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A special type of BL-algebra

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Abstract. In this paper, we introduce a special case of BL-algebras. We study this structure by stating and proving some theorems which give the relationship between this structure and other algebraic structures. Finally we introduce a special filter and study it in detail.

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1. Introduction

BL-algebra is the algebraic structure for Hájek basic logic (BL-Logic) [8], arising from the continuous triangular norms (t-norm), familiar in the framework of fuzzy set theory. The language of propositional Hájek basic logic [8] contains the binary connectives $\circ$, $\Longrightarrow$ and the constant $\overline{0}$.

Axiom of BL-logic are:
$(A1)$ $(\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow \omega))$.
$(A2)$ $(\varphi \circ \psi) \Rightarrow \varphi$.
$(A3)$ $(\varphi \circ \psi) \Rightarrow (\psi \circ \varphi)$.
$(A4)$ $(\varphi \circ (\varphi \Rightarrow \psi)) \Rightarrow (\psi \circ (\psi \Rightarrow \varphi))$.
$(A5a)$ $(\varphi \Rightarrow (\psi \Rightarrow \omega)) \Rightarrow ((\varphi \circ \psi) \Rightarrow \omega)$.
$(A5b)$ $((\varphi \circ \psi) \Rightarrow \omega) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \omega))$.
$(A6)$ $((\varphi \Rightarrow \psi) \Rightarrow \omega) \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \omega) \Rightarrow \omega)$.
$(A7)$ $\overline{0} \Rightarrow \omega$.

BL-algebras rise as Lindenbaum algebras from certain logical axioms in a similar manner that Boolean algebras or MV-algebras do from Classical logic or Lukasiewicz logic, respectively. MV-algebras are BL-algebras while the converse, in general, is not true. Indeed, BL-algebras with involutory complement are MV-algebras.

Moreover, Boolean algebras are MV-algebras and MV-algebras with idempotent product are Boolean algebras (for details, see e.g. [16]). Filter theory play an important role in studying these logical algebras. From logical point of view, various filters correspond to various sets of provable formula. Hájek introduced the concepts of filters and prime filters in BL-algebras. Using prime filters of BL-algebras, Hájek proved the completeness of Basic Logic BL. Turunen studied some properties of the prime filters of BL-algebras in [15]. Haveshki et al. in [9] continued the algebraic analysis of BL-algebras and introduced (positive) implicative and fantastic filters of BL-algebras. Borumand Saeid and Motamed defined the notions of normal filters and obstinate filters in [1] and [2], respectively.

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In continuing our study in BL-algebra, we define an algebraic structure which is weaker than BL-algebra and is a good step for better understanding this algebraic structure.

The structure of the paper is as follows: In section 2, we recall some definitions and facts about BL-algebras that we use in the sequel. In section 3, we introduce special kind of BL-algebras and we investigate some of its properties. This part of paper contains characterizations for dense elements of a special BL-algebra A^* and we prove that a BL-algebra A is special iff $\neg a = 0$, for ever $0 \neq a$. In section 4, we introduce special filter of BL-algebras and prove some theorems which determine the relationship between these notion and another types of filters in BL-algebra.

2. Preliminaries

Definition 2.1. *[8] A BL-algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:*
(BL1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
(BL2) $(A, *, 1)$ *is a commutative monoid,*
(BL3) $*$ *and* $\rightarrow$ *form an adjoint pair i.e,* $c \leq a \rightarrow b$ *if and only if* $a * c \leq b$,
(BL4) $a \wedge b = a * (a \rightarrow b)$,
(BL5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$, *for all* $a, b, c \in A$.

A BL-algebra is A called a Gödel algebra if $a * a = a$, for all $a \in A$. A BL- algebra A is called an MV-algebra if $\neg(\neg x) = x$ or equivalently $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for all $x, y \in A$, where $\neg x = x \rightarrow 0$.

Lemma 2.1. *[9] In each BL-algebra A, the following relations hold for all $x, y, z \in A$:*
(1) $x * (x \rightarrow y) \leq y$,
(2) $x \leq (y \rightarrow (x * y))$,
(3) $x \leq y$ *iff* $x \rightarrow y = 1$,
(4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(5) If $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$,
(6) $y \leq (y \rightarrow x) \rightarrow x$,
(7) $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$,
(8) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(9) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$,
(10) $x \leq y$ *implies* $x * z \leq y * z$,
(11) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \leq y \rightarrow x$, $x \rightarrow 1 = 1$,
(12) $x * \neg x = 0$,
(13) $x * y = 0$ *iff* $x \leq \neg y$ *and* $x \leq y$ *implies* $\neg y \leq \neg x$,
(14) $x \vee y = 1$ *implies* $x * y = x \wedge y$,
(15) $(x \rightarrow y) \rightarrow (x \rightarrow z) = (x \wedge y) \rightarrow z$,
(16) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,
(17) $x \rightarrow y \leq (x * z) \rightarrow (y * z)$,
(18) $x * (y \rightarrow z) \leq y \rightarrow (x * z)$,
(19) $(y \rightarrow z) * (x \rightarrow y) \leq (x \rightarrow z)$,
(20) $x \leq \neg\neg x$, $\neg 1 = 0$, $\neg 0 = 1$, $\neg\neg\neg x = \neg x$, $\neg\neg x \leq \neg x \rightarrow x$
(21) $\neg\neg(x * y) = \neg\neg x * \neg\neg y$,
(22) $x = \neg\neg x * (\neg\neg x \rightarrow x)$,
(23) if $\neg\neg x \leq \neg\neg x \rightarrow x$, *then* $\neg\neg x = x$,
(24) $x \rightarrow \neg y = y \rightarrow \neg x = \neg\neg x \rightarrow \neg y = \neg(x * y)$,
(25) $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$.

Definition 2.2. *[7] In each BL-algebra A, the order of an element $x \in A$, denoted by $ord(x)$, is $n(ord(x) = n)$, if there exist a smallest positive integer n such that $x^n = x * \cdots * x = 0$ and is $\infty(ord(x) = \infty)$, if no such n exist $x^n = 0$.*

Definition 2.3. *[8] A filter of a BL-algebra A is a nonempty subset F of A such that for all $a, b \in A$, we have:*
(1) $a, b \in F$ implies $a * b \in F$,
(2) $a \in F$ and $a \leq b$ imply that $b \in F$.

Definition 2.4. *[17] A proper filter M of a BL-algebra A is called maximal (or ultrafilter) if it is not properly contained in any other proper filter of A.*

Definition 2.5. *[7] Let A be a BL-algebra and F be a filter of A. F is called a prime filter if $x \vee y \in F$ implies $x \in F$ or $y \in F$.*

Theorem 2.1. *[7] Let A be a BL-algebra and F be a filter of A. F is a prime filter iff $x \to y \in F$ or $y \to x \in F$, for all $x, y \in A$.*

For any BL- algebra A, The reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. For any BL-algebra A, $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$).

An element a of A is said to be dense iff $\neg a = 0$. We denote by $D_s(A)$ the set of the dense elements of A [13].

We define dense elements of a filter F of A by $D_s(F) = \{x \in F : \neg x = 0\}$ [12].

Definition 2.6. *[12] The intersection of all maximal filter of a BL-algebra A is called the radical of A and it is denoted by $Rad(A)$ and $Rad(A) = \{a \in A : \neg(a^n) \leq a$, for any $n \in N^*\}$.*

Theorem 2.2. *[8] Let F be a filter of a BL-algebra A. Define:*

$$x \equiv_F y \quad \text{iff} \quad x \to y \in F \quad \text{and} \quad y \to x \in F.$$

Then $\equiv_F$ is a congruence relation on A and congruence classes is denoted by $[x]$ or x/F.

The set of all congruence classes is denoted by A/F, i.e., $A/F := \{[x] | x \in A\}$, where $[x] = \{y \in A | x \equiv_F y\}$.

Define $\bullet, \to, \sqcap, \sqcup$ on A/F, as follows:
$[x] \bullet [y] = [x * y]$,
$[x] \to [y] = [x \to y]$,
$[x] \sqcap [y] = [x \wedge y]$,
$[x] \sqcup [y] = [x \vee y]$,
Therefore $(A/F, \sqcap, \sqcup, \bullet, \to, [1], [0])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

Definition 2.7. *([1],[2],[9],[15]) A nonempty subset F of A is called:*

- *A Boolean filter of A if F is a filter of A and $x \vee (\neg x) \in F$,*
- *An implicative filter of A if $1 \in F$ and $x \to (y \to z) \in F$ and $x \to y \in F$ imply that $x \to z \in F$,*
- *A positive implicative filter of A if $1 \in F$ and $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$,*
- *A fantastic filter of A if $1 \in F$ and $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$,*
- *A normal filter of A if F is a filter of A and $z \to ((y \to x) \to x) \in F$ and $z \in F$ imply that $(x \to y) \to y \in F$,*

- *An obstinate filter of* A *if* F *is a filter of* A *and* $x, y \notin F$ *imply* $x \to y \in F$ *and* $y \to x \in F$,

for all $x, y, z \in A$.

Definition 2.8. *[13] A proper filter* F *of a* BL*-algebra* A *is called:*
(I) *primary iff, for all* $a, b \in A$, $\neg(a * b) \in F$ *implies that there exists* $n \in N^*$ *such that* $\neg a^n \in F$ *or* $\neg b^n \in F$.

(II) *quasi-primary iff, for all* $a, b \in A$, $\neg(a * b) \in F$ *implies that there exist* $u \in A$ *and* $n \in N^*$ *such that* $u \vee \neg u \in B(A), \neg(a^n * u) \in F$ *and* $\neg(b^n * \neg u) \in F$.

Definition 2.9. *[13]* (I) *A residuated lattice* A *is simple iff* $ord(a) < \infty$, *for every* $1 \neq a \in A$.
(II) *A residuated is said to be local iff it has exactly one maximal filter.*

3. Special BL-algebra

from now on $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra unless otherwise specified.

Definition 3.1. *A* BL*-algebra* A *is called special if it satisfies the following condition:*
(A_1^*) *for all* $0 \neq a, b \in A, \neg(a \to b) = \neg(b \to a)$.
Denoting a special BL*-algebra* A *by* A^*.

By the following example we show the relationship between special BL-algebra and other algebraic structures.

Example 3.1. (a) *Let* $A = \{0, a, b, c, 1\}$. *Define on* A *the following operations:*

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

We have $\neg(a \to b) = \neg(b \to a)$, *for all* $0 \neq a, b \in A$, *then* A *is a special* BL*-algebra.*

(b) *Let* $A = \{0, a, b, c, d, 1\}$. *Define on* A *the following operations:*

$\to$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

It is clear that A *is a* BL*-algebra. The condition* $\neg(x \to y) = \neg(y \to x)$*, for all* $0 \neq x, y \in A$ *dose not hold, since* $d = \neg(a \to b) \neq \neg(b \to a) = 0$*, hence* A *is not a special* BL*-algebra.*

(c) *Let* $A = \{0, a, b, 1\}$*. Define on* A *the following operations:*

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

$*$	1	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

We can see that A *is special* BL*-algebra, but* A *is not a Gödel algebra, since* $b^2 = a \neq b$*.*

Proposition 3.1. *For a* BL*-algebra* A *the following conditions are equivalent:*
(i) A *is a special* BL*-algebra,*
(ii) $\neg a = 0$*, for any* $0 \neq a \in A$*.*

Proof. $(i) \Rightarrow (ii)$ Let A be a special BL-algebra. Then we have $\neg(a \to b) = \neg(b \to a)$, for all $0 \neq a, b \in A$. Consider $b = 1$, therefore we have $\neg(a \to 1) = \neg(1 \to a)$ for all $0 \neq a \in A$. It is clear that $\neg a = 0$, for all $0 \neq a \in A$.

$(ii) \Rightarrow (i)$ Let $\neg a = 0$, for every $0 \neq a \in A$. Then there exist $0 \neq c, d \in A$ such that $\neg(a \to b) = \neg c = 0$ and $\neg(b \to a) = \neg d = 0$, for all $0 \neq a, b \in A$, since if $a \to b = 0$, we conclude that $a * b \leq b \leq a \to b = 0$, thus $a \to \neg b = 1$, then $a \leq \neg b = 0$, therefore $a = 0$, which is a contradiction. Hence $\neg(a \to b) = \neg(b \to a) = 0$, for every $0 \neq a, b \in A$. $\square$

Remark 3.1. *In every special* BL*-algebra we can see* F *is a filter of* A^* *iff is a filter of* A*.*

Remark 3.2. *If* A *is a non trivial* MV*-algebra, then* $\neg\neg a = a$*, for all* $a \in A$*. Hence* $D_s(A) = \{1\}$*. Therefore* $\neg a \neq 0$*, for all* $1 \neq a \in A$*, then* A *is not a special* BL*-algebra. If* A *is special* BL*-algebra, then* $\neg\neg a = 1$*, for all* $0 \neq a \in A$*. Therefore* A *is not an* MV*-algebra.*

Corollary 3.1. *Let* A *be a* BL*-algebra. Then* $A/D_s(A)$ *is not a special* BL*-algebra.*

Proof. We show that $A/D_s(A)$ is an MV-algebra, suppose that $A/D_s(A)$ is not an MV-algebra, then there exists $x \in A$ such that $\neg\neg(x/D_s(A)) \neq x/D_s(A)$. We have $x/D_s(A) \leq \neg\neg(x/D_s(A))$. Hence $\neg\neg(x/D_s(A)) \nleq x/D_s(A)$, implies that $\neg\neg(x/D_s(A)) \to x/D_s(A) \neq 1/D_s(A)$, implies $\neg\neg x \to x \notin D_s(A)$. Therefore we conclude that $\neg(\neg\neg x \to x) \neq 0$. Which is a contradiction. Since A is a BL-algebra, then $(A/D_s(A))$ is an MV-algebra. Then $(A/D_s(A))$ is not special BL-algebra. $\square$

Remark 3.3. *Let A be a special BL-algebra. Then we have A/F is a special BL-algebra, for all filter F of A. Therefore $A/D_s(A)$ is special BL-algebra.*

Proposition 3.2. *In any special BL-algebra A^*, the following properties hold:*

(1) $\neg\neg(\neg\neg a \to a) = 1$, for all $a \in A^*$,

(2) $a * b \neq 0$, for all $0 \neq a, b \in A^*$ such that $a \neq \neg b$ and $b \neq \neg a$,

(3) The unique maximal filter of A^* is $D(A^*) = \{a \in A^* : ord(a) = \infty\}$, so $D(A^*) = Rad(A^*) = A^* \backslash \{0\}$,

(4) $D_s(F) = D_s(A^*) \cap F = A^* \backslash \{0\} \cap F = F$, for all filter F of A^*,

(5) $A^*/Rad(F)$ and $A^*/D_s(A^*)$ and $A^*/Rad(A^*)$ are MV-algebra, for all filter F of A^*.

Proof. (1) We have $a \leq \neg\neg a$, for all $a \in A^*$, then

$$\begin{aligned} (a \to \neg\neg a) = 1 &\implies \neg(a \to \neg\neg a) = 0 \\ &\implies \neg(\neg\neg a \to a) = 0 \\ &\implies \neg\neg(\neg\neg a \to a) = 1. \end{aligned}$$

(2),(3) and (4) are clear.
(5) By Proposition 3.7 [13] and (3) we have $A^*/D_s(A^*)$, $A^*/Rad(A^*)$ and $A^*/Rad(F)$ are MV-algebra for all filter F of A^* $\square$

In the following we show that the converse of above proposition is not correct.

Example 3.2. (a) *Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define on A the following operation:*

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then A is a BL-algebra. It is clear that $\neg\neg(\neg\neg a \to a) = 1$, for all $a \in A$. But it is not a special BL-algebra, since $a = \neg(b \to a) \neq \neg(a \to b) = 0$.

(b) *Consider above BL-algebra, we can see that the unique maximal filter of A is $\{1, b\} = D(A) = \{a \in A : ord(a) = \infty\}$ but it is not a special BL-algebra.*

(c) *Let $A = \{0, a, b, 1\}$. Define on A the following operations:*

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

It is clear that for all filter of A, $D_s(F) = F$, but it is not a special BL-algebra.

(d) Let $A = \{0, a, b, c, d, 1\}$. Define on A the following operations:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

It is clear that $A/Rad(F)$ and $A/D_s(F)$ and $A/Rad(A)$ are MV-algebra, for all filter F of A but it is not a special BL-algebra.

Theorem 3.1. *In any BL-algebra A, the following conditions are equivalent:*
(1) A is special BL-algebra,
(2) $a \to \neg\neg b = b \to \neg\neg a$, for all $0 \neq a, b \in A$,
(3) $\neg\neg a \to \neg\neg b = \neg\neg b \to \neg\neg a$, for all $a, b \in A \setminus \{0, 1\}$,
(4) $ord(a) = \infty$ and $ord(\neg a) = 1$, for all $0 \neq a \in A$,

Proof. (1)⇒(2) By Proposition 3.2 and Lemma 2.1, for all $a, b \in A^*$, we have

$$\begin{aligned} 1 = \neg\neg(\neg\neg b \to b) &\leq \neg\neg((a \to \neg\neg a) \to (a \to b)), \\ &\leq \neg\neg((a \to \neg\neg b) \to \neg\neg(a \to b)), \\ &= \neg((a \to \neg\neg b) * \neg(a \to b)), \\ &= (a \to \neg\neg b) \to \neg\neg(a \to b). \end{aligned}$$

Hence $(a \to \neg\neg b) \leq \neg\neg(a \to b)$.

Thus by Lemma 2.1. (6), (5) and (4) have

$$\begin{aligned} \neg\neg(a \to b) &\leq \neg\neg(a \to \neg\neg b), \\ &= \neg\neg(\neg(a * \neg b)), \\ &= \neg(a * \neg b), \\ &= a \to \neg\neg b. \end{aligned}$$

Then we have $\neg\neg(a \to b) = (a \to \neg\neg b)$.

Hence

$$\begin{aligned} a \to \neg\neg b &= \neg\neg(a \to b), \\ &= \neg\neg(b \to a), \\ &= b \to \neg\neg a. \end{aligned}$$

(2)⇒(1) Let A be a BL-algebra and $a \to \neg\neg b = b \to \neg\neg a$, for all $a, b \in A$. We have

$$\begin{aligned} \neg(a \to b) &= \neg\neg\neg(a \to b), \\ &= \neg(\neg\neg a \to \neg\neg b), \\ &= \neg\neg(a * \neg b), \\ &= \neg(a \to \neg\neg b), \\ &= \neg(b \to \neg\neg a), \\ &= \neg\neg(b * \neg a), \\ &= \neg(\neg\neg b \to \neg\neg a), \\ &= \neg(b \to a). \end{aligned}$$

Hence A is special BL-algebra.

(3)⇒(1) Let A be a BL-algebra and $\neg\neg a \to \neg\neg b = \neg\neg b \to \neg\neg a$, for all $0 \neq a, b \in A$. We have

$$\begin{aligned} \neg(a \to b) &= \neg\neg\neg(a \to b) \\ &= \neg(\neg\neg a \to \neg\neg b) \\ &= \neg(\neg\neg b \to \neg\neg a) \\ &= \neg(b \to a). \end{aligned}$$

Hence A is a special BL-algebra.

(1)⇒(3) In every BL-algebra A, we have $\neg\neg(a \to b) = \neg\neg a \to \neg\neg b$. Then:

$$\begin{aligned} \neg\neg a \to \neg\neg b &= \neg\neg(a \to b) \\ &= \neg\neg(b \to a) \\ &= \neg\neg b \to \neg\neg a. \end{aligned}$$

(1)⇒(4) It is clear.

(4)⇒(1) If $ord(\neg a) = 1$, for all $0 \neq a \in A$. Then we have $\neg a = 0$, for all $0 \neq a \in A$. Hence by Proposition 3.1 we can conclude that A is a special BL-algebra. □

We recall that a SBL-algebra is a BL-algebra that satisfy $x \wedge \neg x = 0$.

Proposition 3.3. *(1) If A is a special BL-algebra, then A is a SBL-algebra.*

(2) If A is a linear SBL-algebra, then A is a special BL-algebra.

In the following example we show that the converse of part (1) of above proposition is not correct.

Example 3.3. *Let $A = \{0, a, b, 1\}$. Define on A the following operations:*

$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

It is clear that A is a SBL-algebra, but it is not a special BL-algebra.

Proposition 3.4. *A^*/F is not MV-algebra, for all proper filter F of A^*.*

Proof. Suppose that there exists a proper filter F of A^* such that A^*/F is an MV-algebra. By definition of an MV-algebra we get that $\frac{\neg\neg x}{F} = \frac{x}{F}$, for all $x \in A^*$. Therefore $(\neg\neg x \rightarrow x) \in F$ and $(x \rightarrow \neg\neg x) \in F$. Then $x \in F$ and $1 \in F$, for all $x \in A^*$, since $\neg a = 0$, for all $a \in A^*$. Thus $A^* = F$, which is a contradiction. □

Proposition 3.5. *Let A be an MV-algebra. Then A/F is a special BL-algebra iff F is a maximal filter of A.*

Proof. A/F is special BL-algebra iff $\neg\neg a \in F$, for all $0 \neq a \in A$ iff $a \in F$, for all $0 \neq a \in A$ iff $F = A \backslash \{0\} = M$. □

By the following example we study the relationship between special BL-algebra and simple BL-algebra.

Example 3.4. *Consider BL-algebra $A = \{0, a, b, 1\}$ in Example 3.2, part (c) it is clear that A is simple but it is not a special BL-algebra.*

Let A be a special BL-algebra. Then by Theorem 3.1, part (4) we have $ord(a) = \infty$, for all $0 \neq a \in A$. Hence A is not a simple BL-algebra.
Consider BL-algebra $A = \{0, a, b, c, 1\}$ in Example 3.1 (a) it is clear that A is a special BL-algebra but it is not a simple BL-algebra.

Proposition 3.6. *Every special BL-algebra is a local BL-algebra.*

Proof. Let A be a special BL-algebra. By Theorem 3.1, part (4) we have $ord(a) = \infty$, for all $0 \neq a \in A$, then A has exactly one maximal filter which is $D(A) = \{a \in A : ord(a) = \infty\}$. □

But by the following example we show that every local BL-algebra is not a special BL-algebra.

Example 3.5. *Let $A = \{0, a, b, c, d, 1\}$. Define on A the following operations:*

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

It is clear that A is a local BL-algebra but it is not a special BL-algebra.

4. Filter theory in A^*

Theorem 4.1. *Let F be a filter of A^*. Then F is a positive implicative filter of A^* if and only if F is a Boolean filter of A^* if and only if F is an obstinate filter of A^* if and only if F is a maximal filter of A^* if and only if $F = A^*\backslash\{0\}$.*

Example 4.1. *Let $A = \{0, a, b, c, 1\}$. Define on A the following operations:*

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

Then A is a special BL-algebra and $F = \{1, b\} \neq A\backslash\{0\}$ is an implicative filter of A^*, but it is not a maximal filter of A^*.

Proposition 4.1. *If F is maximal (positive, implicative, obstinate, normal, fantastic) filter of A^*, then A^*/F is a Boolean algebra.*

Corollary 4.1. *Any proper filter in special BL-algebra A is primary and quasi-primary filter.*

Proposition 4.2. *Let A/P be a special BL-algebra. Then P is a primary filter of A.*

Proof. Assume that A/P is a special BL-algebra and $\neg(x * y) = (y \rightarrow \neg x) \in P$, for some $x, y \in A$. Then $y/P \rightarrow \neg x/P = (y \rightarrow \neg x)/P = 1/P$, so $y/P \leq \neg x/P$. Assume that $\neg(x^n) \notin P$, for all $n \in N$. Then $\neg(x^n)/P \neq 1/P$, hence $(x^n)/P \neq 0/P$. Since A/P is a special BL-algebra $\neg x/P = 0/P$. Therefore also $(y^m)/P \leq (\neg x)^m/P = 0/P$, for some $m \in N$. Whence $(y^m)/P = 1/P$, i.e. $\neg(y^m) \in P$. Thus P is primary. □

Remark 4.1. *Consider BL-algebra in Example 3.2 (a), it is clear that $F = \{1, b\}$ is a primary filter, but A/F is not special BL-algebra because $\neg\neg a = a \notin F$.*

In the following example we show the relationship between F and A/F.

Example 4.2. *(a) Let* $A = \{0, a, b, c, 1\}$*. Define on* A *the following operations:*

$\rightarrow$	1	0	a	b	c
1	1	0	a	b	c
0	1	1	1	1	1
a	1	0	1	1	1
b	1	0	c	1	c
c	1	0	b	b	1

$*$	1	0	a	b	c
1	1	0	a	b	c
0	0	0	0	0	0
a	a	0	a	a	a
b	b	0	a	b	a
c	c	0	a	a	a

It is clear that A *is a special* BL*-algebra. We get that* $F = \{1\}$ *is a proper filter of* A^* *but it is not a prime filter because* $b \vee c = 1 \in F$ *but* $b, c \notin F$*.*

(b) Consider BL*-algebra* $A = \{0, a, b, 1\}$ *in Example 3.2 part (a), it is clear that* $F = \{1, b\}$*, is prime, primary, normal and maximal, but* A/F *is not special* BL*-algebra.*
(c) Consider BL*-algebra* $A = \{0, a, b, c, d, 1\}$ *in Example 3.2, part (d) it is clear that* $F = \{1, c, d\}$*, is positive implicative and Boolean, but* A/F *is not special* BL*-algebra.*
(d) Consider BL*-algebra* $A = \{0, a, b, c, d, 1\}$ *in Example 3.2, part (b) it is clear that* $F = \{1, c\}$*, is fantastic filter of* A*, but* A/F *is not a special* BL*-algebra.*

Proposition 4.3. *Let* F *be a proper obstinate filter of* A *and* $\neg x \notin F$*, for all* $0 \neq x \in A$*. Then* A/F *is a special* BL*-algebra.*

Proof. If $\neg x \notin F$, for all $0 \neq x \in A$ and F is an obstinate filter, we can get that $\neg\neg x \in F$, then $\neg x/F = 0/F$, hence A/F is a special BL-algebra. ☐

Definition 4.1. *A proper filter* F *of a* BL*-algebra* A *is called special filter iff* $\neg(a \rightarrow b) = \neg(b \rightarrow a)$*, for all* $a, b \in F$*.*

Example 4.3. *Consider* BL*-algebra* $A = \{0, a, b, 1\}$ *in Example 3.2 (a). It is clear that* $F = \{1, b\}$ *is a special filter of* A*.*

Proposition 4.4. F *is a special filter of* A *iff* $D_s(F) = \{x \in F : \neg x = 0\} = F$*.*

Proof. It is clear that $D_s(F) \subseteq F$. If $a \in F$, then $\neg(a \rightarrow 1) = \neg(1 \rightarrow a)$, thus $\neg a = 0$. Therefore $a \in D_s(F)$.

Conversely, if $F = D_s(F)$, then $\neg a = \neg b = 0$, for all $a, b \in F$. In the other hand we have $a \leq b \rightarrow a$, then $\neg(b \leq a) = \neg a = 0$. Hence $\neg(a \rightarrow b) = \neg(b \rightarrow a) = 0$, for all $a, b \in F$. Therefore F is a special filter of A. ☐

Proposition 4.5. *For all proper filter* F *of* A*,* $D_s(F) = F$ *iff* A *is a special* BL*-algebra.*

Proof. If $D_s(F) = F$, for all filter F of A, hence A is a special BL-algebra.

Conversely, if A is a special BL-algebra, then we have $\neg a = 0$, for all $0 \neq a \in A$. Therefore $D_s(A) = F$, for all filter F of A. ☐

Proposition 4.6. *Let F be a filter of A. Then F is special iff* $\{[x] \in A/F : \neg\neg[x] = [1]\} = \{1\}$.

Proof. Let F be a special filter of A, then by Proposition 4.4, we have $F = D_s(F)$ and let $[x] \in A/F$ such that $\neg\neg[x] = [1]$. Then we have $[\neg\neg x] = \neg\neg[x] = [1]$. Hence $\neg\neg x \in F$, so $x \in D_s(F)$. By hypothesis we get that $x \in F$, therefore $[x] = [1]$. Hence $\{[x] \in A/F : \neg\neg[x] = [1]\} = \{1\}$.

Conversely, let $x \in D_s(F)$. Then $\neg\neg x \in F$, thus $\neg\neg[x] = [\neg\neg x] = [1]$. Hence $[x] \in \{[x] \in A/F : \neg\neg[x] = [1]\} = \{1\}$. Thus by hypothesis $[x] = [1]$. So $x \in F$. Therefore $D_s(F) \subseteq F$. Let $x \in F$ by Lemma 2.1, we have $x \leq \neg\neg x$. So $\neg\neg x \in F$ and then $x \in D_s(F)$. Therefore $F \subseteq D_s(F)$ and we conclude that $F = D_s(F)$, hence F is special filter of A. □

We determine the relationship between the special filter and the other types of filters in BL-algebra.

Proposition 4.7. *If F is a maximal filter of A, then F is special filter.*

Proof. It is clear that $F \subseteq D_s(F) \subset A$, since F is a maximal filter of A we get that $F = D_s(F)$. Therefore F is special. □

Corollary 4.2. *If A be special BL-algebra, then $Rad(A/D_S(F)) = Rad(F)/F$, for all filter F of A.*

Proof. By Proposition 3.8 [12], we have $Rad(A/D_S(F)) = Rad(F)/D_S(F)$, then we conclude that $Rad(A/D_S(F)) = Rad(F)/F$. □

By the following example we show that F be special filter of A, but A/F is not special BL-algebra.

Example 4.4. *Consider BL-algebra $A = \{0, a, b, 1\}$ in Example 3.2 part (a), it is clear that $F = \{1, b\}$, is special filter, but A/F is not special BL-algebra.*

In the following example we show that extension property dose not hold for special filters.

Example 4.5. *Consider BL-algebra $A = \{0, a, b, c, 1\}$ in Example 3.2, part (a). It is clear that $G = \{1, b\}$ and $F = \{1\}$, are filter such that $F \subseteq G$. Therefore F can not extended to G, since F is special but G is not a special filter.*

5. Conclusion

In this paper, we introduced a special case of BL-algebras and named it A^*. We presented a characterization and many important properties of A^*. Moreover, we gave some example for A^* and showed the relationship between special BL-algebra and other algebraic structures. In addition we proved that the unique maximal filter of A^* is $D(A^*)$. In any A^* we had $A^*/Rad(A^*)$, $A^*/Rad(F)$ and $A^*/D_s(A^*)$ are MV-algebra, for all filter F of A^* but A^*/F is not MV-algebra, for all proper filter F of A^*. Also we studied some types of filters in A^* and proved some theorems that determined relationship between this notion and other types of filters of A^*.

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