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# Strict partial orders between $\omega$-trees

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Abstract. We consider the set $OBT(\omega)$ of the $\omega$-labeled trees ([4]) and the equivalence relation $\simeq$ on this set introduced in [5]. In this paper we define and study a strict partial order $\prec$ on the set $OBT(\omega)$ and a strict partial order $\sqsubset$ on the factor set $OBT(\omega)/_{\simeq}$. We characterize these relations and finally we show that $t_1 \prec t_2$ if and only if $[t_1] \sqsubset [t_2]$, where $[t]$ denotes the equivalence class of $t$.

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## 1. Introduction

More and more the mechanisms of universal algebras are implied in computer science. Most times these mechanisms are associated with other mechanisms such as those offered by graph theory. This paper is developed in considering this circumstances. The starting point in this research was given by the need to formalize cooperating systems based on semantic schemas ([2], [3]). From this point of view in this paper we develop the series of results published in [4], [5] and [6].

This paper is organized as follows: In Section 2 we recall the main notions and results that are used in subsequent sections. In Section 3 we prove two helpful results that are used in the subsequent sections of this paper. Section 4 defines a strict order on the set $OBT(\omega)$ of the $\omega$-trees and we study this relation. In Section 5 we study a strict order on the factor set $OBT(\omega)/\simeq$. A connection between these two strict orders is given also in Section 5. The Section 6 contains the conclusions of our study.

## 2. Basic concepts and results

We consider a finite set $L$ and a decomposition $L = L_N \cup L_T$, where $L_N \cap L_T = \emptyset$. The elements of $L_N$ are called *nonterminal labels* and those of $L_T$ are called *terminal labels.* The elements of $L$ are called *labels.* A **split mapping** on $L$ ([4]) is a function $\omega : L_N \longrightarrow L \times L$.

The relation parent-child in a directed graph can be defined by means of a list $[(i, i_1), \ldots, (i, i_n)]$, where $i_1, \ldots, i_n$ are all children of $i$ ([1]). In this case the children are ordered by the place of a child in the sequence $i_1, \ldots, i_n$. Particularly we can obtain an ordered tree as a pair $(A, D)$, where $A$ is the set of nodes and $D$ is the set of all lists describing the relation parent-child.

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An $\omega$**-tree** ([4]) is a tuple $t = (A, D, h)$, where
- $(A, D)$ is an ordered tree and every element of $D$ is of the form $[(i, i_1), (i, i_2)]$;
- $h : A \longrightarrow L$ is a mapping such that

$$[(i, i_1), (i, i_2)] \in D \Rightarrow h(i) \in L_N \ \& \ \omega(h(i)) = (h(i_1), h(i_2)) \tag{1}$$

For each $i \in A$ the element $h(i)$ is called the **label** of the node $i$. The mapping $h$ is named the **labeling mapping** of $t$. By $OBT(\omega)$ we denote the set of all $\omega$-trees.

Let $t_1 = (A_1, D_1, h_1)$ and $t_2 = (A_2, D_2, h_2)$ be two elements of $OBT(\omega)$ and an arbitrary mapping $\alpha : A_1 \longrightarrow A_2$. For every $u = [(i, i_1), (i, i_2)] \in D_1$ we denote

$$\overline{\alpha}(u) = [(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))]$$

If $t = (A, D, h)$ is an $\omega$-tree then we denote by $root(t)$ the element of $A$ designated by the root of $t$.

If $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ then we define the relation $t_1 \preceq t_2$ if there is a mapping $\alpha : A_1 \longrightarrow A_2$ such that:

$$u \in D_1 \Longrightarrow \overline{\alpha}(u) \in D_2 \tag{2}$$

$$h_1(root(t_1)) = h_2(\alpha(root(t_1))) \tag{3}$$

Such a mapping $\alpha$ is an **embedding mapping** of $t_1$ into $t_2$ ([4]). The relation $\preceq$ is reflexive and transitive, but is not antisymmetric. Thus $\preceq$ is not a partial order on the set $OBT(\omega)$.

If $t = (A, D, h) \in OBT(\omega)$ then a pair $(i, i_j)$ appearing in a list of $D$ is an arc of $t$. A sequence $(j_0, j_1, \ldots, j_k)$ is a path of $t$ if $(j_r, j_{r+1})$ is an arc of $t$ for every $r \in \{0, \ldots, k-1\}$. We denote by $Path(t)$ the set of all paths of $t$.

**Proposition 2.1.** ([4]) *Suppose that* $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$ *and* $\alpha : A_1 \longrightarrow A_2$ *is a mapping such that (2) is satisfied. Then*
(1) *If* $(m, n)$ *is an arc in* $t_1$ *then* $(\alpha(m), \alpha(n))$ *is an arc in* $t_2$.
(2) *If* $d = (n_0, n_1, \ldots, n_k) \in Path(t_1)$ *then* $\alpha(d) = (\alpha(n_0), \alpha(n_1), \ldots, \alpha(n_k)) \in Path(t_2)$.

We consider $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ and $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. We define $t_1 \simeq t_2$ if $t_1 \preceq t_2$ and $t_2 \preceq t_1$. The relation $\simeq$ is reflexive, symmetric and transitive therefore it is an equivalence relation ([5]). We denote by $[t]$ the equivalence class of $t$.

Let us consider $[t_1] \in OBT(\omega)/_{\simeq}$ and $[t_2] \in OBT(\omega)/_{\simeq}$. We define the relation $[t_1] \sqsubseteq [t_2]$ if $t_1 \preceq t_2$. The relation $\sqsubseteq$ does not depend on representatives, it is a reflexive, antisymmetric and transitive binary relation ([5]). As a consequence, the pair $(OBT(\omega)/_{\simeq}, \sqsubseteq)$ becomes a partial ordered set.

Finally we relieve the following useful result.

**Proposition 2.2.** ([5]) *Suppose that* $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$ *and* $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. *The following two conditions are equivalent:*
(1) $t_1 \simeq t_2$
(2) *There is a bijective mapping* $\alpha : A_1 \longrightarrow A_2$ *such that the conditions (4), (5) and (6) are satisfied:*

$$\alpha(root(t_1)) = root(t_2) \tag{4}$$

$$u \in D_1 \Longleftrightarrow \overline{\alpha}(u) \in D_2 \tag{5}$$

$$h_1(root(t_1)) = h_2(\alpha(root(t_1))) \tag{6}$$

*Moreover, if the bijective mappings $\alpha : A_1 \longrightarrow A_2$ and $\beta : A_1 \longrightarrow A_2$ satisfy conditions (4), (5) and (6) then $\alpha = \beta$.*

## 3. Preliminary results

In this section we prove two helpful results that are used in the subsequent sections of this paper. First we prove the following help result.

**Proposition 3.1.** *Suppose that $t_1 \in OBT(\omega)$ and $t_2 \in OBT(\omega)$. If $t_1 \preceq t_2$ and there is a bijective embedding mapping of $t_1$ onto $t_2$ then $t_1 \simeq t_2$.*

*Proof.* Suppose that $t_1 = (A_1, D_1, h_1)$, $t_2 = (A_2, D_2, h_2)$ and $\alpha : A_1 \longrightarrow A_2$ is a bijective embedding mapping of $t_1$ onto $t_2$. In order to prove that $t_1 \simeq t_2$ we verify that (4), (5) and (6) are satisfied.

• Let us prove (4). We suppose by contrary that $\alpha(root(t_1)) \neq root(t_2)$. There is $i \in A_1 \setminus \{root(t_1)\}$ such that $\alpha(i) = root(t_2)$ because $\alpha$ is a bijective mapping. Denote $p = \alpha(root(t_1))$. There is a path and only one $(root(t_1), p_1, \ldots, p_r, i) \in Path(t_1)$. By Proposition 2.1 we have $(\alpha(root(t_1)), \alpha(p_1), \ldots, \alpha(p_r), \alpha(i)) \in Path(t_2)$. In other words there is a path $(p, \ldots, root(t_2))$, which is not true because the graph associated to $(A_2, D_2)$ is a tree. It follows that our assumption is false and thus (4) is true.

• In order to prove (5) we prove first the following useful property: *$i \in A_1$ is a leaf of $t_1$ if and only if $\alpha(i) \in A_2$ is a leaf of $t_2$.*

Suppose that $\alpha(i)$ is a leaf of $t_2$ and that $i$ is not a leaf of $t_1$. There is $[(i, k_1), (i, k_2)] \in D_1$, therefore $[(\alpha(i), \alpha(k_1)), (\alpha(i), \alpha(k_2))] \in D_2$. This shows that $\alpha(i)$ is not a leaf of $t_2$, which is not true.

Conversely, suppose that $i$ is a leaf of $t_1$. By contrary, suppose that $\alpha(i)$ is not a leaf of $t_2$. It follows that there is $[(\alpha(i), p), (\alpha(i), q)] \in D_2$. The mapping $\alpha$ is surjective, therefore there are $m \in A_1$ and $r \in A_1$ such that $\alpha(m) = p$ and $\alpha(r) = q$. There is a path $(root(t_1), r_1, \ldots, r_s, r) \in Path(t_1)$. Because we proved before that $\alpha(root(t_1)) = root(t_2)$ and $\alpha(r) = q$, we obtain a path $(root(t_2), \alpha(r_1), \ldots, \alpha(r_s), q) \in Path(t_2)$. If $\alpha(r_s) = \alpha(i)$ then $r_s = i$ and this case is not possible because $(r_s, r)$ is an arc of $t_1$, $r_s = i$ and $i$ is a leaf of $t_1$. It follows that $\alpha(r_s) \neq \alpha(i)$, therefore $r_s \neq i$. But in this case $(\alpha(i), q)$ and $\alpha(r_s), q)$ are arcs in $t_2$, which is not possible because the graph associated to $(A_2, D_2)$ is a tree. This shows that $\alpha(i)$ is a leaf of $t_2$.

• Let us prove (5). The implication from left to right is true because $\alpha$ is an embedding mapping of $t_1$ into $t_2$. It remains to prove the implication from right to left. Suppose that $i, i_1, i_2 \in A_1$ and $[(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2$. As we shown before, $i \in A_1$ is not a leaf of $t_1$ because $\alpha(i)$ is not a leaf of $t_2$. It follows that there is $[(i, j_1), (i, j_2)] \in D_1$. We have $[(\alpha(i), \alpha(j_1)), (\alpha(i), \alpha(j_2))] \in D_2$ because $\alpha$ is an embedding mapping of $t_1$ into $t_2$. But we have chosen $i, i_1, i_2 \in A_1$ such that $[(\alpha(i), \alpha(i_1)), (\alpha(i), \alpha(i_2))] \in D_2$. It follows that $\alpha(i_1) = \alpha(j_1)$ and $\alpha(i_2) = \alpha(j_2)$. The mapping $\alpha$ is injective and this implies $i_1 = j_1$ and $i_2 = j_2$. It results that $[(i, i_1), (i, i_2)] \in D_1$.

• Finally we remark that (6) is nothing else than (3), which is true because $\alpha$ is an embedding mapping of $t_1$ into $t_2$. $\square$

**Proposition 3.2.** *Suppose that $t_1 = (A_1, D_1, h_1) \in OBT(\omega)$, $t_2 = (A_2, D_2, h_2) \in OBT(\omega)$. If $t_1 \preceq t_2$ then one and only one of the following conditions is satisfied:*

(1) $t_1 \simeq t_2$

(2) $\alpha(A_1) \subset A_2$ *for every embedding mapping $\alpha$ of $t_1$ into $t_2$*

*Proof.* Suppose that $\alpha : A_1 \longrightarrow A_2$ is an embedding mapping of $t_1$ into $t_2$. This is an injective mapping, therefore $\alpha(A_1) \subseteq A_2$. It follows that either $\alpha(A_1) = A_2$ or $\alpha(A_1) \subset A_2$. In the first case the mapping $\alpha$ is bijective. From Proposition 3.1 we know that $t_1 \simeq t_2$. It remains to consider the case $\alpha(A_1) \subset A_2$. Consider another embedding mapping $\beta : A_1 \longrightarrow A_2$ of $t_1$ into $t_2$. If we would have $\beta(A_1) = A_2$ then $\beta$ is a bijective mapping, therefore by Proposition 3.1 we should have $t_1 \simeq t_2$. It follows that $\beta = \alpha$. This is not possible because $\alpha(A_1) \subset A_2$ and $\beta(A_1) = A_2$. It remains the case $\beta(A_1) \subset A_2$. $\square$

## 4. The strict partial order $\prec$ on $OBT(\omega)$

First we recall the concept of strict partial order. A binary relation $\rho \subseteq X \times X$ on the set $X$ is a *strict partial order* if the following conditions are fulfilled:
(1) $\rho$ is irreflexive: $x\rho x$ does not hold for any $x \in X$;
(2) $\rho$ is transitive: $x\rho y$ and $y\rho z$ implies $x\rho z$.

**Remark 4.1.** *A strict partial order is asymmetric. In other words, if $x\rho y$ then $y\rho x$ does not hold. Really, if by contrary we suppose that there $x$ and $y$ such that $x\rho y$ and $y\rho x$ then by transitivity we obtain $x\rho x$.*

We can define now the following binary relation $\prec \subseteq OBT(\omega) \times OBT(\omega)$.

**Definition 4.1.** *We write $t_1 \prec t_2$ if the following conditions are satisfied:*
(1) $t_1 \preceq t_2$
(2) *Every embedding mapping of $t_1$ into $t_2$ is not surjective.*

**Proposition 4.1.** *The relation $\prec$ is a strict partial order on $OBT(\omega)$.*

*Proof.* • The relation $\prec$ is irreflexive: $t_1 \prec t_1$ does not hold for any $t_1 \in OBT(\omega)$. Really, if $\alpha : A_1 \longrightarrow A_1$ is an embedding mapping of $t_1$ into $t_1$ then $\alpha$ is surjective because every embedding mapping is injective and $A_1$ is a finite set. This case is not possible.

• The relation $\prec$ is transitive: $t_1 \prec t_2$ and $t_2 \prec t_3$ implies $t_1 \prec t_3$. From Definition 4.1 we have $t_1 \preceq t_2$ and $t_2 \preceq t_3$ therefore $t_1 \preceq t_3$ because $\preceq$ is transitive. By Proposition 3.2 we have one and only one of the following two properties:

- $t_1 \simeq t_3$
- every embedding mapping of $t_1$ into $t_3$ is not surjective.

Let us suppose that we have the first property, $t_1 \simeq t_3$. If $\alpha : A_1 \longrightarrow A_2$ is an embedding mapping of $t_1$ into $t_2$ and $\beta : A_2 \longrightarrow A_3$ is an embedding mapping of $t_2$ into $t_3$ then $\alpha \circ \beta : A_1 \longrightarrow A_3$ is an embedding mapping of $t_1$ into $t_3$. We supposed that $t_1 \simeq t_3$, therefore $\alpha \circ \beta$ is a bijective mapping. We have also $\alpha(A_1) \subset A_2$ and $\beta(A_2) \subset A_3$ because $t_1 \prec t_2$ and $t_2 \prec t_3$. For every $z \in A_3$ there is $x \in A_1$ such that $\beta(\alpha(x)) = z$ because $\alpha \circ \beta$ is a surjective mapping. It follows that $\beta$ is a surjective mapping, which is not true because $\beta(A_2) \subset A_3$. It follows that the condition $t_1 \simeq t_3$ can not be satisfied, therefore embedding mapping of $t_1$ into $t_3$ is not surjective. In conclusion the conditions from Definition 4.1 are satisfied and we have $t_1 \prec t_3$. $\square$

## 5. The strict partial order $\sqsubset$ on the set $OBT(\omega)/\simeq$

In this section we study a strict partial order on the set $OBT(\omega)/_{\simeq}$. This is generated by means of the relation $\sqsubseteq$.

Every partial order $\rho \subseteq X \times X$ induces a strict order $\rho_s \subseteq X \times X$ defined as follows:

$$x\rho_s y \Longleftrightarrow x\rho y \ \& \ x \neq y$$

This method can be applied for the partial order $\sqsubseteq$. We denote by $\sqsubset$ the strict order induced by $\sqsubseteq$. We have

$$[t_1] \sqsubset [t_2] \Longleftrightarrow [t_1] \sqsubseteq [t_2] \ \& \ [t_1] \neq [t_2]$$

We can give the following characterization of this relation.

**Proposition 5.1.** *The following sentences are equivalent:*

(1) $[t_1] \sqsubset [t_2]$

(2) $[t_1] \sqsubseteq [t_2]$ *and there is an embedding mapping of* $t_1$ *into* $t_2$ *which is not surjective.*

(3) $[t_1] \sqsubseteq [t_2]$ *and every embedding mapping of* $t_1$ *into* $t_2$ *is not surjective.*

*Proof.* We prove the following implications

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (1)$$

• Suppose that $[t_1] \sqsubset [t_2]$. This means that $[t_1] \sqsubseteq [t_2]$ and $[t_1] \neq [t_2]$. Suppose by contrary that (2) is not true. This means that every embedding mapping of $t_1$ into $t_2$ is surjective. Consider such a mapping $\alpha$. From Proposition 3.1 we deduce that $t_1 \simeq t_2$, therefore $[t_1] = [t_2]$. This contradicts the property $[t_1] \neq [t_2]$.

• Suppose that (2) is true, but (3) is not true. Therefore there is a surjective embedding mapping $\beta$ of $t_1$ into $t_2$. By Proposition 3.1 we have $t_1 \simeq t_2$ and therefore if $\alpha$ is an embedding mapping of $t_1$ into $t_2$ then $\beta = \alpha$. This shows that (2) is not true.

• Suppose that (3) is true. We prove that $[t_1] \neq [t_2]$. Really, if we suppose that $[t_1] = [t_2]$ then there is only one embedding mapping of $t_1$ into $t_2$ and this is a bijective mapping. This contradicts (3). □

The next proposition establishes the connection between $\sqsubset$ and $\prec$.

**Proposition 5.2.** *We have* $t_1 \prec t_2$ *is and only if* $[t_1] \sqsubset [t_2]$.

*Proof.* Suppose that $t_1 \prec t_2$. From Definition 4.1 we obtain $t_1 \preceq t_2$, therefore $[t_1] \sqsubseteq [t_2]$. The same definition shows that every embedding mapping of $t_1$ into $t_2$ is not surjective. The third condition from Proposition 5.1 is satisfied, therefore $[t_1] \sqsubset [t_2]$.

Conversely, suppose that $[t_1] \sqsubset [t_2]$. We apply again Proposition 5.1 and the third condition of this proposition is satisfied. It follows that $t_1 \preceq t_2$ and every embedding mapping of $t_1$ into $t_2$ is not surjective. From Definition 4.1 we obtain $t_1 \prec t_2$. □

**Proposition 5.3.** *If* $t_1 \prec t_2$, $t_1^* \simeq t_1$ *and* $t_2^* \simeq t_2$ *then* $t_1^* \prec t_2^*$

*Proof.* We have $t_1^* \in [t_1]$, $t_2^* \in [t_2]$ and $[t_1] \sqsubseteq [t_2]$, therefore $t_1^* \preceq t_2^*$. It remains to prove that every embedding mapping of $t_1^*$ into $t_2^*$ is not surjective. By contrary, we suppose that there is a surjective embedding mapping of $t_1^*$ onto $t_2^*$ denoted by $\alpha$. There is a bijective mapping $\beta_1$, which is the embedding mapping of $t_1$ into $t_1^*$ because $t_1^* \in [t_1]$. We have $t_2^* \in [t_2]$ and therefore there is a bijective mapping $\beta_2$, which gives the embedding mapping of $t_2^*$ onto $t_2$. If $\alpha : A_1 \longrightarrow A_2$ is an embedding mapping of $t_1$ into $t_2$ and $\beta : A_2 \longrightarrow A_3$ is an embedding mapping of $t_2$ into $t_3$ then $\alpha \circ \beta : A_1 \longrightarrow A_3$ is an embedding mapping of $t_1$ into $t_3$. It follows that the mapping $\beta_1 \circ \alpha \circ \beta_2$ is an embedding mapping of $t_1$ into $t_2$. This is a surjective mapping, which contradict the fact that $t_1 \prec t_2$ and □

## 6. Conclusions

In this section we defined and studied two strict partial orders: one relation is defined for $\omega$-trees and other order for equivalence classes of $\omega$-trees. The results presented in this paper complements some results presented in the paper [6], where an extension based on nonterminal leaves of an $\omega$-tree is studied.

## References


[1] A.V. Aho and J.D. Ullman, *The Theory of Parsing, Translation and Compiling*, Prentice-Hall, Vol. I, 1972.

[2] N. Ţăndăreanu, Master-Slave Systems of Semantic Schemas and Applications, *The 10th IASTED International Conference on Intelligent Systems and Control* (ISC 2007), November 19-21, (2007), Cambridge, Massachusetts, 150–155.

[3] N. Ţăndăreanu, An Extension of Semantic Schemas to Represent Multiple Meanings, *International Multi-Conference on Engineering and Technological Innovation* (IMETI 2008), June 29th- July 2nd (2008), Vol.I, 143–148.

[4] N. Ţăndăreanu and C.Zamfir, Algebraic properties of $\omega$-trees (I), *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), No. 1, 80–89.

[5] N. Ţăndăreanu and C.Zamfir, Algebraic properties of $\omega$-trees (II), *Annals of the University of Craiova, Mathematics and Computer Science Series* **37** (2010), No. 2, 7–17.

[6] N. Ţăndăreanu and C.Zamfir, Slices and extensions of $\omega$-trees, *Annals of the University of Craiova, Mathematics and Computer Science Series* **38** (2011), No. 1, 72–82.



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