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On some properties of Jensen-Steffensen's functional

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Abstract. Motivated by results of S.S. Dragomir, related to superadditivity and monotonicity of discrete Jensen's functional, in this paper we consider Jensen-Steffensen's functional, i.e. Jensen's functional with the conditions derived from Jensen-Steffensen's inequality. We state and prove results for this functional, similar to those of Dragomir, as well as their integral versions. Finally, some applications concerning means are given.

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1. Introduction

Let us denote with $\mathcal{P}_n$ the set of all real n-tuples $\mathbf{p} = (p_1, \ldots, p_n)$ such that $P_k := \sum_{i=1}^{k} p_i$, $k = 1, \ldots, n$, with $0 \leq P_k \leq P_n$, $k = 1, \ldots, n-1$, and $P_n > 0$. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a convex function. If $\mathbf{x} = (x_1, \ldots, x_n)$ is a monotonic (increasing or decreasing) n-tuple in I^n and $\mathbf{p}$ is in $\mathcal{P}_n$, then Jensen-Steffensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds. Since $\sum_{i=1}^{n} p_i x_i \in I$, (see the proof of corresponding theorem in [7, p.57], functional

$$J(f, \mathbf{x}, \mathbf{p}) := \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right), \tag{2}$$

introduced by observing the difference between the right side and the left side of (1), is well defined. We call it discrete Jensen-Steffensen's functional. For a fixed function f and n-tuple $\mathbf{x}$, $J(f, \mathbf{x}, \cdot)$ can be considered as a function on the set $\mathcal{P}_n$. Also, because of (1) we have that $J(f, \mathbf{x}, \mathbf{p}) \geq 0$, for all $\mathbf{p}$ in $\mathcal{P}_n$.

Of course, (1) can be considered under stricter conditions on $\mathbf{p}$, that is, requiring that $\mathbf{p}$ is a non-negative n-tuple, such that $P_n = \sum_{i=1}^{n} p_i > 0$, while $\mathbf{x}$ is any n-tuple in I^n. These assumptions rename (1) into well known Jensen's inequality. In this case, observing the difference between the right side and the left side of Jensen's inequality, Dragomir et al. (see [3]) introduced and investigated discrete Jensen's functional

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right). \tag{3}$$

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Now, let $\mathcal{P}_n^0$ denote the set of all non-negative n-tuples of real numbers with $P_n = \sum_{i=1}^n p_i > 0$. Obviously, $\mathcal{P}_n^0 \subseteq \mathcal{P}_n$. For fixed function f and n-tuple $\mathbf{x}$, $J_n(f, \mathbf{x}, \cdot)$ can be considered as a function on the set $\mathcal{P}_n^0$. Dragomir proved that $J_n(f, \mathbf{x}, \cdot)$ is superadditive on $\mathcal{P}_n^0$, that is, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^0$, then

$$J_n(f, \mathbf{x}, \mathbf{p}+\mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) + J_n(f, \mathbf{x}, \mathbf{q}), \tag{4}$$

and is also increasing on $\mathcal{P}_n^0$, that is,

$$\text{if} \quad \mathbf{p} \geq \mathbf{q}, \quad \text{then} \quad J_n(f, \mathbf{x}, \mathbf{p}) \geq J_n(f, \mathbf{x}, \mathbf{q}) \geq 0. \tag{5}$$

(Here $\mathbf{p} \geq \mathbf{q}$ means $p_i \geq q_i$, $i = 1, \ldots, n$.) However, monotonicity property (5) had been obtained by Pečarić, (see [4, p.717]), even before Dragomir unified both properties. Furthermore, in a recent paper, (see [2]), Dragomir gave comparative inequalities - this time the result is stated for normalized Jensen's functional. Namely, he proved that for a convex function $f : K \to \mathbb{X}$ defined on a closed convex subset K of linear space X and non-negative n-tuples $\mathbf{p}$ and $\mathbf{q}$, such that $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$ inequalities

$$\max_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\} J_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq \min_{1\leq i\leq n}\left\{\frac{p_i}{q_i}\right\} J_n(f, \mathbf{x}, \mathbf{q}) \geq 0, \tag{6}$$

hold for any $\mathbf{x} = (x_1, \ldots, x_n) \in K^n$.

Barić et al. gave (in [1]) an alternative proof of (6). That proof allowed them to prove another result, analogous to (6), in the case when $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and, for our issue more important: n-tuples $\mathbf{p}$ and $\mathbf{q}$ satisfy conditions for Jensen-Steffensen's inequality. We cite this result.

Theorem 1.1. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two n-tuples satisfying*

$$0 \leq P_k, Q_k \leq 1, \quad k = 1, \ldots, n-1, \quad P_n = Q_n = 1.$$

For $k \in \{1, \ldots, n\}$ denote $P_k := \sum_{i=1}^k p_i$, $Q_k := \sum_{i=1}^k q_i$. Let m and Mbe any real constants such that

$$P_k - mQ_k \geq 0, \quad (1-P_k) - m(1-Q_k) \geq 0, \quad k = 1, \ldots, n-1$$

and

$$MQ_k - P_k \geq 0, \quad M(1-Q_k) - (1-P_k) \geq 0, \quad k = 1, \ldots, n-1.$$

If $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and if $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then

$$MJ_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq mJ_n(f, \mathbf{x}, \mathbf{q}). \tag{7}$$

It was also shown in [1] that Theorem 1.1 provides improvement of (6) in the case of $X = \mathbb{R}$. The following corollary of Theorem 1.1 was also given in [1] and will be of interest in the sequel. It considers the uniform distribution $\mathbf{u} = (\frac{1}{n}, \ldots, \frac{1}{n})$ and corresponding non-weighted functional

$$J_n(f, \mathbf{x}) := J_n(f, \mathbf{x}, \mathbf{u}) = \frac{1}{n}\sum_{i=1}^n f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^n x_i\right).$$

Corollary 1.1. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ be n-tuple satisfying*

$$0 \leq P_k \leq 1, \quad k = 1, \ldots, n-1, \quad P_n = 1.$$

For $k \in \{1, \ldots, n\}$ denote $P_k := \sum_{i=1}^k p_i$ and define

$$\tilde{m}_0 := n \cdot \min\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1, \ldots, n-1\right\},$$

$$\tilde{M}_0 := n \cdot \max\left\{\frac{P_k}{k}, \frac{1-P_k}{n-k} : k = 1, \dots, n-1\right\}.$$

If $f : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ and if $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ is any monotonic n-tuple, then

$$\tilde{M}_0 J_n(f, \mathbf{x}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq \tilde{m}_0 J_n(f, \mathbf{x}). \tag{8}$$

In the following section we are going to show that superadditivity property holds for Jensen-Steffensen's functional, too, and that monotonicity property is adjustable to it. By means of these, we are going to give the alternative proofs of the results from [1]. In Section 3 we are going to prove the integral versions of our results, including those related to Boas' generalization of Jensen-Steffensen's inequality. Finally, the last section contains some applications and improvements concerning weighted quasi-arithmetic means.

2. Main results

First we are concerned with the superadditivity property of the functional (2).

Theorem 2.1. *Let $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$ be two n-tuples from $\mathcal{P}_n$. If $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1, \dots, x_n) \in I^n$ is any monotonic n-tuple, then $J(f, \mathbf{x}, \cdot)$ defined by (2) is superadditive on $\mathcal{P}_n$, i.e.*

$$J(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq J(f, \mathbf{x}, \mathbf{p}) + J(f, \mathbf{x}, \mathbf{q}) \geq 0. \tag{9}$$

Proof. Regarding definition we have

$$\begin{aligned} J(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) &= \sum_{i=1}^{n} (p_i + q_i) f(x_i) - (P_n + Q_n) f\left(\frac{\sum_{i=1}^{n} (p_i + q_i) x_i}{P_n + Q_n}\right) \\ &= \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} q_i f(x_i) \\ &\quad - (P_n + Q_n) f\left(\frac{\sum_{i=1}^{n} (p_i + q_i) x_i}{P_n + Q_n}\right), \end{aligned} \tag{10}$$

while convexity of f and Jensen's inequality yield

$$\begin{aligned} f\left(\frac{\sum_{i=1}^{n} (p_i + q_i) x_i}{P_n + Q_n}\right) &= f\left(\frac{\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} q_i x_i}{P_n + Q_n}\right) \\ &= f\left(\frac{P_n}{P_n + Q_n} \frac{\sum_{i=1}^{n} p_i x_i}{P_n} + \frac{Q_n}{P_n + Q_n} \frac{\sum_{i=1}^{n} q_i x_i}{Q_n}\right) \\ &\leq \frac{P_n}{P_n + Q_n} f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right) + \frac{Q_n}{P_n + Q_n} f\left(\frac{\sum_{i=1}^{n} q_i x_i}{Q_n}\right) \end{aligned} \tag{11}$$

Finally, combining relation (10) and inequality (11) we get

$$\begin{aligned} &J(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq \\ &\quad \geq \sum_{i=1}^{n} p_i f(x_i) + \sum_{i=1}^{n} q_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right) - Q_n f\left(\frac{\sum_{i=1}^{n} q_i x_i}{Q_n}\right) \\ &\quad = J(f, \mathbf{x}, \mathbf{p}) + J(f, \mathbf{x}, \mathbf{q}). \end{aligned}$$

Because of (1) we have that $J(f, \mathbf{x}, \mathbf{p}) \geq 0$ and $J(f, \mathbf{x}, \mathbf{q}) \geq 0$, so the proposed right inequality in (9) holds. □

In order to adjust monotonicity property (5) to functional (2), we are going to impose some extra conditions on n-tuples $\mathbf{p}$ and $\mathbf{q}$, as follows.

Theorem 2.2. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two n-tuples from $\mathcal{P}_n$. Let $P_k \geq Q_k$, $P_n - P_k \geq Q_n - Q_k$, $k = 1, \ldots, n-1$, and $P_n > Q_n$, where $P_k = \sum_{i=1}^k p_i$ and $Q_k = \sum_{i=1}^k q_i$. If $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then for functional $J(f, \mathbf{x}, \cdot)$ defined by (2) inequality*

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{q}) \tag{12}$$

holds on $\mathcal{P}_n$.

Proof. Write $J(f, \mathbf{x}, \mathbf{p}) = J(f, \mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q})$. Now, if we could apply superadditivity property (9) to $\mathbf{p} - \mathbf{q}$ and $\mathbf{q}$, monotonicity property would also be proved. And that would be the case if the n-tuple $\mathbf{p} - \mathbf{q} = (p_1 - q_1, \ldots, p_n - q_n)$ belonged to $\mathcal{P}_n$. Hence the following conditions need to be satisfied: $0 \leq P_k - Q_k \leq P_n - Q_n$, $k = 1, \ldots, n-1$, and $P_n - Q_n > 0$, which yields: $0 \leq P_k - Q_k$, $P_k - Q_k \leq P_n - Q_n$, $k = 1, \ldots, n-1$, and $P_n - Q_n > 0$. Now, taking into account that $J(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) \geq 0$, we have

$$J(f, \mathbf{x}, \mathbf{p}) = J(f, \mathbf{x}, \mathbf{p} - \mathbf{q} + \mathbf{q}) \geq J(f, \mathbf{x}, \mathbf{p} - \mathbf{q}) + J(f, \mathbf{x}, \mathbf{q}) \geq J(f, \mathbf{x}, \mathbf{q}).$$

□

Remark 2.1. *We can easily obtain the result from Theorem 1.1 from [1] by means of Theorem 2.2. Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n$ and let m and M be real constants such that $\mathbf{p} - m\mathbf{q}$ and $M\mathbf{q} - \mathbf{p}$ are in $\mathcal{P}_n$. If $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, is a convex function and if $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then by Theorem 2.2 is*

$$J(f, \mathbf{x}, \mathbf{p}) = J(f, \mathbf{x}, \mathbf{p} - m\mathbf{q} + m\mathbf{q}) \geq J(f, \mathbf{x}, \mathbf{p} - m\mathbf{q}) + J(f, \mathbf{x}, m\mathbf{q}) \geq mJ(f, \mathbf{x}, \mathbf{q}).$$

Similarly we get

$$J(f, \mathbf{x}, \mathbf{p}) \leq MJ(f, \mathbf{x}, \mathbf{q}),$$

that is

$$MJ(f, \mathbf{x}, \mathbf{q}) \geq J(f, \mathbf{x}, \mathbf{p}) \geq mJ(f, \mathbf{x}, \mathbf{q}). \tag{13}$$

Since $\mathbf{p} - m\mathbf{q} \in \mathcal{P}_n$ implies $P_k \geq mQ_k$ and $(P_n - P_k) \geq m(Q_n - Q_k)$, and since $M\mathbf{q} - \mathbf{p} \in \mathcal{P}_n$ implies $P_k \leq MQ_k$ and $(P_n - P_k) \leq M(Q_n - Q_k)$, $k = 1, \ldots, n-1$, which are the assumptions of Theorem 1.1 (only in a non-normalized form), by obtaining (13), we proved Theorem 1.1.

Applying Theorem 2.2, we are able to give the result on bounding the functional (2) with a non-weighted functional. But, almost the same result, only in a slightly specialized form, is given in Corollary 1.1. Our proof would then be the alternative one, obtained via Theorem 2.2. Hence the detailed analysis is given in the form of a remark.

Remark 2.2. *In order not to derange our former consideration, we write Corollary 1.1 in a slightly different form, namely, for $P_n > 0$:*

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be an n-tuple from $\mathcal{P}_n$. Define

$$m = \min_{1 \leq k \leq n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n - k} \right\}, \quad M = \max_{1 \leq k \leq n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n - k} \right\},$$

where $P_k = \sum_{i=1}^{k} p_i$ *and* $P_n = \sum_{i=1}^{n} p_i$. *If* $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, *is a convex function and if* $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ *is any monotonic* n*-tuple, then*

$$MJ_{\mathcal{N}}(f, \mathbf{x}) \geq J(f, \mathbf{x}, \mathbf{p}) \geq mJ_{\mathcal{N}}(f, \mathbf{x}),$$

where $J_{\mathcal{N}}(f, \mathbf{x}) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$.

Alternative proof of Corollary 1.1*:* Let $\mathbf{q}_{\min} \in \mathcal{P}_n^0$ be a constant n-tuple, i.e. $q_{\min} = (\alpha, \alpha, \ldots, \alpha)$, where $\alpha > 0$, for $Q_n := \sum_{i=1}^{n} q_i > 0$ must be satisfied. Provided $P_k \geq Q_k = k\alpha$, $P_n - P_k \geq Q_n - Q_k = (n-k)\alpha$, $k = 1, \ldots, n-1$, and $P_n > Q_n = n\alpha$, Theorem 2.2 can be applied. Further, these imply corresponding conditions concerning α :

(i) $\alpha \leq \frac{P_k}{k}$, $\quad k = 1, \ldots, n-1$,

(ii) $\alpha \leq \frac{P_n - P_k}{n-k}$, $\quad k = 1, \ldots, n-1$,

(iii) $\alpha < \frac{P_n}{n}$.

In order to prove the right inequality, let us first denote

$$m = \min_{1 \leq k \leq n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n-k} \right\}.$$

Obviously, m satisfies conditions (i) and (ii), and is a candidate for the choice of α. However, (iii) needs some extra considerations. Fix $k \in \{1, \ldots, n\}$. Then (i) and (ii) imply $n\alpha \leq P_n$, i.e. $\alpha \leq \frac{P_n}{n}$. Now we distinguish two cases:

1° $\alpha < \dfrac{P_n}{n}$. Condition (iii) is instantly satisfied and Theorem 2.2 yields

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{q}_{\min}).$$

2° $\alpha = \dfrac{P_n}{n}$, i.e. $P_n = n\alpha$. From (ii) we get $n\alpha - P_k \geq n\alpha - k\alpha$, i.e. $P_k \leq k\alpha$. But from (i) is also $P_k \geq k\alpha$, hence $P_k = k\alpha$, $k = 1, \ldots, n-1$. Since in that case is $\mathbf{p} = (\alpha, \alpha, \ldots, \alpha) = \mathbf{q}_{\min}$, inequality

$$J(f, \mathbf{x}, \mathbf{p}) \geq J(f, \mathbf{x}, \mathbf{q}_{\min})$$

holds again.

So m is a good choice for α. Now, respecting notation from the corollary statement we get

$$J(f, \mathbf{x}, \mathbf{q}_{\min}) = m\left(\sum_{i=1}^{n} f(x_i) - nf\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right) = mJ_{\mathcal{N}}(f, \mathbf{x}).$$

Lower bound provided by the non-weighted functional is then

$$J(f, \mathbf{x}, \mathbf{p}) \geq mJ_{\mathcal{N}}(f, \mathbf{x}).$$

Upper bound is obtained similarly, by exchanging the roles of $\mathbf{p}$ and $\mathbf{q}$, and with

$$M = \max_{1 \leq k \leq n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n-k} \right\}.$$

□

3. Integral versions

One of the integral analogues of Jensen-Steffensen's inequality was given by R. P. Boas.

Theorem 3.1. *(Steffensen-Boas) Let $x : [\alpha, \beta] \to \langle a, b\rangle$ be a continuous and monotonic function, where $-\infty < \alpha < \beta < \infty$ and $-\infty \le a < b \le \infty$, and let $f : \langle a, b\rangle \to \mathbb{R}$ be a convex function. If $\lambda : [\alpha, \beta] \to \mathbb{R}$ is either continuous or of bounded variation satisfying*

$$\lambda(\alpha) \le \lambda(t) \le \lambda(\beta) \quad \text{for all} \quad t \in [\alpha, \beta], \qquad \lambda(\beta) - \lambda(\alpha) > 0, \tag{14}$$

then

$$f\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta x(t) d\lambda(t)\right) \le \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta f(x(t)) d\lambda(t).$$

The condition (14) on λ can be regarded as a very week version of monotonicity, but the monotonicity condition on x is very restrictive. So Boas proved that one can strengthen the hypothesis on λ and correspondingly weaken the hypothesis on x, so that (16) still holds:

Theorem 3.2. *(Boas) Let $\lambda : [\alpha, \beta] \to \mathbb{R}$ be either continuous or of bounded variation and such that there exist $k \ge 2$ points $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_k = \beta$ so that*

$$\begin{aligned} &\lambda(\alpha) \le \lambda(t_1) \le \lambda(\gamma_1) \le \lambda(t_2) \le \cdots \le \lambda(\gamma_{k-1}) \le \lambda(t_k) \le \lambda(\beta), \\ &\text{for all} \quad t_i \in [\gamma_{i-1}, \gamma_i], \quad i = 1, \ldots, k, \quad \lambda(\beta) - \lambda(\alpha) > 0. \end{aligned} \tag{15}$$

If $x : [\alpha, \beta] \to \langle a, b\rangle$ is a continuous and monotone function on each of the intervals $[\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$, then inequality

$$f\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta x(t) d\lambda(t)\right) \le \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta f(x(t)) d\lambda(t).$$

holds for any convex function $f : \langle a, b\rangle \to \mathbb{R}$.

Regarding corresponding assumptions of Theorem 3.1 or Theorem 3.2, inequality

$$f\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta x(t) d\lambda(t)\right) \le \frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta f(x(t)) d\lambda(t). \tag{16}$$

is called Jensen-Steffensen's integral inequality or Jensen-Steffensen-Boas' integral inequality, respectively. In the similar way, we consider the functional

$$J(f, x, \lambda) := \int_\alpha^\beta f(x(t)) d\lambda(t) - (\lambda(\beta) - \lambda(\alpha)) f\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta x(t) d\lambda(t)\right), \tag{17}$$

which is called Jensen-Steffensen's or Jensen-Steffensen-Boas' depending on which conditions of the previous theorems are taken into consideration. Also, under appropriate assumptions on f, x and λ, either for Jensen-Steffensen's or Jensen-Steffensen-Boas' integral inequality, we have that $J(f, x, \lambda) \ge 0$. For the sake of simplicity, denote with

$\Lambda_{[\alpha,\beta]}, -\infty < \alpha < \beta < \infty$: the class of all functions $\lambda : [\alpha, \beta] \to \mathbb{R}$ which are either continuous or of bounded variation and satisfy the conditions (14);

$\tilde{\Lambda}_{[\alpha,\beta]}$: the subclass of $\Lambda_{[\alpha,\beta]}$, containing every $\lambda \in \Lambda_{[\alpha,\beta]}$ satisfying the conditions (15).

Now we can state and prove the integral analogues of the results from the previous section. First we give results related to functional (17) under conditions given in Theorem 3.1.

Theorem 3.3. *Let λ and μ be functions from $\Lambda_{[\alpha,\beta]}$, either both continuous or both of bounded variation. If $x:[\alpha,\beta]\to\langle a,b\rangle$, $a,b\in\overline{\mathbb{R}}$, is a continuous and monotonic function and if $f:\langle a,b\rangle\to\mathbb{R}$ is a convex function, then functional $J(f,x,\cdot)$ defined by* (17) *is superadditive on $\Lambda_{[\alpha,\beta]}$, i.e.*

$$J(f,x,\lambda+\mu)\geq J(f,x,\lambda)+J(f,x,\mu)\geq 0. \tag{18}$$

Proof. Let us first denote: $\lambda(\beta)-\lambda(\alpha):=\lambda_\alpha^\beta$ and $\mu(\beta)-\mu(\alpha):=\mu_\alpha^\beta$. Regarding definition we have

$$\begin{aligned}
&J\left(f,x,\lambda+\mu\right)=\\
&=\int_\alpha^\beta f(x(t))d(\lambda+\mu)(t)-\left(\lambda_\alpha^\beta+\mu_\alpha^\beta\right)\cdot f\left(\frac{1}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\int_\alpha^\beta x(t)d(\lambda+\mu)(t)\right)\\
&=\int_\alpha^\beta f(x(t))d\lambda(t)+\int_\alpha^\beta f(x(t))d\mu(t)-\left(\lambda_\alpha^\beta+\mu_\alpha^\beta\right)\cdot\\
&\quad\cdot f\left(\frac{1}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\int_\alpha^\beta x(t)d(\lambda+\mu)(t)\right),
\end{aligned} \tag{19}$$

while convexity of f and (integral) Jensen's inequality yield

$$\begin{aligned}
&f\left(\frac{1}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\int_\alpha^\beta x(t)d(\lambda+\mu)(t)\right)=\\
&\quad=f\left(\frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\lambda_\alpha^\beta}+\frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot\frac{\int_\alpha^\beta x(t)d\mu(t)}{\mu_\alpha^\beta}\right)\\
&\quad\leq\frac{\lambda_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\lambda_\alpha^\beta}\right)+\frac{\mu_\alpha^\beta}{\lambda_\alpha^\beta+\mu_\alpha^\beta}\cdot f\left(\frac{\int_\alpha^\beta x(t)d\mu(t)}{\lambda_\alpha^\beta}\right).
\end{aligned} \tag{20}$$

Finally, combining (19) and (20) we get

$$\begin{aligned}
&J\left(f,x,\lambda+\mu\right)\geq\\
&\quad\geq\int_\alpha^\beta f(x(t))d\lambda(t)+\int_\alpha^\beta f(x(t))d\mu(t)-\lambda_\alpha^\beta\cdot f\left(\frac{\int_\alpha^\beta x(t)d\lambda(t)}{\lambda_\alpha^\beta}\right)-\\
&\quad-\mu_\alpha^\beta\cdot f\left(\frac{\int_\alpha^\beta x(t)d\mu(t)}{\lambda_\alpha^\beta}\right)=J\left(f,x,\lambda\right)+J\left(f,x,\mu\right).
\end{aligned}$$

Because of integral inequality (16), under Jensen-Steffensen's conditions, we have that $J\left(f,x,\lambda\right)\geq 0$ and $J\left(f,x,\mu\right)\geq 0$ so the proposed right inequality in (18) holds. □

The integral version of Theorem 2.2 is given in the form of the result that follows.

Theorem 3.4. *Let λ and μ be functions from $\Lambda_{[\alpha,\beta]}$, either both continuous or both of bounded variation. Let*

$$\lambda(\alpha)-\mu(\alpha)\leq\lambda(t)-\mu(t)\leq\lambda(\beta)-\mu(\beta),\quad t\in[\alpha,\beta],\quad\lambda(\beta)-\mu(\beta)>\lambda(\alpha)-\mu(\alpha).$$

If $x:[\alpha,\beta]\to\langle a,b\rangle$, $a,b\in\overline{\mathbb{R}}$, is a continuous and monotonic function and if $f:\langle a,b\rangle\to\mathbb{R}$ is a convex function, then for functional $J(f,x,\cdot)$ defined by (17) *inequality*

$$J(f,x,\lambda)\geq J(f,x,\mu) \tag{21}$$

holds on $\Lambda_{[\alpha,\beta]}$.

Proof. Write $J(f,x,\lambda) = J(f,x,\lambda-\mu+\mu)$. If we could apply superadditivity property (18) to $\lambda-\mu$ and μ, monotonicity property would also be proved. And that would be the case if $\lambda-\mu$, also continuous or of bounded variation, belonged to $\Lambda_{[\alpha,\beta]}$, which, according to the assumptions of the theorem, is the case. Now, since by (16) is $J(f,x,\lambda-\mu) \geq 0$, we have

$$J(f,x,\lambda) = J(f,x,\lambda-\mu+\mu) \geq J(f,x,\lambda-\mu) + J(f,x,\mu) \geq J(f,x,\mu),$$

which was to prove. □

Remark 3.1. *Theorem* 3.4 *provides an alternative proof of Theorem* 5 *in* [1]. *It follows the same lines as in the discrete case, in Remark* 2.1.

In the sequel we lean on Remark 2.2, and in this setting that means - bounding of functional (17) by a non-weighted functional. As before, we only give an alternative proof of the integral version of Corollary 1.1 from [1], (that is, Corollary 6 in [1]), so our result is given within another remark.

Remark 3.2. *With a slightly altered notation from that in* [1], *according to our former considerations, the result reads:*

Let λ be a function from $\Lambda_{[\alpha,\beta]}$. Let $x : [\alpha,\beta] \to \langle a,b\rangle$, $a,b \in \overline{\mathbb{R}}$, be a continuous and monotonic function and let $f : \langle a,b\rangle \to \mathbb{R}$ be a convex function. If m and M are defined by

$$m := \inf_{\alpha<t<\beta} \left\{ \frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} \right\},$$

$$M := \sup_{\alpha<t<\beta} \left\{ \frac{\lambda(t)-\lambda(\alpha)}{t-\alpha}, \frac{\lambda(\beta)-\lambda(t)}{\beta-t} \right\},$$

then

$$MJ(f,x) \geq J(f,x,\lambda) \geq mJ(f,x), \tag{22}$$

where $J(f,x) := \int_\alpha^\beta f(x(t))dt - (\beta-\alpha) f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right)$.

Proof. Let us prove the right inequality in (22). According to definition of m we have that $m \leq \dfrac{\lambda(t)-\lambda(\alpha)}{t-\alpha}$, $m \leq \dfrac{\lambda(t)-\lambda(\beta)}{t-\beta}$ and $m \leq \dfrac{\lambda(\beta)-\lambda(\alpha)}{\beta-\alpha}$. Hence the following inequalities hold: $\lambda(\alpha) - m\alpha \leq \lambda(t) - mt \leq \lambda(\beta) - m\beta$ and $\lambda(\beta) - \lambda(\alpha) > m\beta - m\alpha$. Let μ be a function from $\Lambda_{[\alpha,\beta]}$, such that $\mu(\alpha) < \mu(t) < \mu(\beta)$, for all $\alpha < t < \beta$ and $\mu(t) = mt$, $\mu(\alpha) = m\alpha$ and $\mu(\beta) = m\beta$. Then we write $\lambda(\alpha) - \mu(\alpha) \leq \lambda(t) - \mu(t) \leq \lambda(\beta) - \mu(\beta)$. Hence by Theorem 3.4 we have that

$$J(f,x,\lambda) \geq J(f,x,\mu). \tag{23}$$

On the other hand, we have

$$\begin{aligned} J(f,x,\mu) &= \int_\alpha^\beta f(x(t))d(mt) - (m\beta - m\alpha) f\left(\frac{1}{m\beta - m\alpha}\int_\alpha^\beta x(t)d(mt)\right) \\ &= m\left(\int_\alpha^\beta f(x(t))dt - (\beta-\alpha) f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right)\right) \\ &= mJ(f,x). \end{aligned} \tag{24}$$

Now, combining (23) and (24) we have the right inequality in (22) proved. The left inequality is obtained similarly, by exchanging the roles of λ and μ.

□

Next we consider functional (17) under conditions of Theorem 3.2, i.e. the ones related to Jensen-Steffensen-Boas' inequality. We give corresponding results, following the same lines as before.

Theorem 3.5. *Let λ and μ be functions from $\tilde{\Lambda}_{[\alpha,\beta]}$, either both continuous or both of bounded variation. Let $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_k = \beta$, $k \geq 2$, be points in $[\alpha, \beta]$. If $x : [\alpha, \beta] \to \langle a, b\rangle$, $a, b \in \overline{\mathbb{R}}$, is a continuous and monotonic function on each of the intervals $[\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$, and if $f : \langle a, b\rangle \to \mathbb{R}$ is a convex function, then functional $J(f, x, \cdot)$ defined by (17) is superadditive on $\tilde{\Lambda}_{[\alpha,\beta]}$, i.e.*

$$J(f, x, \lambda + \mu) \geq J(f, x, \lambda) + J(f, x, \mu) \geq 0. \tag{25}$$

Proof. Follows the same lines as in Theorem 3.1, except for the right inequality in (25). Namely, here we apply inequality (16), but under Jensen-Steffensen-Boas' conditions. □

Theorem 3.6. *Let λ and μ be functions from $\tilde{\Lambda}_{[\alpha,\beta]}$, either both continuous or both of bounded variation. Let $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_k = \beta$, $k \geq 2$. Assume that $x : [\alpha, \beta] \to \langle a, b\rangle$, $a, b \in \overline{\mathbb{R}}$, is a continuous and monotonic function on each of the intervals $[\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$, and that $f : \langle a, b\rangle \to \mathbb{R}$ is a convex function. If for all $\alpha < t < \beta$*

$$\lambda(\gamma_{i-1}) - \mu(\gamma_{i-1}) \leq \lambda(t) - \mu(t) \leq \lambda(\gamma_i) - \mu(\gamma_i), \textit{ for } \quad t \in [\gamma_{i-1}, \gamma_i],\ i = 1, \ldots, k \tag{26}$$

and

$$\mu(\alpha) \leq \mu(t_1) \leq \mu(\gamma_1) \leq \cdots \leq \mu(t_k) \leq \mu(\beta), \textit{ for all } \quad t_i \in [\gamma_{i-1}, \gamma_i],\ i = 1, \ldots, k, \tag{27}$$

then for functional $J(f, x, \cdot)$ defined by (17) inequality

$$J(f, x, \lambda) \geq J(f, x, \mu) \tag{28}$$

holds on $\tilde{\Lambda}_{[\alpha,\beta]}$.

Proof. We consider the function $\rho : [\alpha, \beta] \to \mathbb{R}$ defined by

$$\rho(t) := \lambda(t) - \mu(t), \quad t \in [\alpha, \beta].$$

If λ and μ are both continuous or both of bounded variation, then ρ is continuous or of bounded variation, too. It is obvious that condition (26) is equivalent to the following condition:

$$\rho(\alpha) \leq \rho(t_1) \leq \rho(\gamma_1) \leq \cdots \leq \rho(t_k) \leq \rho(\beta), \text{ for all } \quad t_i \in [\gamma_{i-1}, \gamma_i],\ i = 1, \ldots, k. \tag{29}$$

Since $\lambda = \rho + \mu$, it follows from (27) and (29) that both functions λ and μ satisfy Boas' conditions with the same prescribed points γ_i, $i = 0, \ldots, k$, so that $J(f, x, \lambda)$ and $J(f, x, \mu)$ are well defined (and non-negative, by (16)). Applying Theorem 3.5 to functions ρ and μ, we obtain inequality (28). □

Remark 3.3. *Theorem 3.6 provides an alternative proof of Theorem 6 in [1]. It follows the same lines as in Remark 2.1.*

In the following remark we give yet another alternative proof. It is the proof of Corollary 8 in [1], related to bounding of Jensen-Steffensen-Boas' functional (17) by a non-weighted functional.

Remark 3.4. *With a slightly altered notation from that in [1], according to our former considerations, the result reads:*

Let λ be a function from $\tilde{\Lambda}_{[\alpha,\beta]}$. Let $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_k = \beta$, $k \geq 2$. Assume that $x : [\alpha, \beta] \to \langle a, b\rangle$, $a, b \in \overline{\mathbb{R}}$, is a continuous and monotonic function on each of the intervals $[\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$, and that $f : \langle a, b\rangle \to \mathbb{R}$ is a convex function. If m and M are defined by

$$m := \min_{i=1,\ldots,k} \left\{ \inf \left\{ \frac{\lambda(t) - \lambda(\gamma_{i-1})}{t - \gamma_{i-1}}, \frac{\lambda(\gamma_i) - \lambda(t)}{\gamma_i - t} : \gamma_{i-1} < t < \gamma_i \right\} \right\},$$

$$M := \max_{i=1,\ldots,k} \left\{ \sup \left\{ \frac{\lambda(t) - \lambda(\gamma_{i-1})}{t - \gamma_{i-1}}, \frac{\lambda(\gamma_i) - \lambda(t)}{\gamma_i - t} : \gamma_{i-1} < t < \gamma_i \right\} \right\},$$

then

$$MJ(f, x) \geq J(f, x, \lambda) \geq mJ(f, x), \tag{30}$$

where $J(f, x) := \int_\alpha^\beta f(x(t))dt - (\beta - \alpha) f\left(\frac{1}{\beta-\alpha}\int_\alpha^\beta x(t)dt\right).$

Proof. Let us prove the right inequality in (30). According to definition of m we have that $m \leq \dfrac{\lambda(t) - \lambda(\gamma_{i-1})}{t - \gamma_{i-1}}$ and $m \leq \dfrac{\lambda(t) - \lambda(\gamma_i)}{t - \gamma_i}$. Hence the following inequalities hold: $\lambda(\gamma_{i-1}) - m\gamma_{i-1} \leq \lambda(t) - mt \leq \lambda(\gamma_i) - m\gamma_i$, for all $t \in [\gamma_{i-1}, \gamma_i]$, $i = 1, \ldots, k$. Let μ be a function from $\tilde{\Lambda}_{[\alpha,\beta]}$, such that $\mu(\gamma_{i-1}) < \mu(t) < \mu(\gamma_i)$, for all $\gamma_{i-1} < t < \gamma_i$ and $\mu(t) = mt$, $\mu(\gamma_{i-1}) = m\gamma_{i-1}$ and $\mu(\gamma_i) = m\gamma_i$. Then we write $\lambda(\gamma_{i-1}) - \mu(\gamma_{i-1}) \leq \lambda(t) - \mu(t) \leq \lambda(\gamma_i) - \mu(\gamma_i)$ and apply Theorem 3.6. The rest of the proof is as in Remark 3.2.

□

4. Some further applications

In this section we consider Jensen-Steffensen's functional in relation to certain means. Recall, weighted quasi-arithmetic mean $M_{[\varphi]}$ of $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$, $I \subseteq \mathbb{R}$, with weights $\mathbf{p} = (p_1, \ldots, p_n)$, $p_i \geq 0$, $\sum_{i=1}^n p_i = P_n$, is defined by

$$M_{[\varphi]}(\mathbf{x}; \mathbf{p}) = \varphi^{-1}\left(\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i)\right),$$

where $\varphi : I \to J$, $J \subseteq \mathbb{R}$, is a continuous, monotonic and bijective function. Particularly, $\varphi(x) = x$ yields the expression for weighted arithmetic mean and $\varphi(x) = \log x$ yields the expression for weighted geometric mean. Furthermore, if we consider two continuous, monotonic and bijective functions $\varphi : I \to I$ and $\psi : J \to J$, then the function $f : I \to J$ is called $(M_{[\varphi]}, M_{[\psi]})$ − convex if for every two points $a, b \in I$ and for all $\lambda \in [0, 1]$ the following inequality holds:

$$f(\varphi^{-1}((1 - \lambda)\varphi(a) + \lambda\varphi(b))) \leq \psi^{-1}((1 - \lambda)\psi(f(a)) + \lambda\psi(f(b))).$$

Particularly, $\varphi(x) = \psi(x) = x$ yields the expression for common convexity and $\varphi(x) = x$, $\psi(x) = \log x$ yields the expression for log-convexity. If $f : I \subset \langle 0, \infty\rangle \to \langle 0, \infty\rangle$ is an $(M_{[\varphi]}, M_{[\psi]})$ − convex function, then $g := \psi \circ f \circ \varphi^{-1}$ is convex. One can find a more detailed discussion on this issue in the paper [6] of C.P. Niculescu. We are going to consider the case of $M_{[\varphi]}$ − convex functions, having $\psi(x) = x$. In this setting, $f \circ \varphi^{-1}$ is a convex function. If $\mathbf{x} = (x_1, \ldots, x_n)$ is a monotonic n-tuple, and if we denote $\varphi(\mathbf{x}) := (\varphi(x_1), \ldots, \varphi(x_n))$, we have that $\varphi(\mathbf{x})$ is a monotonic n-tuple, too. Hence inequality (1), with $\mathbf{p} = (p_1, \ldots, p_n)$ satisfying corresponding Jensen-Steffensen's conditions, becomes

$$(f \circ \varphi^{-1}) \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(x_i) \right) \leq \frac{1}{P_n} \sum_{i=1}^{n} p_i (f \circ \varphi^{-1})(\varphi(x_i)) = \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i).$$

Similarly as before, we consider the functional

$$T(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - P_n f \left(\varphi^{-1} \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(x_i) \right) \right). \tag{31}$$

Functional (31) was recently analyzed in the paper [5] of F.C. Mitroi, whose one result we intend to improve here. Note that functional (31) is actually functional (2), obtained by $f \leftrightarrow f \circ \varphi^{-1}$ and $x_i \leftrightarrow \varphi(x_i)$. Hence the results corresponding to those in Section 2, starting with Theorem 2.1, are given in the following corollaries.

Corollary 4.1. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two n-tuples from $\mathcal{P}_n$ and let $\varphi : I \to I$, $I \subseteq \mathbb{R}$, be a continuous, monotonic and bijective function. If $f : I \to \mathbb{R}$ is an $M_{[\varphi]}$ − convex function and if $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then $T(f, \mathbf{x}, \cdot)$ defined by* (31) *is superadditive on $\mathcal{P}_n$, i.e.*

$$T(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \geq T(f, \mathbf{x}, \mathbf{p}) + T(f, \mathbf{x}, \mathbf{q}) \geq 0. \tag{32}$$

Proof. Since $f \circ \varphi^{-1}$ is a convex function and $\varphi(\mathbf{x})$ is a monotonic n-tuple, the proof follows the same lines as in Theorem 2.1. □

Regarding Theorem 2.2 we have the following related result.

Corollary 4.2. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two n-tuples from $\mathcal{P}_n$, satisfying the same conditions as in Theorem* 2.2. *Let $\varphi : I \to I$, $I \subseteq \mathbb{R}$, be a continuous, monotonic and bijective function. If $f : I \to \mathbb{R}$ is an $M_{[\varphi]}$ − convex function and if $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then for functional $T(f, \mathbf{x}, \cdot)$ defined by* (31) *inequality*

$$T(f, \mathbf{x}, \mathbf{p}) \geq T(f, \mathbf{x}, \mathbf{q}) \tag{33}$$

holds on $\mathcal{P}_n$.

Proof. Since $f \circ \varphi^{-1}$ is a convex function and $\varphi(\mathbf{x})$ is a monotonic n-tuple, the proof follows the same lines as in Theorem 2.2. □

In the following remark we consider the bounding of the functional (31) with a non-weighted functional. We lean on Remark 2.2, using the same arguments as in the previous corollaries.

Remark 4.1. *Let $\mathbf{p}$, m and M be as in Remark* 2.2 *and let $\varphi : I \to I$, $I \subseteq \mathbb{R}$, be a continuous, monotonic and bijective function. If $f : I \to \mathbb{R}$ is an $M_{[\varphi]}$ − convex function and if $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ is any monotonic n-tuple, then*

$$M T_{\mathcal{N}}(f, \mathbf{x}) \geq T(f, \mathbf{x}, \mathbf{p}) \geq m T_{\mathcal{N}}(f, \mathbf{x}), \tag{34}$$

where $T_{\mathcal{N}}(f, \mathbf{x}) = \sum_{i=1}^{n} f(x_i) - n f \left(\varphi^{-1} \left(\frac{\sum_{i=1}^{n} \varphi(x_i)}{n} \right) \right)$.

Remark 4.2. *Let us compare the left inequality in* (34) *with the one given in Theorem* 3.1 *in* [5]. *As in Remark* 2.2, *we have that* $M = \max\limits_{1 \leq k \leq n-1} \left\{ \frac{P_k}{k}, \frac{P_n - P_k}{n-k} \right\}$. *Let* $\alpha = \max\{p_1, \ldots, p_n\}$. *Obviously $P_k \leq k\alpha$ i.e. $\frac{P_k}{k} \leq \alpha$ and $\frac{P_n - P_k}{n-k} \leq \alpha$. This implies that $M \leq \alpha$, which is better estimate than the corresponding bound in* [5].

In order to obtain integral versions of the previous results, we transform the functional (17) by means of $M_{[\varphi]}$ − convex function f and the substitutions $f \leftrightarrow f \circ \varphi^{-1}$ (convex function) and $x \leftrightarrow \varphi \circ x$ (monotonic function), and obtain the functional

$$T(f, x, \lambda) := \int_\alpha^\beta f(x(t))d\lambda(t) - (\lambda(\beta) - \lambda(\alpha)) f\left(\varphi^{-1}\left(\frac{1}{\lambda(\beta) - \lambda(\alpha)} \int_\alpha^\beta \varphi(x(t))d\lambda(t)\right)\right). \quad (35)$$

Remark 4.3. *For a monotonic function φ and an $M_{[\varphi]}$ − convex function f, and according to the fore mentioned substitutions ($f \leftrightarrow f \circ \varphi^{-1}$ (convex function) and $x \leftrightarrow \varphi \circ x$ (monotonic function)), functional* (35) *can be applied to all integral results from Section* 3*, under Jensen-Steffensen's or Jensen-Steffensen-Boas' conditions.*

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