of the order of nearly regular induced subgraphs in random graph $G(n, 1/2)$.

Theorem 2.1. *Let $\delta \geq 2/3$ and $\delta > \varepsilon$. Then almost surely every induced (c, ε)-nearly regular subgraph of $G(n, 1/2)$ has at most $2n^{\delta}$ vertices.*

In particular, if $\delta = 2/3$ and $c = 0$, Theorem 2.1 reproduces the upper bound derived in [9, Theorem 1.1], which is mentioned in Section 1.

Recall that a balanced bipartite graph is a bipartite graph with two partition parts having equal cardinality. The analogous bound for random bipartite graph $G(n, m, 1/2)$ is the following.

Theorem 2.2. *Let $\delta > \varepsilon$ and $n \geq m = \Omega(n^{\delta})$. Then almost surely every induced (c, ε)-nearly regular balanced bipartite subgraph of $G(n, m, 1/2)$ has at most $\Theta(n^{\delta})$ vertices.*

In the above statement we only consider bipartite subgraphs since the two independent sets (i.e. the bipartition sets) in a bipartite graph are trivial regular subgraphs. In addition, if a bipartite graph is regular, then it must be balanced. Hereby, we only treat balanced bipartite subgraphs.

3. Proofs

In this section, we provide the proofs of Theorem 2.1 and 2.2 with a similar line of reasoning in [9]. We show the results through a series of lemmas.

We will use Landau notations (e.g. [8]) for asymptotic behavior, such as $O, o, \Omega, \sim$, etc., when the underlying parameter tends to infinity. Let $G(\mathbf{d})$ denote the number of labeled simple graphs on k vertices with degree sequence $\mathbf{d} = (d_1, d_2, \cdots, d_k)$, with the degree of vertex v_i being d_i. The following lemma is a useful asymptotic enumeration result.

Lemma 3.1. *([10]) Let $d_j = d_j(k)$, $1 \leq j \leq k$ be integers such that $\sum_{j=1}^{k} d_j = \lambda k(k-1)$ is an even integer where $1/3 < \lambda < 2/3$, and $|\lambda k - d_j| = O(k^{1/2+\varepsilon})$ uniformly over j, for some sufficiently small fixed $\varepsilon > 0$. Then*

$$G(\mathbf{d}) = f(\mathbf{d})(\lambda^{\lambda}(1-\lambda)^{1-\lambda})^{\binom{k}{2}} \prod_{j=1}^{k} \binom{k-1}{d_j} \tag{1}$$

where $f(\mathbf{d}) = O(1)$, and if $\max_j\{|\lambda k - d_j|\} = o(\sqrt{k})$, then $f(\mathbf{d}) \sim \sqrt{2}e^{1/4}$, uniformly over the choice of such a degree sequence $\mathbf{d}$.

For an integer $k > 0$, let $l(k)$ be the largest even integer not over $(k-1)/2$. Denote the probability $p_k = P(G(k, 1/2)$ is $l(k)$-regular$)$. To prove Theorem 2.1 we will need the following lemma.

Lemma 3.2. *([9]) For every degree sequence* $\mathbf{d} = (d_1, \cdots, d_k)$ *we have*

$$P(G(k,1/2) \text{ has degree sequence } \mathbf{d}) = O(p_k) = O\big(((1+o(1))\sqrt{\pi k/2})^{-k}\big).$$

Noting that $p_k = G(\mathbf{d})2^{-\binom{k}{2}}$ where $d_i = l(k)$ for $1 \le i \le k$, we can prove the above lemma by employing Lemma 3.1.

Proof of Theorem 2.1. By Lemma 3.2, it follows that the probability that a fixed set V_0 of k vertices spans a (c,ε)-nearly regular subgraph in $G(n,1/2)$ is $O\big(k\binom{k}{n^\varepsilon}(2c)^{n^\varepsilon}p_k\big)$. Hence, the probability that $G(n,1/2)$ contains an induced (c,ε)-nearly regular subgraph on at least $2n^\delta$ vertices is

$$\sum_{k=2n^\delta}^{n}\binom{n}{k}O\left(k\binom{k}{n^\varepsilon}(2c)^{n^\varepsilon}p_k\right)$$
$$\le \sum_{k=2n^\delta}^{n}\left(\frac{en}{k}\right)^k k\left(\frac{ek}{n^\varepsilon}\right)^{n^\varepsilon}(2c)^{n^\varepsilon}\big((1+o(1))\sqrt{\pi k/2}\big)^{-k} \quad (2)$$

by using the Stirling formula.

Since the first and last terms after the summation sign on the right-hand side of (2) are decreasing with k, we obtain that (2) is bounded above by

$$\begin{aligned}\sum_{k=2n^\delta}^{n}k\left(\frac{ek}{n^\varepsilon}\right)^{n^\varepsilon}(2c)^{n^\varepsilon}\left(\frac{(1+o(1))\sqrt{2}en}{\sqrt{\pi}(2n^\delta)^{3/2}}\right)^{2n^\delta} &= \left(\frac{2ce}{n^\varepsilon}\right)^{n^\varepsilon}\Big(\sum_{k=2n^\delta}^{n}k^{1+n^\varepsilon}\Big)\\ &\quad\cdot\left(\frac{e}{2\sqrt{\pi}n^{3\delta/2-1}}\right)^{2n^\delta}\\ &\le n^{n^\varepsilon+2}\left(\frac{2ce}{n^\varepsilon}\right)^{n^\varepsilon}\left(\frac{e}{2\sqrt{\pi}n^{3\delta/2-1}}\right)^{2n^\delta}\\ &= O\Bigg((2ce)^{n^\varepsilon}n^{(1-\varepsilon)n^\varepsilon+2}\\ &\quad\cdot\left(\frac{e}{2\sqrt{\pi}n^{3\delta/2-1}}\right)^{2n^\delta}\Bigg). \qquad (3)\end{aligned}$$

Since $e < 2\sqrt{\pi}$ and by our assumptions in Theorem 2.1, we have the right-hand side of (3) tends to 0 as $n \to \infty$. The proof is then complete. □

Next, we analyze the order of induced subgraphs in random bipartite graph model $G(n,m,1/2)$. For a bipartite graph, throughout the paper we label the vertices in one of its independent set by $\{v_{2k-1}\}_{k\ge1}$ and the other by $\{v_{2k}\}_{k\ge1}$. Similarly, let $G(\mathbf{d})$ denote the number of labeled simple graphs on $2k$ vertices with degree sequence $\mathbf{d} = (d_1, d_2, \cdots, d_{2k})$.

For an integer $k > 0$, let $\tilde{l}(k)$ be the largest even integer not over $(2k-1)/2$. Denote the probability $\tilde{p}_k = P(G(k,k,1/2)$ is $\tilde{l}(k)$-regular). Clearly, $\tilde{p}_k = G(\mathbf{d})2^{-k^2}$, with all d_i being set equal to $\tilde{l}(k)$. The following lemma is a consequence of Lemma 3.1.

Lemma 3.3.

$$\tilde{p}_k = \big((1+o(1))2^{-k/2}\sqrt{\pi k}\big)^{-2k} \quad \textit{and} \quad \tilde{p}_{k-1}/\tilde{p}_k = \Theta(k).$$

Proof. Replacing k with $2k$ in Lemma 3.1, taking $\lambda = 1/2$ and $d_j = \tilde{l}(k)$ for $1 \le j \le 2k$, we derive that

$$\begin{aligned}\tilde{p}_k &= (1+o(1))\sqrt{2}e^{1/4}2^{-k^2-\binom{2k}{2}}\binom{2k-1}{\tilde{l}(k)}^{2k}\\ &= (1+o(1))\sqrt{2}e^{1/4}2^{-3k^2}\Big(\frac{2^{2k}}{\sqrt{\pi k}}\Big)^{2k}\\ &= \big((1+o(1))2^{-k/2}\sqrt{\pi k}\big)^{-2k}\end{aligned}$$

by utilizing the Stirling formula. Therefore, it is routine to check that $\tilde{p}_{k-1}/\tilde{p}_k = \Theta(k)$ holds. □

We will also need the following lemma to prove Theorem 2.2.

Lemma 3.4. *For every degree sequence* $\mathbf{d} = (d_1, \cdots, d_{2k})$ *we have*

$$P(G(k,k,1/2) \textit{ has degree sequence } \mathbf{d}) = O\big(\tilde{p}_k(\sqrt{\pi k})^k 2^{k^2/2+k}\big).$$

Proof. Let $\mathbf{d}$ be a degree sequence of length $2k$ for which $G(\mathbf{d})$ is maximal. Therefore, $\mathbf{d}$ is the most likely degree sequence in $G(k,k,1/2)$. Consider the following two cases separately.

Case 1: For all i, $|d_i - k/2| \le \Theta(k^{1/2+\varepsilon})$ for some fixed $\varepsilon > 0$.

Therefore, by Lemma 3.1 we get

$$\begin{aligned}P(G(k,k,1/2) \text{ has degree sequence } \mathbf{d}) &= \frac{G(\mathbf{d})}{2^{k^2}}\\ &= O\bigg(2^{-k^2-\binom{k}{2}}\binom{k-1}{k/2}^k\bigg)\\ &= O\Big(2^{-k^2/2+k}(\sqrt{\pi k})^{-k}\Big)\\ &= O\big(\tilde{p}_k(\sqrt{\pi k})^k 2^{k^2/2+k}\big).\end{aligned}$$

Case 2: There exists some d_i, say d_{2k}, satisfying $|d_{2k} - k/2| > \Theta(k^{1/2+\varepsilon})$ for some fixed $\varepsilon > 0$.

Then we expose the edges from vertex v_{2k} to the rest of the graph. Since d_{2k} obeys the binomial distribution $\mathrm{Bin}(k, 1/2)$, we have

$$P(|d_{2k} - k/2| > \Theta(k^{1/2+\varepsilon})) \le 2e^{-2k^{2\varepsilon}/3} \tag{4}$$

by a standard concentration inequality (c.f. [8] pp.27). Write

$$P(G(k,k,1/2) \text{ has degree sequence } \mathbf{d}) = a_k\tilde{p}_k.$$

Recall the labels of the vertices, and the edges exposed induce a new degree sequence on vertices $1, \cdots, 2k-2$. Hence, the probability that the random bipartite graph $G(k-1, k-1, 1/2)$ has this degree sequence is at most $a_{k-1}\tilde{p}_{k-1}$ in our notation. Accordingly, from (4) we have $a_k\tilde{p}_k \le a_{k-1}\tilde{p}_{k-1}O(e^{-k^{2\varepsilon}})$. Dividing $\tilde{p}_k$ on both sides, we have $a_k \le a_{k-1}O(e^{-k^{\varepsilon}})$ by Lemma 3.3. Induction starting from some k_0 for which the above factor $O(e^{-k^{\varepsilon}}) \le 1$ yields $a_k \le a_{k_0}$. The proof is then complete. □

Proof of Theorem 2.2. By Lemma 3.4, it follows that the probability that a fixed set $V_1 \cup V_2$ with $|V_1| = |V_2| = k$ vertices spans a (c,ε)-nearly regular subgraph in $G(n,m,1/2)$ is $O\big(k\binom{2k}{n^{\varepsilon}}(2c)^{n^{\varepsilon}}\tilde{p}_k(\sqrt{\pi k})^k 2^{k^2/2+k}\big)$. Therefore, the probability that

$G(n, m, 1/2)$ contains an induced (c, ε)-nearly regular balanced bipartite subgraph on at least $\Theta(n^\delta)$ vertices is given by

$$\begin{aligned}
&\sum_{k=\Theta(n^\delta)}^{m} \binom{n}{k}\binom{m}{k} O\bigg(k\binom{2k}{n^\varepsilon}(2c)^{n^\varepsilon}\tilde{p}_k(\sqrt{\pi k})^k 2^{k^2/2+k}\bigg) \\
&\quad = O\bigg(\sum_{k=\Theta(n^\delta)}^{n} \Big(\frac{en}{k}\Big)^{2k} k\Big(\frac{2ek}{n^\varepsilon}\Big)^{n^\varepsilon}(2c)^{n^\varepsilon}2^{-k^2/2+k}(\sqrt{\pi k})^{-k}\bigg) \\
&\quad = O\bigg(\Big(\frac{2ce}{n^\varepsilon}\Big)^{n^\varepsilon}\sum_{k=\Theta(n^\delta)}^{n} k^{1+n^\varepsilon}\Big(\frac{e^2 n^2 2^{1-k/2}}{\sqrt{\pi}k^{5/2}}\Big)^k\bigg) \qquad (5)
\end{aligned}$$

using Lemma 3.3 and the Stirling formula.

With the similar reasoning in the proof of Theorem 2.1, the right-hand side of (5) is bounded above by

$$\begin{aligned}
&O\bigg(\Big(\frac{2ce}{n^\varepsilon}\Big)^{n^\varepsilon} n^{n^\varepsilon+2}\Big(\frac{e^2n^2 2^{1-n^\varepsilon}}{\sqrt{\pi}n^{5\delta/2}}\Big)^{n^\delta}\bigg) \\
&\qquad\qquad = O\bigg((2ce)^{n^\varepsilon} n^{(1-\varepsilon)n^\varepsilon+2}\Big(\frac{2e^2}{2^{n^\delta}\sqrt{\pi}n^{5\delta/2-2}}\Big)^{n^\delta}\bigg). \qquad (6)
\end{aligned}$$

Since $\delta > \varepsilon$ and $2e^2 < 2^{n^\delta} n^{5\delta/2-2}$ for any $\delta > 0$, (6) tends to 0 as $n \to \infty$. The proof is complete. □

4. Concluding Remarks

In this paper, we studied the order of nearly regular induced subgraphs in typical random graph models $G(n, 1/2)$ and $G(n, m, 1/2)$. Upper bounds are provided by using combinatorial and probabilistic techniques. Compared with Theorem 1.1 in [9], our upper bounds seem to be tight. A natural question would be the corresponding lower bounds. One might be tempted to modify the proofs in [9] for the lower bounds. However, we tried this and our lower bounds derived are quite conservative. We believe substantial different techniques are needed to get matching lower bounds. On the other hand, the induced subgraphs in sparse random graphs are treated in [6] using switching techniques.

Finally, we mention another interesting problem: Can we estimate the edge density p given the order of regular induced subgraphs? A study in this direction for connected components can be found in [12].

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(Yilun Shang) Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China
E-mail address: shylmath@hotmail.com