Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 38(2), 2011, Pages 32–42
ISSN: 1223-6934, Online 2246-9958

Local Greatest Equivalence Classes of ω-trees

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Abstract. In [18] we defined the concept of ω-labeled tree as a binary, ordered and labeled tree with several features concerning the labels and order between the direct descendants of a node. In [19] we introduced an equivalence relation $\simeq$ on the set $OBT(\omega)$ of ω-trees and a partial order on the factor set $OBT(\omega)/_{\simeq}$. In this paper we decompose the factor set $OBT(\omega)/_{\simeq}$ into disjoint "local" subsets K, we show that if the relation defined by the mapping ω is a noetherian one then every local subset K has a greatest element, we define an increasing operator on the set $OBT(\omega)/_{\simeq}$, which allows to obtain the greatest element of a local subset. In order to relieve the local features of a subset K we give an example which shows that the greatest element of K is not necessarily a maximal element of the factor set.

2000 *Mathematics Subject Classification.* Primary 05C20; Secondary 18A10.
Key words and phrases. directed ordered graph, tree, ordered tree, partial order, embedding mapping, noetherian relation, greatest element.

1. Introduction

More and more the algebraic structures are used in the computer science domain. This can be explained by the fact that the sets without any explicit operations are not of interest. An algebraic structure links the sets and their operations. Frequently the algebraic structures are viewed as universal algebras to obtain an uniform characterization.

The methods of the graph theory were fully implied in the domain of knowledge representation. The graph theory was combined with the mathematical logic and the universal algebras to obtain improved methods of knowledge representation. In order to enumerate only some of them we relieve the aspects treated in [2] [5], [6], [9], [12], [14], [15] [16], [17]. The implications of the algebraic methods into deductive systems is also a fruitful research area ([3], [4], [10]). Various algebraic structures of trees were used to obtains models in artificial intelligence ([7], [8], [11], [13], [18], [19], [20]).

In this paper we develop the ideas initiated in [18] and [19]. The final task of this research line is to build a mathematical description of the process of communication between the entities of a cooperating system. These results will be used to describe the valuation process in master-slave systems based on semantic schemas ([17]).

In [18] we defined the concept of ω-labeled tree. In [19] we introduced an equivalence relation $\simeq$ on the set $OBT(\omega)$ of ω-trees and a partial order $\sqsubseteq$ on the factor set $OBT(\omega)/_{\simeq}$ such that $(OBT(\omega)/_{\simeq}, \sqsubseteq)$ becomes a partial ordered set. We define a decomposition $OBT(\omega)/_{\simeq} = \bigcup_i K_i$ and K_i is named a local subset of $OBT(\omega)/_{\simeq}$. Two distinct local subsets are disjoint. We show that if the relation defined by the mapping ω is a noetherian one then every local subset K_i has a greatest element. We define an increasing operator with respect to $\sqsubseteq$ on the set $OBT(\omega)/_{\simeq}$. This operator allows to obtain the greatest element of a local subset. In order to relieve the local

Received February 10, 2011. Revision received April 02, 2011.

feature of a subset K_i we show by an example that the greatest element of K_i is not necessarily a maximal element of the set $OBT(\omega)/_{\simeq}$.

The paper is organized as follows. In section 2 we recall the basic concepts and results used in this paper. In Section 3 we define the increasing operator that allows to build the greatest element of a local subset. In Section 4 we present the aspects connected by the existence of the local greatest element and we give a method to obtain such an element. The last section contains the conclusions and future works.

2. Basic concepts

A *directed ordered graph* ([1]) is a pair $G = (A, D)$, where

- A is a finite set of elements called *nodes*
- D is a finite set of elements of the form $[(i, i_1), \ldots, (i, i_n)]$, where $n \geq 1$ and $i, i_1, \ldots, i_n \in A$
- D satisfies the following condition: if $[(i, i_1), \ldots, (i, i_n)] \in D$ and $[(j, j_1), \ldots, (j, j_s)] \in D$ then $i \neq j$.

If $G = (A, D)$ is a directed ordered graph then we can associate to G a *directed graph* $G' = (A, D')$, where $D' = \{(i, j) \mid \exists[(i, i_1), \ldots, (i, i_n)] \in D, \exists r \in \{1, \ldots, n\} : j = i_r\}$. An *ordered tree* is a directed ordered graph $G = (A, D)$ such that G' is a tree and the following property is satisfied:

$$[(i, i_1), \ldots, (i, i_n)] \in D, j, r \in \{1, \ldots, n\}, j \neq r \Rightarrow i_j \neq i_r \tag{1}$$

Let $L = L_N \cup L_T$ be a set of labels such that $L_N \cap L_T = \emptyset$. The elements of L_N are called *nonterminal labels* and those of L_T are called *terminal labels*. The elements of L are called *labels*. A **split mapping** on L ([18]) is a function $\omega : L_N \longrightarrow L \times L$. An ω**-tree** ([18]) is a tuple $t = (A, D, h)$, where

- (A, D) is an ordered tree and every element of D is of the form $[(i, i_1), (i, i_2)]$;
- $h : A \longrightarrow L$ is a mapping such that if $[(i, i_1), (i, i_2)] \in D$ then

$$\begin{cases} h(i) \in L_N \\ \omega(h(i)) = (h(i_1), h(i_2)) \end{cases}$$

For each $i \in A$ the element $h(i)$ is called the **label** of the node i. The mapping h is named the **labeling mapping** of t.

By $OBT(\omega)$ we denote the set of all ω-trees. An element $t = (A, D, h)$ such that $D = \emptyset$ is named a *degenerate element* of $OBT(\omega)$.

Let $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$ be two elements of $OBT(\omega)$ and an arbitrary mapping $\alpha : A_1 \longrightarrow A_2$. For every $u = [(i, i_1), (i, i_2)]$, where $i, i_1, i_2 \in A_1$, we denote $\overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))]$.

If $t = (A, D, h)$ is an ω-tree then we denote by $root(t)$ the element of A designated by the root of t. If $i \in A$ then by $t_{(i)}$ we denote the subtree of t such that $root(t_{(i)}) = i$.

If $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ then we write $t_1 \preceq t_2$ ([18]) if there is a mapping $\alpha : A_1 \longrightarrow A_2$ such that:

$$\overline{\alpha}(D_1) \subseteq D_2 \tag{2}$$

$$h_1(root(t_1)) = h_2(\alpha(root(t_1))) \tag{3}$$

Such a mapping α is an **embedding mapping** of t_1 into t_2. The relation $\preceq$ is reflexive and transitive, but is not antisymmetric.

We define the binary relation $\simeq$ on the set $OBT(\omega)$ as follows: $t_1 \simeq t_2$ if $t_1 \preceq t_2$ and $t_2 \preceq t_1$. This is an equivalence relation ([19]). We denote by $OBT(\omega)/_{\simeq}$ the factor set. If $t \in OBT(\omega)$ then by $[t]$ we denote the equivalence class of t.

Let us consider $[t_1] \in OBT(\omega)/_{\simeq}$ and $[t_2] \in OBT(\omega)/_{\simeq}$. We define the relation $[t_1] \sqsubseteq [t_2]$ if $t_1 \preceq t_2$. The relation $\sqsubseteq$ does not depend on representatives. The pair $(OBT(\omega)/_{\simeq}, \sqsubseteq)$ is a partial ordered set ([19]).

We define recursively the mapping $F : OBT(\omega) \longrightarrow L^*$ as follows ([20]):

- If $t = (\{i\}, \emptyset, h)$ is a degenerate element of $OBT(\omega)$ then $F(t) = h(i)$.
- If $t = (A, D, h) \in OBT(\omega)$, $[(root(t), i), (root(t), j)] \in D$ then by means of the concatenation operation on L^* we define

$$F(t) = F(t_{(i)})F(t_{(j)})$$

For every $k \geq 1$ we define the operator $T_k : OBT(\omega) \longrightarrow OBT(\omega)$ as follows: $T_k(t)$ is obtained from t by deleting all nodes which are reachable from $root(t)$ by a path of length greater than k. T_k is named the **slicing operator** ([20]). The operator T_k is well defined ([20]). In other words, if $t \in OBT(\omega)$ then $T_k(t) \in OBT(\omega)$. If $t_1 \simeq t_2$ then for every $k \geq 1$ we have $T_k(t_1) \simeq T_k(t_2)$ and the embedding mapping of $T_k(t_1)$ into $T_k(t_2)$ is the restriction of the embedding mapping of t_1 into t_2 ([20]). If $t_1 \simeq t_2$ then $F(t_1) = F(t_2)$ ([20]).

Consider an element $t = (A, D, h) \in OBT(\omega)$ such that $F(t) = w_1 h(i_1) w_2 \ldots w_n h(i_n) w_{n+1} \notin L_T^*$, where $w_1, \ldots, w_{n+1} \in L_T^*$ and $h(i_1), \ldots, h(i_n) \in L_N$. An **immediate extension** ([20]) of t is an element $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ such that

$$A_1 = A \cup \bigcup_{i \in \{i_1, \ldots, i_n\}} \{j_{i,1}, j_{i,2}\} \tag{4}$$

$$D_1 = D \cup \bigcup_{i \in \{i_1, \ldots, i_n\}} \{[(i, j_{i,1}), (i, j_{i,2})]\} \tag{5}$$

$$h_1(x) = h(x) \ for\ x \in A \tag{6}$$

We denote by $E(t)$ the set of all immediate extensions of t. If $t \in OBT(\omega)$ and $F(t) \in L_T^*$ then we take $E(t) = [t]$.

3. An increasing operator on $OBT(\omega)/_{\simeq}$

In [20] we proved that $\bigcup_{t_0 \in [t]} E(t_0)$ is an equivalence class. More precisely, we have the following proposition:

Proposition 3.1. ([20]) *If $t \in OBT(\omega)$ then*

$$\bigcup_{t_0 \in [t]} E(t_0) \in OBT(\omega)/_{\simeq} \tag{7}$$

Moreover, if $t_1 \in E(t)$ then $\bigcup_{t_0 \in [t]} E(t_0) = [t_1]$.

Another useful result is specified in the next proposition.

Proposition 3.2. ([20]) *If $t = (A, D, h) \in OBT(\omega)$ and $F(t) \notin L_T^*$ then*

(1) *$t \prec t_1$ for all $t_1 \in E(t)$*
(2) *if $t_1 \in E(t)$ and $t_2 \in E(t)$ then $t_1 \simeq t_2$*

This property allows us to define an operator on the factor set $OBT(\omega)/_{\simeq}$ as we specify in the next definition.

Definition 3.1. *We define the operator $U : OBT(\omega)/_{\simeq} \longrightarrow OBT(\omega)/_{\simeq}$ as follows*

$$U([t]) = \begin{cases} [t] \text{ if } F(t) \in L_T^* \\ \bigcup_{t_0 \in [t]} E(t_0) \text{ if } F(t) \notin L_T^* \end{cases} \tag{8}$$

Proposition 3.3. *For every $t \in OBT(\omega)$ we have $[t] \sqsubseteq U([t])$.*

Proof. If $F(t) \in L_T^*$ then $U([t]) = [t]$. But $[t] \sqsubseteq [t]$ because $\sqsubseteq$ is reflexive. It follows that in this case we have $[t] \sqsubseteq U([t])$. Consider now that $F(t) \notin L_T^*$. In this case we have $U([t]) = \bigcup_{t_0 \in [t]} E(t_0)$. By Proposition 3.1 we have $U([t]) = [t_1]$, where $t_1 \in E(t)$ is an arbitrary element. By Proposition 3.2 we have $t \prec t_1$, therefore $[t] \sqsubseteq [t_1]$. In other words we have $[t] \sqsubseteq U([t])$. □

The following property is a useful one to obtain other properties of the operator U.

Proposition 3.4. *If $t_1 \preceq t_2$, $t_1^* \in E(t_1)$ and $t_2^* \in E(t_2)$ then $t_1^* \preceq t_2^*$.*

Proof. Denote $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. Consider an embedding mapping $\alpha : A_1 \longrightarrow A_2$ of t_1 into t_2. This means that

$$u \in D_1 \Longrightarrow \overline{\alpha}(u) \in D_2 \tag{9}$$

$$h_1(root(t_1) = h_2(\alpha(root(t_1))) \tag{10}$$

From the properties of an embedding mapping we know that $h_1(i) = h_2(\alpha(i))$ for every $i \in A_1$.
Take $t_1^* = (A_1^*, D_1^*, h_1^*) \in E(t_1)$ and $t_2^* = (A_2^*, D_2^*, h_2^*) \in E(t_2)$. From (4), (5) and (6) we can suppose that the components of t_1^* and t_2^* satisfy the following conditions:

$$A_1^* = A_1 \cup \bigcup_{i \in \{i_1, \ldots, i_n\}} \{j_{i,1}, j_{i,2}\}$$

$$D_1^* = D_1 \cup \bigcup_{i \in \{i_1, \ldots, i_n\}} \{[(i, j_{i,1}), (i, j_{i,2})]\}$$

We define the mapping $\alpha^* : A_1^* \longrightarrow A_2^*$ as follows:
1) $\alpha^*(x) = \alpha(x)$ for $x \in A_1$;
2) We have

$$A_1^* \setminus A_1 = \{j_{i_1,1}, j_{i_1,2}, \ldots, j_{i_n,1}, j_{i_n,2}\} \tag{11}$$

and

$$D_1^* \setminus D_1 = \{[(i_1, j_{i_1,1}), (i_1, j_{i_1,2})], \ldots, [(i_n, j_{i_n,1}), (i_n, j_{i_n,2})]\} \tag{12}$$

Take an element $z \in A_1^* \setminus A_1$. From (11) we can suppose that $z = j_{i_m,1}$ for some $m \in \{1, \ldots, n\}$. From (12) we deduce that $[(i_m, j_{i_m,1}), (i_m, j_{i_m,2})] \in D_1^* \setminus D_1$. We have $h_2(\alpha(i_m)) = h_1(i_m)$ and $h_1(i_m) \in L_N$, therefore $h_2(\alpha(i_m)) \in L_N$. We have to consider the following two cases:
1) $\alpha(i_m)$ is not a leaf of t_2.
In this case there is $[(\alpha(i_m), r_1), (\alpha(i_m), r_2)] \in D_2$.
2) $\alpha(i_m)$ is a leaf of t_2.
In this case there is $[(\alpha(i_m), r_1), (\alpha(i_m), r_2)] \in D_2^* \setminus D_2$.
Both in the first case and in the second case we take $\alpha^*(z) = \alpha^*(j_{i_m,1}) = r_1$ and $\alpha^*(j_{i_m,2}) = r_2$. Now we can prove that $\alpha^*(D_1^*) \subseteq D_2^*$.
Take an arbitrary element $u \in D_1^*$. If $u \in D_1$ then $\overline{\alpha}(u) \in D_2$, therefore $\overline{\alpha^*}(u) \in D_2$ because α^* extends α. If $u \in D_1^* \setminus D_1$ then $u = [(i_m, j_{i_m,1}), (i_m, j_{i_m,2})]$. If $\alpha(i_m)$ is not a leaf of t_2 then $\overline{\alpha^*}(u) \in D_2$. If $\alpha(i_m)$ is a leaf of t_2 then $\overline{\alpha^*}(u) \in D_2^* \setminus D_2$. In fact we have $\overline{\alpha^*}(u) \in D_2^*$ and the property is proved. □

Proposition 3.5. *$U : (OBT(\omega)/_{\simeq}, \sqsubseteq) \longrightarrow (OBT(\omega)/_{\simeq}, \sqsubseteq)$ is an increasing operator.*

Proof. Suppose that $[t_1] \sqsubseteq [t_2]$. We have to prove that

$$U([t_1]) \sqsubseteq U([t_2]) \tag{13}$$

If $F(t_1) \in L_T^*$ then $U([t_1]) = [t_1]$. From Proposition 3.3 we obtain $[t_2] \sqsubseteq U([t_2])$. It follows that $U([t_1]) = [t_1] \sqsubseteq [t_2] \sqsubseteq U([t_2])$. It follows that in this case (13) is true. It remains to consider the case $F(t_1) \notin L_T^*$. Denote $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$. The relation $[t_1] \sqsubseteq [t_2]$ can be written equivalently as $t_1 \preceq t_2$. Take $t_1^* = (A_1^*, D_1^*, h_1^*) \in E(t_1)$ and $t_2^* = (A_2^*, D_2^*, h_2^*) \in E(t_2)$. From Proposition 3.1 we have $U([t_1]) = [t_1^*]$ and $U([t_2]) = [t_2^*]$. Now we apply Proposition 3.4 and deduce that $t_1^* \preceq t_2^*$. In other words we have (13). □

Definition 3.2. *We define the mapping $\mathcal{F} : OBT(\omega)/_{\simeq} \longrightarrow L^*$ by $\mathcal{F}([t]) = F(t)$.*

The next proposition shows that the above definition is a correct one.

Proposition 3.6. *The definition of the mapping $\mathcal{F}$ does not depend on representatives.*

Proof. Really, if $t_1 \simeq t_2$ then $F(t_1) = F(t_2)$ ([20]). □

It is known that a binary relation $\rho \subseteq X \times X$ is named a *noetherian relation* on X if can not find an infinite sequence

$$x_1, x_2, \ldots, x_n, \ldots$$

such that

$$x_1 \rho x_2, x_2 \rho x_3, \ldots, x_i \rho x_{i+1}, \ldots$$

Obviously a reflexive binary relation is not a noetherian relation because the sequence $x, x, \ldots$ satisfies the condition $x \rho x$.

Remark 3.1. *If a finite sequence $x_1, x_2, x_3, \ldots, x_n, x_{n+1}$ contains two identical elements and $x_i \rho x_{i+1}$ for $i \in \{1, \ldots, n\}$ then ρ is not a noetherian relation. Really, if $i < j$ and $x_i = x_j$ then $x_1, \ldots, x_i, x_{i+1}, \ldots, x_j, x_{i+1}, \ldots, x_j, \ldots$ is an infinite sequence of consecutive elements belonging to ρ.*

Definition 3.3. *The binary relation ρ_ω generated by ω is the binary relation $\rho_\omega \subseteq L \times L$ defined as follows: $x \rho_\omega y$ if and only if there is $z \in L$ such that $\omega(x) = (y, z)$ or $\omega(x) = (z, y)$.*

Proposition 3.7. *Suppose that ρ_ω is a noetherian relation and $t_0 \in OBT(\omega)$. The sequence $(r_n)_{n \geq 1}$ defined by*

$$\begin{cases} r_1 = [t_0] \\ r_{n+1} = U(r_n), \ n \geq 1 \end{cases} \tag{14}$$

satisfies the following properties:

- *$r_1 \sqsubseteq r_2 \sqsubseteq \ldots \sqsubseteq r_n \sqsubseteq \ldots$*
- *There is a natural number $k_0 \geq 1$ such that $r_1 \sqsubset r_2 \sqsubset \ldots \sqsubset r_{k_0} = r_{k_0+j}$ for every $j \geq 1$.*
- *$\mathcal{F}(r_{k_0}) \in L_T^*$*
- *If $t_0 = (A_0, D_0, h_0)$ and $t_j = (A_j, D_j, h_j) \in r_{j+1}$ then $h_0(root(t_0)) = h_j(root(t_j))$ for every $j \geq 1$.*

Proof. 1) We consider the first case, $F(t_0) \in L_T^*$.
We prove by induction on $i \geq 0$ that $\mathcal{F}(r_i) \in L_T^*$. For $i = 0$ this property is true because $\mathcal{F}(r_1) = F(t_0)$ and $F(t_0) \in L_T^*$. Suppose that this property is true for some $i \geq 0$. But $r_{i+1} = U(r_i)$ and from Definition 3.1 we have $U(r_i) = r_i$ because $\mathcal{F}(r_i) \in L_T^*$ by the inductive assumption. It follows that $\mathcal{F}(r_{i+1}) \in L_T^*$. As a consequence we have $r_1 = r_2 = \ldots$ and the proposition is proved and $k_0 = 1$.

2) We consider the second case, $F(t_0) \notin L_T^*$.
The property

$$r_1 \sqsubseteq r_2 \sqsubseteq \ldots \sqsubseteq r_n \sqsubseteq \ldots \tag{15}$$

is obtained from Proposition 3.3 and Proposition 3.5. We prove that in the sequence (15) there is $n_0 \geq 1$ such that $r_{n_0} = r_{n_0+1}$.
In order to prove this property we suppose, by contrary, that $r_i \sqsubset r_{i+1}$ for every $i \geq 1$. In other words, the relation (15) is an infinite sequence of the form

$$r_1 \sqsubset r_2 \sqsubset \ldots \sqsubset r_n \sqsubset \ldots \tag{16}$$

From (16) we deduce that for every $i \geq 1$ we have $\mathcal{F}(r_i) \notin L_T^*$. Really, if for some $i \geq 1$ we have $\mathcal{F}(r_i) \in L_T^*$ then $U(r_i) = r_i$. But $U(r_i) = r_{i+1}$, therefore $r_i = r_{i+1}$ and this fact contradicts (16).
Using (8) we obtain for every $i \geq 1$:

$$r_{i+1} = \bigcup_{t \in r_i} E(t) \tag{17}$$

For each $i \geq 1$ take an arbitrary, but fixed element $t_i \in E(t_{i-1})$. We verify by induction on $i \geq 1$ the property $t_i \in r_{i+1}$. We have $r_1 = [t_0]$, therefore $t_0 \in r_1$. From (17) we have $t_1 \in E(t_0) \subseteq r_2$ and thus the property is true for $i = 1$. If we suppose that $t_i \in r_{i+1}$ then $t_{i+1} \in E(t_i) \subseteq r_{i+2}$. It follows that $t_i \in r_{i+1}$ for every $i \geq 0$. But $F(t_i) = \mathcal{F}(r_{i+1})$ and therefore $F(t_i) \notin L_T^*$ because $\mathcal{F}(r_{i+1}) \notin L_T^*$.
We obtained an infinite sequence $t_0 \prec t_1 \prec \ldots \prec t_n \prec \ldots$ of successive immediate extensions. It follows that for every $i \geq 0$ there is a leaf q_{n_i} of t_i such that $h_i(q_{n_i}) \in L_N$. Consider a natural number $i > Card(L_N)$. Denote $root(t_0) = q_0$. By the general properties of a tree and because t_{i+1} is an immediate extension of t_i we deduce that there is

$$(q_0, p_1, \ldots, p_s, q_{n_0}, q_{n_1}, \ldots q_{n_i}) \in Path(t_i)$$

such that

- $(q_0, p_1, \ldots, p_s, q_{n_0}) \in Path(t_0)$
- $(q_0, p_1, \ldots, p_s, q_{n_0}, q_{n_1}, \ldots q_{n_j}) \in Path(t_j)$ for $j \in \{1, \ldots, i\}$.

Moreover, every node of these paths is labeled by a nonterminal in the corresponding tree:
$\{h_i(q_0), h_i(p_1), \ldots, h_i(q_{n_0}), h_i(q_{n_1}), \ldots, h_i(q_{n_i})\} \subseteq L_N$

To simplify the notation we denote $m_0 = q_0$, $m_1 = q_1$, ..., $m_s = p_s$, $m_{s+1} = q_{n_0}$, $m_{s+2} = q_{n_1}$, ..., $m_{s+i+1} = q_{n_i}$. Tacking into account the definition of ρ_ω we obtain $h_i(m_j)\rho_\omega h_i(m_{j+1})$ for $j \in \{0, \ldots, s+i\}$. Because $i > Card(L_N)$ and L_N is a finite set, in the sequence $\{h_i(m_j)\}_{j \in \{0,\ldots,s+i\}}$ there are j_1 and j_2 such that $h_i(m_{j_1}) = h_i(m_{j_2})$. By Remark 3.1 this is not possible because ρ_ω is a noetherian binary relation. It results that our assumption (16) is false.

Let us denote by k_0 the least natural number such that $r_k = r_{k+1}$, therefore $r_1 \sqsubset t_2 \sqsubset \ldots \sqsubset r_{k_0} = r_{k_0+1}$. In this case we have $r_{k_0} = U(r_{k_0})$. In other words we have $\mathcal{F}(r_{k_0}) \in L_T^*$.

The last sentence of the proposition is proved by induction on j. We verify this sentence for $j = 1$. We have $t_0 \in r_1$, $t_1 \in r_2$ and $r_1 \sqsubset r_2$. It follows that $t_1 \in U([t_0])$

and more precisely we have $t_1 \in E(t)$ for some $t \in [t_0]$. From $t_1 \in E(t)$ we obtain $root(t_1) = root(t)$ and $h_1(root(t_1)) = h(root(t))$, where h is the labeling mapping of t. From $t \in [t_0]$ we have $h(root(t)) = h_0(root(t_0))$. It follows that $h_1(root(t_1)) = h_0(root(t_0))$. Thus the property is verified for $j = 1$.

We suppose that the property is true for every $j \in \{0, \ldots, n-1\}$ and take $t_n \in r_{n+1}$. If $r_{n+1} = r_n$ then the property is proved. It remains to study the case $r_n \sqsubset r_{n+1}$. By the inductive assumption we have $h_0(root(t_0)) = h_{n-1}(root(t_{n-1}))$. Obviously $t_n \in E(t_{n-1})$, therefore $root(t_n) = root(t_{n-1})$. Moreover, $h_n(root(t_n)) = h_{n-1}(root(t_{n-1}))$. It follows that $h_0(root(t_0)) = h_n(root(t_n))$, therefore the property is true for $j = n$. □

4. Local partial ordered sets of ω-trees and their greatest elements

For every $a \in L_N$ we consider the set

$$OBT_a(\omega) = \{\, t \in OBT(\omega) \mid t = (A, D, h), h(root(t)) = a\}$$

We remark that

$$OBT(\omega)/_{\simeq} = \bigcup_{a \in L_N} OBT_a(\omega)/_{\simeq}$$

$$OBT_a(\omega)/_{\simeq} \cap OBT_b(\omega)/\simeq = \emptyset$$

for $a \neq b$. In this way we obtained a partition of the set $OBT(\omega)/_{\simeq}$.

For each $a \in L_N$ we can consider the partial ordered set $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$. Such a structure is named *local partial ordered set* of ω-trees.

In this section we study the greatest element of a local partial ordered set of ω-trees.

Proposition 4.1. *Suppose that ρ_ω is a noetherian relation and $a \in L_N$. Let us consider $t_1, t_2 \in OBT_a(\omega)$. If $t_1 \preceq t_2$ and α is an embedding mapping of t_1 into t_2 then $\alpha(root(t_1)) = root(t_2)$.*

Proof. Suppose that $t_1 = (A_1, D_1, h_1)$, $t_2 = (A_2, D_2, h_2)$, $\alpha(root(t_1)) = j$ and $j \neq root(t_2)$. There is a path $(root(t_2), q_1, \ldots, q_m, j) \in Path(t_2)$. Take the sequence $h_2(root(t_2)), h_2(q_1), \ldots, h_2(q_m), h_2(j)$.

We have $h_2(root(t_2)) = a$ because $t_2 \in OBT_a(\omega)$ and $h_2(j) = a$. Really, $h_2(\alpha(i)) = h_1(i)$ for every i therefore we have $h_2(j) = h_2(\alpha(root(t_1))) = h_1(root(t_1)) = a$. It follows that we have the sequence $a\rho_\omega h(q_1), h(q_1)\rho_\omega h(q_2), \ldots, h(q_m)\rho_\omega h(j)$, where $h(j) = a$ and thus there is an infinite sequence of elements from L_N such that two by two belong to ρ_ω. This is not possible because ρ_ω is a noetherian relation. □

Proposition 4.2. *Suppose that ρ_ω is a noetherian relation and $a \in L_N$. If $t_0 \in OBT_a(\omega)$ then the element r_{k_0} given by Proposition 3.7 is the greatest element of the partial ordered set $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$.*

Proof. We have to prove the following two properties:

(i) $r_{k_0} \in OBT_a(\omega)/_{\simeq}$

(ii) For every $[t] \in OBT_a(\omega)/_{\simeq}$ we have $[t] \sqsubseteq r_{k_0}$.

Consider a representative $t_{k_0} = (A_{k_0}, D_{k_0}, h_{k_0})$ of r_{k_0}. The sentence (i) is obtained immediately from Proposition 3.7 because

$$a = h_0(root(t_0)) = h_{k_0}(root(t_{k_0}))$$

Let us prove the sentence (ii). Take an arbitrary element $t = (A, D, h) \in OBT_a(\omega)$. We shall build an embedding mapping $\alpha : A \longrightarrow A_{k_0}$ of t into t_{k_0} such that α is an embedding mapping of $T_k(t)$ into $T_k(t_{k_0})$ for every $k \geq 1$, where T_k is the slicing operator.

The first step in this construction is to define

$$\alpha(root(t)) = root(t_{k_0}) \tag{18}$$

We have $h_{k_0}(\alpha(root(t))) = h_{k_0}(root(t_{k_0})) = a$ and $h(root(t)) = a$ because $t_{k_0}, t \in OBT_a(\omega)$. It follows that $h_{k_0}(\alpha(root(t))) = h(root(t))$.

Because an ω-tree can not contain a single node we deduce that there are the elements $[(root(t), i_1), (root(t), i_2)] \in D$ and $[(root(t_{k_0}), j_1), (root(t_{k_0}), j_2)] \in D_{k_0}$. We can define

$$\alpha(i_1) = j_1, \alpha(i_2) = j_2$$

As a consequence we have

$$T_1(t) \preceq T_1(t_{k_0}) \tag{19}$$

where T_k is the slicing operator.

In what follows we prove by induction on $k \geq 1$ that

$$T_k(t) \preceq T_k(t_{k_0}) \tag{20}$$

From (19) we have (20) for $k = 1$. Suppose that (20) is true for $k = p$ and α is the embedding mapping of $T_p(t)$ into $T_p(t_{k_0})$. We prove now that α can be extended so that it becomes an embedding mapping of $T_{p+1}(t)$ into $T_{p+1}(t_{k_0})$. We have the following two cases:

1) If $T_{p+1}(t) = T_p(t)$ then $T_{p+1}(t) = T_s(t)$ for every $s \geq p$. But $T_p(t) \preceq T_p(t_{k_0})$ therefore $T_s(t) \preceq T_p(t_{k_0})$ for every $s \geq p$. In other words $T_{p+i}(t) \preceq T_p(t_{k_0})$ for every $i \geq 1$. But $T_p(t_{k_0}) \preceq T_{p+i}(t_{k_0})$. By transitivity we obtain $T_{p+i}(t) \preceq T_{p+i}(t_{k_0})$ for every $i \geq 1$. Particularly we have (20) for $k = p + 1$.

2) If remains to consider the case $T_{p+1}(t) \neq T_p(t)$. By Proposition 3.2 we have $T_p(t) \prec T_{p+1}(t)$ and consider a maximal path $(root(t), q_1, \ldots, q_p, q_{p+1}) \in Path(T_{p+1}(t))$. In order to make a choice we can suppose that

$$[(q_p, q_{p+1}), (q_p, m_{p+1})] \in D \tag{21}$$

It follows that $h(q_p) \in L_N$. We have $(root(t), q_1, \ldots, q_p) \in Path(T_p(t))$. But α is an embedding mapping of $T_p(t)$ into $T_p(t_{k_0})$ therefore $(\alpha(root(t)), \alpha(q_1), \ldots, \alpha(q_p)) \in Path(T_p(t_{k_0}))$. From $T_p(t) \preceq T_p(t_{k_0})$ we have $h_{k_0}(\alpha(q_p)) = h(q_p)$ therefore $h_{k_0}(\alpha(q_p)) \in L_N$. Because $F(t_{k_0}) \in L_T^*$ we deduce that $\alpha(q_p)$ is not a leaf of t_{k_0}, therefore there is

$$[(\alpha(q_p), j_1), (\alpha(q_p), j_2)] \in D_{k_0} \tag{22}$$

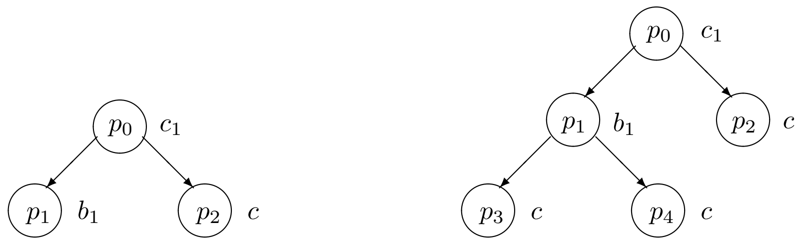
From (21) and (22) we can extend the mapping α taking $\alpha(q_{p+1}) = j_1$ and $\alpha(m_{p+1}) = j_2$. We proceed in this way for every path $(root(t), q_1, \ldots, q_p, q_{p+1}) \in Path(T_{p+1}(t))$ and thus (20) is true for $k = p + 1$. In conclusion we have $t \preceq t_{k_0}$, therefore $[t] \sqsubseteq r_{k_0}$. □

Proposition 4.3. *Suppose that $a \in L_N$, ρ_ω is a noetherian relation and $t \in OBT_a(\omega)$. The following two sentences are equivalent:*

(1) $F(t) \in L_T^*$

(2) $[t]$ *is the greatest element of the set* $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$

Proof. Let us prove first that (2) $\Rightarrow$ (1). Suppose that $[t]$ is the greatest element of the set $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$. Starting with $t_0 = t$ we obtain from (14) the greatest element r_{k_0} of $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$. It follows that $[t] = r_{k_0}$, therefore $[t] = [t_{k_0}]$, where t_{k_0} is a representative of r_{k_0}. It follows that $F(t) = F(t_{k_0})$. But $F(t_{k_0}) \in L_T^*$ by Proposition 3.7, so $F(t) \in L_T^*$. Thus (1) is true.



The element t_0 The element $t_1 \in OBT_{c_1}(\omega)$

FIGURE 1. $t_0, t_1 \in OBT(\omega)$

We prove now the implication (1) $\Rightarrow$ (2). Suppose that $F(t) \in L_T^*$. We apply (14) for $t_0 = t$. Because $F(t) \in L_T^*$ we obtain $r_2 = U(r_1) = r_1$, therefore r_1 is the greatest element. But $r_1 = [t]$, therefore (2) is proved. □

In conclusion, if $a \in L_N$ and ρ_ω is a noetherian relation then there is an element and only one which is the greatest element of the set $(OBT_a(\omega)/_{\simeq}, \sqsubseteq)$. This element can be computed as follows:

Step 1: Take the ω-tree $t_0 = (A_0, D_0, h_0)$, where $A_0 = \{r_0, i_1, i_2\}$, $D_0 = \{[(r_0, i_1), (r_0, i_2)]\}$, $h_0(r_0) = a$, $\omega(a) = (h_0(i_1), h_0(i_2))$.

Step 2: Starting with $i = 0$ we choose $t_{i+1} \in E(t_i)$ while $F(t_i) \notin L_T^*$.

Step 3: If $F(t_i) \in L_T^*$ then t_i is a representative of the greatest element.

This method allows us to prove the following property:

Proposition 4.4. *The local greatest element is not a maximal element of the whole partial ordered set* $(OBT(\omega)/_{\simeq}, \sqsubseteq)$.

Proof. Consider $L_T = \{b, c\}$, $L_N = \{a, b_1, c_1\}$ and the split mapping $\omega(a) = (b, c_1)$, $\omega(c_1) = (b_1, c)$ and $\omega(b_1) = (c, c)$. Obviously ρ_ω is a noetherian binary relation. Consider the element $t_0 = (\{p_0, p_1, p_2\}, \{[(p_0, p_1), (p_0, p_2)]\}, h_0)$, where $h_0(p_0) = c_1$, $h_0(p_1) = b_1$ and $h_0(p_2) = c$. We have $t_0 \in OBT(\omega)$. Take $t_1 \in E(t_0)$, which is represented in Figure 1. We have $F(t_1) \in L_T^*$, therefore by Proposition 4.3 the element $[t_1]$ is the greatest element of the set $(OBT_{c_1}(\omega)/_{\simeq}, \sqsubseteq)$. But $[t_1]$ is not a maximal element of $OBT(\omega)$ because the element t represented in Figure 2 has the property $[t_1] \sqsubset [t]$. □

5. Conclusions

In this paper we develop the initial ideas presented in [18], [19] and [20]. The ultimate goal of this research is to build a formalism for the valuation process in a cooperating system based on semantic schemas. In this paper we defined a decomposition of the factor set into local partial ordered subset such that we can build the

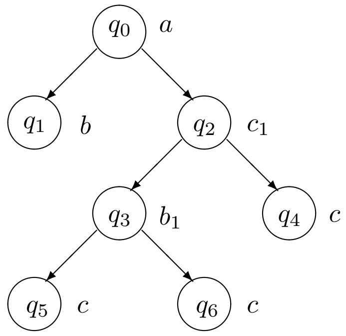


FIGURE 2. $t \in OBT(\omega)$ and $t_1 \prec t$

greatest element of a subset. In a forthcoming paper we show that the greatest element allows to define a "template" for the inference process in a master-slave system based on semantic schemas.

References

[1] A.V. Aho and J.D. Ullman, *The Theory of Parsing, Translation, and Compiling*, Prentice-Hall, Vol. I, 1972.

[2] S. Bhattacharya and S.K. Ghosh, An Artificial Intelligence Based Approach for Risk Management Using Attack Graph, *Proceedings of the 2007 International Conference on Computational Intelligence and Security* (2007), 794–798.

[3] W.J. Blok and J. Rebagliato, Algebraic Semantics for Deductive Systems, *Studia Logica* **74** (2003), 153–180.

[4] D. Busneag and S. Rudeanu, A glimpse of deductive systems in algebra, *Central European Journal of Mathematics* **8** (2010), No. 4, 688–705.

[5] M. Chein and M.L. Mugnier, Graph-based Knowledge Representation- Computational Foundations of Conceptual Graphs, *Series: Advanced Information and Knowledge Processing*, Springer, 2008.

[6] L.C. Ciungu, On pseudo-BCK algebras with pseudo-double negation, *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), No. 1, 19–26.

[7] Q. Ding, M. Khan, A. Roy and W. Perrizo, The P-tree algebra, *Proceedings of the 2002 ACM symposium on Applied computing* (2002), 426–431.

[8] H.V. Jagadish, L.V.S. Lakshmanan, D. Srivastava and K. Thompson, TAX: A Tree Algebra for XML, Database Programming Languages, *Lecture Notes in Computer Science* **2397** (2002), 149–164.

[9] E. Miranda and M. Zaffalon, Coherence graphs, *Artificial Intelligence* **173** (2009), No. 1, 104–144.

[10] D. Piciu, A. Jeflea and R. Cretan, On the lattice of deductive systems of a residuated lattice, *Annals of the University of Craiova, Math. Comp. Sci. Ser.* **35** (2008), 199–210.

[11] S. Paparizos and H.V. Jagadish, Pattern tree algebras: sets or sequences?, *Proceedings of the 31st VLDB Conference* (2005), Trondheim, Norway, 349–360.

[12] J.F. Sowa, Conceptual graphs for a database interface, *IBM Journal of Research and Development* **20** (1976), No. 4, 336–357.

[13] P.J. Tan and D.L. Dowe, Decision Forests with Oblique Decision Trees, MICAI 2006: *Advances in Artificial Intelligence, Lecture Notes in Computer Science, Springer Berlin* **4293** (2006), 593–603.

[14] N. Ţăndăreanu, Collaborations between distinguished representatives for labelled stratified graphs, *Annals of the University of Craiova, Mathematics and Computer Science Series* **30** (2003), No. 2, 184–192.

[15] N. Ţăndăreanu, Distinguished Representatives for Equivalent Labelled Stratified Graphs and Applications, *Discrete Applied Mathematics* **144** (2004), No. 1-2, 183–208.

[16] N. Ţăndăreanu, Knowledge representation by labeled stratified graphs, *Proceedings of the 8th World Multi-Conference on Systemics, Cybernetics and Informatics* **5** (2004), 345–350.

[17] N. Ţăndăreanu, Master-Slave Systems of Semantic Schemas and Applications, *The 10th IASTED International Conference on Intelligent Systems and Control* (ISC 2007), November 19-21, (2007), Cambridge, Massachusetts, 150–155.

[18] N. Ţăndăreanu and C.Zamfir, Algebraic properties of ω-trees (I), *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), No. 1, 80–89.

[19] N. Ţăndăreanu and C.Zamfir, Algebraic properties of ω-trees (II), *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), No. 2, 7–17.

[20] N. Ţăndăreanu and C. Zamfir, Slices and extensions of ω-trees, *Annals of the University of Craiova, Mathematics and Computer Science Series* **38** (2011), No. 1, 72–82.

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