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Characterizations of automorphic loops

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Abstract. The notion of a new class of automorphic loops was introduced in [1] as a quasigroup with right identity element e. In this paper, we extend the study of the class of automorphic loops. We decompose the class of automorphic loops into automorphic abelian and non-abelian subclasses. We search within a subclass of abelian automorphic loops, the families of left Bol/ right Bol and Moufang loops with their characterizations.

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1. Introduction

It is well-known that a loop is a one-operational non-associative generalization of a group. The publications of Moufang [6] and Bol [2] provided a motivation to the theory of loops, which gained a ground to deviate along the research areas of algebra, geometry, topology and combinatorics. Historically, the idea of a loop is contemporary to that of a group but the development of loop theory remained eclipsed under the fast moving research horizon of the theory of groups. After the completion of the list of simple groups, the research environment is new getting more suitability for the structures of non-associative models like those of a loop and quasigroups. In the literature of loop theory, the groups are being used to derive new families of loops. K-loops are generalizations of abelian groups [4]. In the famous paper of Moufang [6], she derived that the alternative rule in algebra implies the well-known four Moufang identities [6]. Then such loops satisfying these identities, were called Moufang loops. In the present research environment it is called a *Bol loop with left and right Bol properties*. The theory of Moufang loops has been developed by Bruck [5]. The theory of loops is expanding in different fields of applied sciences.

The notion of a common loop is a non-associative generalization of the class of a group. The idea on an automorphic loop is a generalization of the class of common loop one side and on the other side quasigroup is generalization of the class of automorphic loops. Loops hold a sandwich position between a quasigroup and that of common loop.

The notion of a new class of automorphic loops was introduced in [1]. A class of automorphic loops was constructed on a given group G by adjoining an induced binary operation $*$ on group G. This system is a quasigroup with the right identity element e and is a member of the generalized class of "common loop" with left and right identity element. In this paper, we decompose the class of automorphic loops

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into automorphic abelian and non-abelian subclasses. We search within a subclass of automorphic loop the families of left Bol and Moufang loops with characterizations.

2. Automorphic loops

Definition 2.1. [1] *Let G be a finite group of order $|G| \geq 3$ and a non identity ψ in Aut(G). Then an automorphic loop is a structure $\mathcal{L} = (G, \cdot, *, \psi)$ on a group G in which induced binary operation $* : G \times G \to L$ is defined by $*(x, y) = x * y = x \cdot \psi(y)$ and satisfies the following axioms:*
(L1) $x * e = x$,
(L2) $e * x = \psi(x)$,
(L3) $x * (y * z) = (x * y) * \psi(z)$,
(L4) $x * (e * x) = x * \psi(x)$,
(L5) $xy = x * \psi^{-1}(y)$,
(L6) $(e * x) * x = \psi(x^2)$
for all x, y, $z \in G$.
If $\psi = I$ then $x * y = xy$ and hence the loop $\mathcal{L}$ is the group G itself which is an improper loop. Thus , a loop $\mathcal{L}$ on G by ψ in Aut(G) is a group if and only if ψ is the identity of Aut(G). We study here only proper automorphic loop where $\psi \neq I$.

Finite automorphic loops

Example 2.1. *Consider the automorphic loop $\mathcal{L}$ on $G = C_6 = \{< x >: x^6 = e\}$ under $\psi = (x\ x^5)(x^2\ x^4) \in Aut(C_6)$ and $*$ is given by the following Cayley's table:*

$*$	e	x	x^2	x^3	x^4	x^5
e	e	x^5	x^4	x^3	x^2	x
x	x	e	x^5	x^4	x^3	x^2
x^2	x^2	x	e	x^5	x^4	x^3
x^3	x^3	x^2	x	e	x^5	x^4
x^4	x^4	x^3	x^2	x	e	x^5
x^5	x^5	x^4	x^3	x^2	x	e

The loop $\mathcal{L}$ is proper.

Example 2.2. *Consider the automorphic loop $\mathcal{L}$ on the symmetric group $S_3 = \{e, a, a^2, b, ab, a^2b\}$, where $x = a^2$, $y = ab$, $z = a^2b$, under $\psi = (b\ y\ z)$ in Aut(G) and $*$ is given by the following Cayley's table:*

$*$	e	a	x	b	y	z
e	e	a	x	y	z	b
a	a	x	e	z	b	y
x	x	e	a	b	y	z
b	b	z	y	x	a	e
y	y	b	z	e	x	a
z	z	y	b	a	e	x

Since $(a * b) * y = e \neq x = a * (b * y)$, the loop $\mathcal{L}$ on S_3 under $\psi = (b\ y\ z)$ in Aut(G)is a proper loop.

Infinite automorphic loops

Example 2.3. *An infinite group $(\mathbb{Z}, +)$ under the automorphism $i : \mathbb{Z} \to \mathbb{Z}$ defined by $i(x) = -x$ for all $x \in \mathbb{Z}$, and $x_1 * x_2 = x_1 * i(x_2) = x_1 - x_2$ for all $x_1, x_2 \in \mathbb{Z}$ forms an automorphic loop $\mathcal{L}$.*

Example 2.4. *The group $(\mathbb{R}^+, \cdot)$ under the automorphism $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\psi(m) = \frac{1}{m}$ for all $m \in \mathbb{R}^+$, and under the operation $m * n = m\ \psi(n) = \frac{m}{n}$ for all m, $n \in \mathbb{R}^+$ forms an automorphic loop $\mathcal{L}$.*

Proposition 2.1. *In the automorphic loop $\mathcal{L}$, the following linear equations are solvable:*
(1) $x * a = b$
(2) $a * x = c$
for all x, a, b, $c \in \mathcal{L}$.

Proof. The proof is easy and hence omitted. □

Corollary 2.1. *In the equation (1), $x = \psi(a^{-1})$ is the left inverse of a. In equation (2), $x = \psi^{-1}(a^{-1})$ is the right inverse of a.*

3. The Structure of Aut($\mathcal{L}$)

In this section, we confine to the study of the group of automorphism of an automorphic loop $\mathcal{L}$ on a group G under an automorphism ψ in the group Aut(G) of all automorphisms of G and characterize the loops $\mathcal{L}$ by varying G and ψ in Aut(G).

Definition 3.1. *A bijective map $\alpha : \mathcal{L} \to \mathcal{L}$ on the loop $\mathcal{L}$ is an automorphism of $\mathcal{L}$ if*

$$\alpha(x * y) = \alpha(x) * \alpha(y) \quad \text{for x, y} \in \mathcal{L}.$$

Lemma 3.1. [1] *Given a group G and $\psi(\neq I)$ in the group Aut(G) of all automorphisms of G. Then $\psi \in$ Aut($\mathcal{L}$).*

Theorem 3.1. *Let $\mathcal{L}$ be a loop on a group G under ψ in Aut(G). Then an automorphism α in Aut(G) and $\alpha \neq \{\psi, I\}$ is an automorphism of the loop $\mathcal{L}$ if and only if $\alpha \circ \psi = \psi \circ \alpha$.*

Proof. Suppose that α in Aut(G) is also in Aut$(\mathcal{L})$. Then

$$\begin{aligned}
\alpha(x) * \alpha(y) &= \alpha(x * y), \quad \text{x, y} \in \mathcal{L} \\
&= \alpha(x\psi(y)) \ \ (definition) \\
&= \alpha(x)\alpha(\psi(y)) \ \ (supposition) \\
&= \alpha(x) * \psi^{-1}(\alpha(\psi(y))) \ \ (by\ L_3) \\
&= \alpha(x) * (\psi^{-1} \circ \alpha \circ \psi)(y).
\end{aligned}$$

It implies that

$$\alpha = \psi^{-1}\alpha\psi \Longrightarrow \psi \circ \alpha = \alpha \circ \psi.$$

For converse, suppose that $\alpha \in$ Aut$(\mathcal{L})$ such that $\alpha \circ \psi = \psi \circ \alpha$ in $\mathcal{L}$. Then $\alpha \in$ Aut(G) since x, $y \in G$ and

$$\begin{aligned}
\alpha(xy) &= \alpha(x * \psi^{-1}(y)) \ \ (\mathrm{L}_3) \\
&= \alpha(x) * \alpha(\psi^{-1}(y)) \ \ (\text{supposition}) \\
&= \alpha(x)\psi(\alpha(\psi^{-1}(y))) \ \ (\text{definition}) \\
&= \alpha(x)\alpha(y) \ \ (\text{supposition}).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *For ψ in Aut(G), the group of automorphism of the loop $\mathcal{L}$ on G is isomorphic to the centralizer of ψ in G.*

Example 3.1. *Consider the automorphic loop $\mathcal{L}$ on the Klein four group $G = V_4 = \{< x, y : x^2 = e = y^2 = z^2; xy = z = yx\}$ under $\psi = (x \ \ y) \in Aut(V_4)$ and $*$ is given by the following Cayley's table:*

$*$	e	x	y	z
e	e	y	x	z
x	x	z	e	y
y	y	e	z	x
z	z	x	y	e

The loop $\mathcal{L}$ on G under $\psi = (x\ y)$ is proper. The automorphism $\psi = (x\ y)$ an automorphism of the loop $\mathcal{L}$ since $\psi(x{*}y) = e = \psi(x){*}\psi(y) = y{*}x$. Since Aut($V_4$)$\cong S_3$ and $\psi_1 = (x\ z)$, $\psi_2 = (y\ z)$, $\psi_3 = (x\ y\ z)$ and $\psi_3^{-1} = (x\ z\ y)$ as other non-identity automorphisms of S_3 which do not commute with $\psi = (x\ y)$ and are easily verified that they are not automorphisms of the loop $\mathcal{L}$. Thus the subgroup $\{< \psi >: \psi^2 = I\}$ of Aut(V_4) is the group of automorphisms of $\mathcal{L}$, i.e., Aut($\mathcal{L}$)$\cong \{< \psi : \psi^2 = I\} = C_2$.

Example 3.2. *Consider the automorphic loop $\mathcal{L}$ on cyclic group $G = \{< x >: x^5 = e\}$ under $\psi = (x\ x^2\ x^4\ x^3)$ and $*$ is given by the following Cayley's table:*

$*$	e	x	x^2	x^3	x^4
e	e	x^2	x^4	x	x^3
x	x	x^3	e	x^2	x^4
x^2	x^2	x^4	x	x^3	e
x^3	x^3	e	x^2	x^4	x
x^4	x^4	x	x^3	e	x^2

The loop $\mathcal{L}$ is proper and abelian automorphic loop. Since $Aut(C_5) \cong C_4$ and hence ψ, $\psi^2 = (x\ x^4)(x^2\ x^3)$ and $\psi^{-1} = (x\ x^3\ x^4\ x^2)$ are three non-identity automorphisms of C_5, where $\psi^2(x * x^2) = \psi^2(e) = e = \psi^2(x) * \psi^2(x^2) = x^4 * x^3$, ete. Thus ψ^2 is an automorphisms of the loop $\mathcal{L}$. Similarly, $\psi^{-1}(x * x^2) = \psi^{-1}(e) = e = \psi^{-1}(x) * \psi^{-1}(x^2) = x^3 * x$ is an automorphisms of the loop $\mathcal{L}$. Thus Aut($\mathcal{L}$)= Aut(C_5).

Thus we conclude that:

Theorem 3.3. *Let $\mathcal{L}$ be an automorphic loop on a cyclic group G of order p under ψ such that $\psi(g) = g^{-1}$ for all $g \in G$ and $\psi \in Aut(G)$. Then Aut(G)=Aut($\mathcal{L}$).*

Theorem 3.4. *Let G be a finite group with $\psi \in Aut(G)$ and let $\mathcal{L}$ be an automorphic loop. Then Aut(G)=Aut($\mathcal{L}$) if and only if G is a cyclic group.*

4. Abelian Automorphic Loop

In this section, we extend the study of an abelian loop which is already introduced in [1]. Since the structure of such a loop is based on a group G and a non-identity automorphism ψ of G, the role of the class of abelian loops $\mathcal{L}$ is highlighted to get access to the study of the class of well-known Bol and Moufang loops in the next section.

Definition 4.1. *An automorphic loop $\mathcal{L}$ on a group G is abelian if and only if $\psi(x) * y = \psi(y) * x$ for all $x, y \in \mathcal{L}$.*

It is important to realize and connect it to the class of abelian G. If $x, y \in G$ and G is an abelian group then $xy = yx$ for all $x, y \in G$. Then

$$\begin{aligned} \psi(xy)=\psi(yx) &= \psi(x)\psi(y) \quad (\text{as } \psi \in \text{Aut(G)}) \\ \psi(y)\psi(x) &= \psi(x)\psi(y) \\ \psi(y) * \psi^{-1}(\psi(x)) &= \psi(x) * \psi^{-1}(\psi(y)) \quad (L_3) \\ \Rightarrow \quad \psi(y) * x &= \psi(x) * y. \end{aligned}$$

Since the converse argument remains valid logically, we thus characterize that:

Theorem 4.1. *An automorphic loop $\mathcal{L}$ on a group G under $\psi(\neq I)$ in Aut(G) is abelian if and only if G is abelian.*

We now define a very special subclass of class of automorphic loops in which each of the elements of such a subclass is of order 2. Thus we define:

Definition 4.2. *An automorphic loop $\mathcal{L}$ on a group G under ψ in Aut(G) is called an involutionary loop if $x * x = e$ for all $x \in \mathcal{L}$.*

Since $x * x = e$ for all x, we attach to such kind of loops with special property to the class of abelian automorphic loops via inversion automorphism of G, i.e., $x * x = e \Rightarrow x\psi(x) = e$ in $G \Rightarrow \psi(x) = x^{-1}$ for $x \in G$. It suggest that $e * x = x^{-1}$. Thus ψ ought to be the inversion of G and hence G to be abelian. (Note that the class is proper if G is not an elementary abelian 2-group). Thus we characterize that:

Theorem 4.2. *Let G be a group of order $n \geq 3$ and not an elementary abelian 2-group. Then the loop $\mathcal{L}$ on G under ψ is an involutionary loop if and only if ψ is an inversion in Aut(G).*

Example 4.1. *Consider the automorphic loop $\mathcal{L}$ on cyclic group $G = \{< x >: x^4 = e\}$ under $\psi = (x\ x^3)$ and $*$ is given by the following Cayley's table:*

$*$	e	x	x^2	x^3
e	e	x^3	x^2	x
x	x	e	x^3	x^2
x^2	x^2	x	e	x^3
x^3	x^3	x^2	x	e

It is a proper involutionary abelian loop.

Lemma 4.1. *In an abelian loop $\mathcal{L}$ on a group G under $\psi(\neq I)$ in Aut(G), the following equalities hold for all $x, y, z \in \mathcal{L}$:*
(i) $(x * y) * z = (x * z) * y$
(ii) $x * (y * z) = (x * y) * \psi(z) = (x * \psi(z)) * y$

Proof. (i)
$$(x * y) * z = (x\psi(y))\psi(z) = (x\psi(z))\psi(y) = (x * z) * y.$$
(ii)
$$\begin{aligned} x * (y * z) &= x\psi(y\psi(z)) \\ &= x\psi(y)\psi^2(z) \\ &= (x\psi(y))\psi^2(z) \\ &= (x * y) * \psi(z) \\ &= (x * \psi(z)) * y. \end{aligned}$$
□

Corollary 4.1. $e * (e * z) = \psi^2(z)$.

Corollary 4.2. $x * (x * y) = (x * x) * \psi(y)$.

Corollary 4.3. $(e * y) * z = (e * z) * y$.

Lemma 4.2. *In an abelian loop $\mathcal{L}$ on a group G under $\psi(\neq I)$ in Aut(G) consisting of all automorphisms of G, the following assertions are logically equivalent:*
(a) *the loop $\mathcal{L}$ is abelian with $\psi^2 = I$*
(b) *$e * (x * y) = y * x$ for all $x,\ y \in \mathcal{L}$.*

Proof. (a) $\Rightarrow$ (b): suppose that the loop $\mathcal{L}$ is abelian with $\psi^2 = I$, then

$$\begin{aligned} e * (x * y) &= \psi(x * y) \\ &= \psi(x\psi(y)) \quad \text{(definition)} \\ &= \psi(x)\psi^2(y) \quad (\psi \in \text{Aut(G)}) \\ &= \psi(x)y \quad \text{(supposition)} \\ &= y\psi(x) \quad \text{(abelian property)} \\ &= y * \psi^{-1}(\psi(x)) \quad \text{(definition)} \\ &= y * x. \end{aligned}$$

(b) $\Rightarrow$ (a): suppose that $e * (x * y) = y * x$, then it implies that

$$\begin{aligned} \psi(x * y) &= y * x \\ \psi(x) * \psi(y) &= y * x \\ \psi^2(y) * x &= y * x \\ \psi^2(y) &= y \\ \psi^2 &= I. \end{aligned}$$

This completes the proof. $\square$

Corollary 4.4. *In an abelian loop $\mathcal{L}$ on a group G under ψ in Aut(G) such that $\psi^2 = I$*
(i) $e * (e * z) = z$
(ii) $e * z = z^{-1} = \psi(z)$
(iii) $e * (e * (y * z)) = y * z$
(iv) $e * (y * z) = (e * (e * z) * y$
for all $x, y, z \in \mathcal{L}$.

Example 4.2. *Consider automorphic loop $\mathcal{L}$ on the Klein four group $G = V_4 = \{< x, y >: x^2 = e = y^2 = z^2; xy = z = yx\}$ under $\psi = (x\ y\ z) \in Aut(V_4) \cong S_3$ and $*$ is given by the following Cayley's table:*

$*$	e	x	y	z
e	e	y	z	x
x	x	z	y	e
y	y	e	x	z
z	z	x	e	y

The loop $\mathcal{L}$ is an abelian automorphic loop $\mathcal{L}$ but it does not satisfy identity $e*(x*y) = y * x$ for all $x,\ y \in \mathcal{L}$.

5. Characterization of the Class of Loops

In this section, we shall focus on the study of automorphic loop $\mathcal{L}$ based on a group G and an automorphism ψ in the group $\mathrm{Aut}(G)$ consisting of all automorphism of G under the specified binary operation $*$ such that $x * y = x\psi(y) \in G$. We shall classify the class of loops $\mathcal{L}$ and characterize its subclasses as referred to holding the left-Bol-property (l.b.p) and the right-Bol-property (r.b.p). A well-known class of loops called a Moufong loop is a further common subclass of the subclasses loops of $\mathcal{L}$ holding (l.b.p) and (r.b.p) properties. This study supports to reveal as (l.b.p)-property of the loop $\mathcal{L}$ is particularly based on ψ the choice of ψ in $\mathrm{Aut}(G)$ and not on the abelian or non-abelian group G, where as the (r.b.p)-property is specifically based on the abelian group G as well as ψ in $\mathrm{Aut}(G)$. This way, we elaborate that the loop $\mathcal{L}$ referred to (r. b.p)observes (l.b.p) but the converse is not valid, in general. The common subclass based on an abelian group G is a Moufong loop.

Proposition 5.1. *In a loop $\mathcal{L}$ on a group G under $\psi \in Aut(G)$, the following axioms hold:*
(a) $(x * (y * x)) * z = x * (y * (x * \psi^{-2}(z)))$,
(b) $((z * x) * y) * x = z * (x * \psi^{-1}(yx))$,
(c) $(x * y) * z = x * (y * \psi^{-1}(z))$,
(d) $x * (y * (z * u)) = (x * (y * z) * \psi^2(u))$
for all x, y, z, $u \in \mathcal{L}$.

Proof. (a) For x, y, $z \in \mathcal{L}$, we have

$$\begin{aligned}
(x * (y * x)) * z &= (x\psi(y\psi(x)))\psi(z) \quad \text{(definition)}\\
&= x((\psi(y\psi(x)))\psi(z)) \quad \text{(associativeinG)}\\
&= x\psi((y\psi(x)z)) \quad (\psi \in \mathrm{Aut(G)})\\
&= x * ((y(\psi(x))z)) \quad \text{(definition)}\\
&= x * (y\psi(x\psi^{-1}(z))) \quad \text{(definition)}\\
&= x * (y * (x\psi^{-1}(z))) \quad (\psi \in \mathrm{Aut}(\mathcal{L}))\\
&= x * (y * (x * \psi^{-2}(z))).
\end{aligned}$$

Thus the axiom (a) holds in $\mathcal{L}$. Hence

$$(x * (y * x)) * z = x * (y * (x * \psi^{-2}(z))).$$

(b) For x, y, $z \in \mathcal{L}$, we have

$$\begin{aligned}
((z * x) * y) * x &= ((z\psi(x))\psi(y))\psi(x)\\
&= (z\psi(x))\psi(yx)\\
&= (z * x) * (yx)\\
&= z * (x * \psi^{-1}(yx)).
\end{aligned}$$

Thus the axiom (b) holds in $\mathcal{L}$. Hence

$$((z * x) * y) * x = z * (x * \psi^{-1}(yx)).$$

(c) For x, y, $z \in \mathcal{L}$, we have

$$\begin{aligned}(x*y)*z &= (x\psi(y))\psi(z)\\ &= x(\psi(y)\psi(z)) \quad \text{(associative)}\\ &= x(\psi(yz)) \quad (\psi \in \text{Aut(G)})\\ &= x*(y*\psi^{-1}(z)).\end{aligned}$$

Thus the axiom (c) holds in $\mathcal{L}$. Hence

$$(x*y)*z = x*(y*\psi^{-1}(z)).$$

(d) For x, y, $z \in \mathcal{L}$, we have

$$\begin{aligned}x*(y*(z*u)) &= x*(y\psi(z\psi(u)))\\ &= x*((y\psi(z)\psi^2(u))\\ &= (x\psi(y\psi(z)))\psi^3(u))\\ &= (x*(y*z)*\psi^2(u)).\end{aligned}$$

Thus the axiom (d) holds in $\mathcal{L}$. Hence

$$x*(y*(z*u)) = (x*(y*z)*\psi^2(u)).$$

□

Characterization of a left Bol loop

Definition 5.1. *The automorphic loop $\mathcal{L}$ on a group G under ψ in Aut(G) is called a left Bol loop if it satisfies the (l.b.p), i.e.,*

$$(x*(y*x))*z = x*(y*(x*z). \qquad \cdots (l.b.p)$$

Example 5.1. *(*On an abelian group*)*
If

$$G = V_4 = \{< x, y >: x^2 = y^2 = z^2 = e, z = xy = yx\},$$

Aut(V_4)$\cong S_3$, then the loop $\mathcal{L}$ under $\psi = (x\ y)$ is represented by the following Cayley's table:

$*$	e	x	y	z
e	e	y	x	z
x	x	z	e	y
y	y	e	z	x
z	z	x	y	e

(i) Of course, the loop $\mathcal{L}$ with $\psi = (x\ y)$ in Aut(V_4) is proper since $(x*y)*y = x \neq y = x*(y*y)$.

(ii) The loop $\mathcal{L}$ on an abelian group V_4 under ψ with $\psi^2 = I$ is a left Bol loop since

$$(x*(y*x))*z = y = x*(y*(x*z)).$$

Example 5.2. *(*On an abelian group with $\psi^2 \neq I$*)*
If

$$G = V_4 = \{< x, y >: x^2 = y^2 = z^2 = e, z = xy = yx\},$$

Aut(V_4)$\cong$ S_3, then the loop $\mathcal{L}$ under $\psi = (x\ y\ z)$, where $\psi^3 = I$. The loop $\mathcal{L}$ $\psi = (x\ y\ z)$ is represented by the following Cayley's table:

$*$	e	x	y	z
e	e	y	z	x
x	x	z	y	e
y	y	e	z	x
z	z	x	e	y

(i) We assume that the loop $\mathcal{L}$ is proper since $(x * y) * y = x \neq z = x * (y * y)$.
(ii) The loop $\mathcal{L}$ is not a left Bol loop since it does not fulfill the l.b.p property, i.e.,

$$(x * (y * x)) * z = e \neq y = x * (y * (x * z)).$$

Example 5.3. *(*On non-abelian group*)*
Consider the automorphic loop $\mathcal{L}$ on the symmetric group $S_3 = \{e, a, a^2, b, ab, a^2b\}$, where $x = a^2$, $y = ab$, $z = a^2b$, under $\psi = (a\ x)(y\ z)$ in Aut(G) and $*$ is given by the following Cayley's table:

$*$	e	a	x	b	y	z
e	e	x	a	b	z	y
a	a	e	x	y	b	z
x	x	a	e	z	y	b
b	b	y	z	e	a	x
y	y	z	b	a	x	e
z	z	b	y	x	e	a

(i) The loop $\mathcal{L}$ is proper since

$$(x * b) * z = a \neq e = x * (b * z).$$

(ii) The loop $\mathcal{L}$ is not a left Bol loop since it does not fulfill the l.b.p property, i.e.,

$$(a * (b * a)) * y = a = a * (b * (a * y)).$$

Theorem 5.1. *A loop $\mathcal{L}$ on a group G under ψ $\in$Aut(G) is a left Bol loop if and only if $\psi^2 = I$.*

Proof. For an arbitrary ψ in Aut(G) if the loop $\mathcal{L}$ satisfies the (l.b.p)-property then for x, y, z in $\mathcal{L}$,

$$(x * (y * x)) * z = x * (y * (x * z)).$$

By using the axiom (c) of Proposition 5.1, it implies that

$$x * (y * (x * \psi^{-2}(z)) = x * (y * (x * z))$$
$$\Rightarrow \quad \psi^{-2}(z) = z \quad \text{for each z} \in \mathcal{L}.$$

Thus $\psi^2 = I$.
Conversely, if $\psi^2 = I$ then by definition

$$(x * (y * x)) * z = x\psi(y)x\psi(z) = x * (y * (x * z)).$$

Thus it proves the theorem. □

We characterize the left Bol loop $\mathcal{L}$ on a group G of order ≥ 3 having $\psi^2 = I$.

Theorem 5.2. *Let G be a group G of order $n \geq 3$ having non-identity automorphism ψ of G. Then the automorphic loop $\mathcal{L}$ on G under ψ is a left Bol loop if and only if $\psi^2 = I$.*

Characterization of a right Bol loop

Definition 5.2. *The automorphic loop $\mathcal{L}$ on a group G under ψ in Aut(G) is called a right Bol loop if it satisfies the (r.b.p.), i.e.,*

$$((z * x) * y) * x = z * ((x * y) * x).$$

Example 5.4. *Consider the automorphic loop $\mathcal{L}$ on cyclic group $G = \{< x >: x^5 = e\}$ under $\psi = (x\ x^4)(x^2\ x^3)$ of order 2 in Aut(G) and $*$ is given by the following Cayley's table:*

$*$	e	x	x^2	x^3	x^4
e	e	x^4	x^3	x^2	x
x	x	e	x^4	x^3	x^2
x^2	x^2	x	e	x^4	x^3
x^3	x^3	x^2	x	e	x^4
x^4	x^4	x^3	x^2	x	e

(i) Of course, the loop $\mathcal{L}$ is proper since

$$(e * x) * x^3 = x \neq x^2 = e * (x * x^3).$$

(ii) The loop is abelian which is verified by definition.
(iii) If $z = x^2$, $x = x$, $y = x^4$ then r.b.p is fulfilled since

$$((z * x) * y) * x = x = z * ((x * y) * x).$$

Hence the loop $\mathcal{L}$ is right Bol loop.

Example 5.5. *Consider the automorphic loop $\mathcal{L}$ on cyclic group $G = \{< x >: x^4 = e\}$ under $\psi = (x\ x^3)$ in Aut(G) and $*$ is given by the following Cayley's table:*

$*$	e	x	x^2	x^3
e	e	x^3	x^2	x
x	x	e	x^3	x^2
x^2	x^2	x	e	x^3
x^3	x^3	x^2	x	e

The loop $\mathcal{L}$ is proper since

$$(x * x^2) * x^3 = e \neq x^2 = x * (x^2 * x^3).$$

The loop $\mathcal{L}$ is a right Bol loop, if $z = x^2$, $x = x$, $y = x^3$, i.e.,

$$((x^2 * x) * x^3) * x = x = x^2 * ((x * x^3) * x).$$

Thus the loop $\mathcal{L}$ on $G = C_4$ under $\psi = (x\ x^3)$ in Aut(G) is a proper left and right Bol loop.

Example 5.6. *In Example 5.3 of automorphic loop $\mathcal{L}$ on*

$$G = V_4 = \{< x, y >: x^2 = y^2 = z^2 = e, z = xy = yx\}$$

under $\psi = (x\ y)$ is proper left Bol loop. We notice that it is also right Bol loop. Similarly, the loops $\mathcal{L}$ on $G = V_4$ under $\psi = (x\ z)$ and $(y\ z)$ in Aut(G) are easily verified to be left Bol and right Bol loops.

Theorem 5.3. *An automorphic loop $\mathcal{L}$ on a group G under ψ in Aut(G) is a right Bol loop if and only if G is an abelian group and ψ is inversion auto of G.*

Proof. Suppose that $\mathcal{L}$ on a group G under $\psi(\neq I)$ in $\mathrm{Aut}(G)$ is a right Bol loop. Then for x, y, z in $\mathcal{L}$ on G satisfies the (r.b.p.)-property, i.e.,

$$((z*x)*y)*x = z*((x*y)*x).$$

By using $*$-operation as defined in $\mathcal{L}$, it implies that

$$\begin{aligned} ((z\psi(x))\psi(y))\psi(x) &= (z\psi((x\psi(y))\psi(x))) \\ \Rightarrow\ (z\psi(x))(\psi(y)\psi(x)) &= z\psi(x\psi(y)\psi(x)) \\ \Rightarrow\ z\psi(x)\psi(yx) &= z\psi(x)\psi^2(y)\psi^2(x) \\ \Rightarrow\ (z\psi(x))\psi(yx) &= z\psi(x)\psi^2(yx) \\ \Rightarrow\ \psi(yx) &= \psi^2(yx). \end{aligned}$$

Thus either $\psi = I$, if G is a group (abelian or non-abelian group) and ψ is a non-identity automorphism of G defined by $\psi(yx) = xy$, where $\psi^2(yx) = yx$ for all x, y $\in G$. Hence ψ is inversion (non-identity) automorphism of G, where $\psi^2 = I$.

Conversely, if a loop $\mathcal{L}$ is automorphic on an abelian group and $\psi(x) = x^{-1}$ for all $x \in G$, then the right Bol property is fulfilled where, by definition

$$((z*x)*y)*x \quad = \quad z*((x*y)*x.$$

It completes the proof. □

Corollary 5.1. *$\psi = (x\ y)$ is an automorphism of $\mathcal{L}$ if and only if $\mathcal{L}$ is a right Bol loop.*

Characterization of a Moufang loop

We now complete the characterization process with highlight of the class of the automorphic Moufang loops $\mathcal{L}$ over a group G under $\psi \in \mathrm{Aut}(G)$ from the discussion of the subclasses of the left Bol and right Bol loops of the class of automorphic loops, it is important to conclude that:

Theorem 5.4. *There is one-to-one correspondence from the subclass of left Bol automorphic loops to the subclass of right Bol automorphic loops but the converse does not hold, in general.*

If L (Bol) and R (Bol) are subclasses of left Bol and right Bol of the class of automorphic loops on an abelian group G under ψ in $\mathrm{Aut}(G)$ of order 2, i.e., $\psi^2 = I$. Then $R(\mathrm{Bol}) \subseteq L(\mathrm{Bol})$ and hence $R(\mathrm{Bol}) \cap L(\mathrm{Bol}){=}R(\mathrm{Bol})$. We classify this reality by examples.

Example 5.7. *In example 5.10 of an automorphic right Bol loop $\mathcal{L}$ on $G = \{<x>: x^5 = e\}$ under $\psi = (x\ x^4)(x^2\ x^3)$ in Aut(G) also holds the (l.b.p) property,*

$$(x*(y*x)*z = x*(y*(x*z))$$

for $y = x^4$, $z = x^2$ and $x = x$, i.e.,

$$(x*(x^4*x))*x^2 = x = x*(x^4*(x*x^2))$$

which shows that loop $\mathcal{L}$ on C_5 under ψ is a left Bol loop as well.

Example 5.8. *In example 5.11 of an automorphic right Bol loop $\mathcal{L}$ on $G = \{<x>: x^4 = e\}$ under $\psi = (x\ x^3)$ in Aut(G) also holds the (l.b.p) property, for $x = x$, $y = x^3$ and $z = x^2$, i.e.,*

$$(x*(x^3*x))*x^2 = x = x*(x^3*(x*x^2)).$$

Thus loop $\mathcal{L}$ on C_4 under ψ is a left Bol loop as well.

We conclude that:

Theorem 5.5. *An automorphic loop $\mathcal{L}$ on an abelian group G under the automorphism (inversion) $\psi = i$ of Aut(G) is a left Bol and a right Bol loop as well is a common member of subclass of the left Bol and right Bol loops within the class of automorphic loops.*

Definition 5.3. *An automorphic loop $\mathcal{L}$ on an abelian group G under ψ, inversion i in Aut(G) is called a Moufang automorphic loop.*

Remark 5.1. *Moufang loops form a well known class of common loops which falls within a class of automorphic loops.*

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